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A GENERALIZATION OF THOM CLASSES AND CHARACTERISTIC CLASSES TO NONSPHERICAL FIBRATIONS

REINHARD SCHULTZ

Let X be a polyhedron, and let F_X denote the contravariant functor consisting of fiber homotopy types of Hurewicz fibrations over a given base whose fibers are homotopy equivalent to X. A fundamental theorem on fiber spaces states that F_X is a representable homotopy functor and a universal space for F_X is the classifying space for the topological monoid of self-equivalences of X [**2**; **5**]. Frequently, algebraic topological information about the associated universal fibration yields information about arbitrary fibrations with fiber (homotopy equivalent to) X. However, present knowledge of the algebraic topological properties of the universal base space is extremely limited except in some special cases. Consequently, no systematic method of studying the properties of these universal fibrations exists.

Perhaps the simplest algebraic topological problem involving the universal fibration is the determination of the image of the cohomology of the universal total space in $H^*(X)$; of course, this image is contained in the image of an arbitrary total space by naturality. A variant of the above problem is the determination of the image of the cohomology of the total space of a universal *orientable* fibration induced on a suitable covering of the universal base space. We shall show that the algebraic structure of $H^*(X)$ yields some information on this problem. As illustrations of this principle, we shall give examples of noncontractible 1-connected finite complexes for which the restriction maps are onto; it follows that every fibration whose fibers are (homotopically) these complexes have a fiber which is totally nonhomologous to zero and a collapsing Serre spectral sequence (see Proposition 6). We shall also give examples of closed 1-connected manifolds for which the restriction maps of the induced orientable fibrations are onto (see Proposition 5).

Classes in the image of restriction to $H^*(X)$ arise from suitable characteristic classes defined for oriented sectioned X-fibrations. One motivation for this is the existence of a canonical cross section on the pullback of the universal X-fibration $p : E \to B$ to E; explicitly, this cross section sends $e \in E$ to $(e, e) \in \{(u, v) \in E \times E | p(u) = p(v)\}$. Since the restriction of this fibration to a fiber $X \subseteq E$ is the trivial fibration $\pi_1 : X \times X \to X$ and the canonical cross section pulls back to the diagonal map, computations of the classes for

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this example are often quite simple. It follows that these easily computed classes must extend to $H^*(E')$, where E' is the total space of any oriented X-fibration (compare Proposition 0). In the examples mentioned above, these classes generate $H^*(X)$.

We shall conclude this paper by discussing some applications of these characteristic classes along the lines of [**3**, § 8].

I wish to thank D. Gottlieb for discussing certain aspects of his results in [**3**] and related papers with me.

1. Generalized Thom classes. In this section all cohomology groups are assumed to have coefficients in a fixed commutative ring with unit.

Let

$$X \to E \xrightarrow{p} B$$

be the universal Hurewicz fibration and let $\pi \subseteq \pi_1(B)$ be the ineffective kernel of the natural action on $H^*(X)$. If $q : B' \to B$ is the regular covering associated to π, then q is the smallest covering map for which the induced fibration over B' is H^*-orientable; furthermore, fundamental theorems on covering spaces imply that the classifying map of any H^*-orientable X-fibration factors through g. We define an *orientation* of an X-fibration to be a lifting of the classifying map to B'. Let E' be the total space of the induced fibration over B', and let $p' : E' \to B'$ be the projection. Let $q' : E' \to E$ be the canonical map.

Following standard terminology, we define a sectioned fibration to be a fibration together with a specified cross section. We define oriented sectioned X-fibrations and oriented sectioned fiber homotopy equivalences of such objects in the obvious manner.

There is a canonical oriented sectioned X-fibration ξ' over E' which is induced by the composite $qp' = pq'$. Its orientation is p' and its cross section is induced from the canonical cross section of E via q'. This example has an important naturality property.

PROPOSITION 0. *Let K be a CW complex and let ξ be an oriented sectioned X-fibration. Then there is a well-defined homotopy class $w_\xi \in [K, E']$ such that if $[f] = w_\xi$, then there is an oriented sectioned fiber homotopy equivalence from ξ to $f^*\xi'$.*

Proof. Let $g : K \to B'$ be an orientation, and let $h : K \to E$ be a lifting of qg induced by the cross section. Then there is a factorization of

$$(g, h) : K \to B' \times E$$

through E', and we call this factorization f. By construction, there is an oriented sectioned fiber homotopy equivalence from ξ to $f^*\xi'$; it remains to show that f is well-defined up to homotopy. But this is a straightforward consequence of the covering homotopy property applied to the fibrations p and q.

The following is a simple consequence of the Serre spectral sequence:

PROPOSITION 1. *Let X be q-connected (where $q \geqq 1$), and let $p_0 : E_0 \to B_0$ be an orientable sectioned X-fibration with cross section σ. Then the restriction map from $H^{q+1}(E_0, \sigma(B_0))$ to $H^{q+1}(X)$ is bijective.*

If X is a sphere, the inverse image of the generator of $H^{q+1}(X) = Z$ is the ordinary Thom class with respect to a suitable orientation.

Definition. Let X be as above, and let $u \in H^*(X)$. A system of *generalized Thom classes* associated to u is a map assigning to each oriented sectioned X-fibration $\xi = (E_0 \to B_0)$ a class $U(u, \xi) \in H^r(E_0, \sigma(B_0))$ satisfying the following conditions:

(i) If $j : X \to (E_0, \sigma(B_0))$ is a fiber inclusion, then $j^*U(u, \xi) = u$.

(ii) If $f : B_1 \to B_0$ is a continuous map, then $U(u, f^*\xi) = f^*U(u, \xi)$.

COROLLARY 2. *Let X be q-connected where $q > 1$ and let $u \in H^{q+1}(X)$. Then a unique system of generalized Thom classes associated to u exists.*

Proof. Use Proposition 1 to define the class U in the universal case, and define them in all other cases using rule (ii). By Proposition 1 the class $U(u, \xi)$ is the unique class that restricts to $u \in H^{q+1}(X)$.

2. The characteristic classes of a relation. In this section we shall restrict our attention to a q-connected space X whose cohomology over some fixed commutative ring with unit is a free graded module; all cohomology groups in this section are assumed to have coefficients in this ring.

Let $u_1, \ldots, u_r \in H^{q+1}(X)$, and assume that $v \in H^*(X)$ is expressible as a polynomial in these elements; explicitly, write $v = f(u_1, \ldots, u_r)$. If $U(u_1, \xi), \ldots, U(u_r, \xi)$ are the systems of generalized Thom classes associated to $u_1, \ldots, u_r$, then the expression

$$U(v, \xi) = f(U(u_1, \xi), \ldots, U(u_r, \xi))$$

defines a system of generalized Thom classes associated to v.

Suppose that $v = 0$; in other words, $f(u_1, \ldots, u_r) = 0$ is a nontrivial polynomial relation satisfied by the elements of $H^{q+1}(X)$. Although $U(v, \xi)$ defines a system of generalized Thom classes associated to zero in this case, it need not be zero. For example, if q is odd, $X = S^{q+1}$, $r = 1$, and $u \in H^{q+1}(S^{q+1})$, then $u^2 = 0$ is a nontrivial relation which holds in $H^*(S^{q+1})$ but $U(u, \xi)^2$ vanishes if and only if the Euler class of ξ does. We shall construct similar characteristic classes for other spaces; the vanishing of these classes is a necessary condition for the vanishing of $U(v, \xi)$.

THEOREM 3. *Let $u_1, \ldots, u_r \in H^*(X)$, let $0 = f(u_1, \ldots, u_r) \in H^k(X)$ be a polynomial relation satisfied by the u_i, and let $U(u_i, \xi)$ be a system of generalized Thom classes associated to u_i. Assume that $H^i(X) = 0$ for $k > i > k - m$ and a free basis for $H^{k-m}(X)$ may be constructed using polynomials in the classes*

u_i (*say* $g_1, \ldots, g_s$). *Then there exists a unique family of classes* $z_i(\xi) \in H^m(B_0)$ $(1 \leqq i \leqq s)$ *such that:*

(*i*) $f(U(u_1, \xi), \ldots U(u_r, \xi)) = \sum p^* z_i(\xi) \cdot g_i(U(u_1, \xi) \ldots, U(u_r, \xi))$

modulo terms of filtration $m + 1$ *in the Serre spectral sequence* ($p : E_0 \to B_0$ *is the projection*).

(ii) *If* $h : B_1 \to B_0$ *is continuous, then* $z_i(h^*\xi) = h^* z_i(\xi)$.

We call the classes $z_i(\xi)$ the *characteristic classes of* ξ *associated to* the *generalized Thom classes* $U(u_i, \xi)$, the *relation* $f(u_1, \ldots, u_r) = 0$, and the *basis* $\{g_1, \ldots, g_s\}$.

Proof. The hypothesis on $H^{k-m}(X)$ implies that $E_2^{0,k-m} = E_\infty^{0,k-m}$ in the Serre spectral sequence for ξ, and the existence of a cross section implies that $E_2^{t,0} = E_\infty^{t,0}$ for all t. Thus the multiplicative properties of the Serre spectral sequence imply that $E_2^{t,k-m} = E_\infty^{t,k-m}$; since $H^{k-m}(X)$ is free, the E_2 term is a tensor product, and it follows that $E_\infty^{t,k-m} \cong E_\infty^{0,k-m} \otimes H^t(B)$ under the map induced by the cup product pairing on E_∞. By hypothesis $E_\infty^{0,k-m}$ is freely generated by the images of the classes $g_i(U(u_1, \xi), \ldots, U(u_r, \xi))$.

Since $f(u_1, \ldots, u_r) = 0$, the restrictions of $V = f(U(u_1, \xi), \ldots, U(u_r, \xi))$ to $H^*(X)$ is also zero by naturality; thus V has positive Serre filtration in $H^k(E_0, \sigma(B_0))$. The hypothesis that the last nonzero cohomology of X before dimension k occurs in dimension $k - m$ implies that V has filtration at least m and hence determines a unique element in $E_\infty^{m,k-m}$. Part (i) of the theorem follows from the tensor product representation of $E_\infty^{m,k-m}$ given above. Part (ii) follows from elementary naturality considerations.

The Euler class arises as a special case of the above theorem for which $u \in H^q(S^q)$ is a generator and $f(u) = u^2$. Grothendieck and others have used a construction similar to Theorem 3 to define the Chern classes of a complex vector bundle (see [**1**, pp. 42–47]).

3. Examples and applications. Obviously, the usefulness of Theorem 3 for constructing characteristic classes depends strongly upon knowledge of the ring structure of $H^*(X)$ and the existence of very few (if any) cohomology generators in dimensions greater than q. We shall only consider examples in which $H^*(X)$ is generated by $H^q(X)$ where X is $(q-1)$-connected; the class of all such spaces contains all products $S^q \times \ldots \times S^q$ and (if $q = 2, 4, 8$) appropriate projective spaces.

Assume that X is $(q-1)$-connected and $u \in H^q(X)$; we wish to compute $U(u, \xi)$ for ξ the trivial bundle $\pi_1 : X \times X \to X$ with the diagonal cross section. Denote the diagonal of X in $X \times X$ by Δ_X; the restriction map $H^*(X \times X)$ to $H^*(\Delta_X)$ has a one-sided inverse given by $w \to w \times 1$, and hence

$$H^k(X \times X, \Delta_X)$$

consists of all expressions in

$$H^k(X \times X) \cong \sum H^i(X) \otimes H^{k-i}(X)$$

whose cup products add up to zero. Since $i_2^* U(u, \xi) = u$ and $H^q(X \times X) \cong H^q(X) \otimes Z \oplus Z \otimes H^q(X)$ it follows that $U(u, \xi) = 1 \times u + v \times 1$ for some $v \in H^q(X)$; since the cup products add up to zero, $v = -u$ must hold. Thus we have the following:

Formula 4. If X is $(q - 1)$-connected, $u \in H^q(X)$, and ξ is the above sectioned fibration, then $U(u, \xi) = 1 \times u - u \times 1$.

This formula and Theorem 3 are useful for producing examples of nonzero characteristic classes.

Example 1. Let $X = S^q \times \ldots \times S^q$ (r factors) and let $u_1, \ldots, u_r$ be co-spherical generators of $H^q(X)$; assume q is even. Then $u_i^2 = 0$ for all i and hence there exist characteristic classes $z^i{}_j \in H^2(B_0)$ associated to $u_i^2 = 0$ and the $\{u_i\}$ basis of $H^q(S^q \times \ldots \times S^q)$. If ξ is the above sectioned bundle over X, a routine computation shows that $z^i{}_j = 0$ if $i \neq j$ and $z^i{}_i = -2u_i$.

Example 2. Let A be the complex numbers, quaternions, or Cayley numbers, and let $X = AP^n$ be A-projective n-space ($n \leqq 2$ for the Cayley numbers). Let $v \in H^a(AP^n)$ be a generator ($a = \dim A$). Then there is a characteristic class $z \in H^a(B_0)$ associated to the relation $v^{n+1} = 0$ and the basis $\{v^n\}$ of $H^{an}(AP^n)$. Another routine computation shows that $z(\xi) = (n + 1)v$ in this case.

Example 3. Let X be the connected sum $CP^n \# S^2 \times \ldots \times S^2$ (n factors). Then $\tilde{H}^*(X)$ is isomorphic to $\tilde{H}^*(CP^n) \oplus \tilde{H}^*(S^2 \times \ldots \times S^2)$ modulo the relation $v^n = u_1 \ldots u_n$. In this case one can perform both of the above constructions independently and obtain characteristic classes whose values for ξ are $(n + 1)v$ and $-2u_1, \ldots, -2u_n$. On the other hand, one can consider the relations $u_i v = 0$ and take characteristic classes corresponding to the basis of $H^2(X)$ given by u_i and v; in this case one obtains characteristic classes whose values for ξ are $\pm u_i$ and $\pm v$. It follows that *every element* in $H^*(X)$ is a *polynomial in characteristic classes for* ξ.

We previously noted that ξ is the restriction to X of a sectioned fibration over E', the total space of the universal orientable fibration; let $p' : E' \to B'$ be the projection. Then we have the following application of the above algebraic manipulations:

PROPOSITION 5. *Let $p_0 : E_0 \to B_0$ be an orientable Hurewicz fibration with fiber $S^2 \times \ldots \times S^2 \# CP^n$. Then the integral cohomology Serre spectral sequence for p_0 collapses.*

Proof. By the above discussion, the restriction map

$$H^*(E') \to H^*(S^2 \times \ldots \times S^2 \# CP^n)$$

is surjective in the universal case. Since $H^*(S^2 \times \ldots \times S^2 \# CP^n)$ is a free graded abelian group, the result follows in the universal case by standard considerations. If $p_0 : E_0 \rightarrow B_0$ is an arbitrary oriented fibration with fiber $S^2 \times \ldots \times S^2 \# CP^n$, the inclusion of F in E' factors through the inclusion of F in E_0 by the universality of p', and this factorization yields the result for arbitrary fibrations.

If we modify the above example slightly, we can obtain finite complexes X for which $H^*(E) \rightarrow H^*(X)$ is always onto (orientability being automatic).

PROPOSITION 6. *There exist infinitely many homotopy inequivalent finite complexes X satisfying the following conditions:*

(i) *X is 3-connected and non-contractible.*

(ii) *The integral cohomology Serre spectral sequence for every X-fibration collapses.*

Proof. Let $\nu \in \pi_7(S^4)$ be the Hopf map, let $n \geqq 0$ be a positive integer, and let $C(k\nu)$ denote the mapping cone of $k\nu$. Construct a "connected sum" of $C(\nu)$ and $C((24n + 2)\nu)$ by imitating the construction for manifolds on the top-dimensional cells. This yields 3-connected cell complexes X_n with $\tilde{H}^q(X_n) = 0$ unless $n = 4, 8$ while $\tilde{H}^4(X_n)$ has two infinite cyclic generators x, y and $\tilde{H}^8(X_n)$ has one generator z. These generators satisfy the multiplicative relations $xy = 0$, $x^2 = z$, $y^2 = (24n + 2)z$. An elementary cup product argument shows that X_m and X_n are homotopy equivalent if and only if $m = n$.

As in the proof of Proposition 5, the relation $xy = 0$ yields characteristic classes whose values for the diagonally sectioned trivial fibration are $\pm x$ and $\pm y$. Since $x^2 = z$, (ii) is valid for H^*-orientable fibrations.

It suffices to show that every X_n-fibration is H^*-orientable; equivalently, it suffices to show that every homotopy self-equivalence of X_n induces the identity in cohomology. An elementary cup product argument like the previous one shows that if $f : X_n \rightarrow X_n$ is a homotopy self-equivalence, then $f^*x = \pm x$ and $f^*y = \pm y$. Since we can assume f is cellular, it follows that f induces homotopy self-equivalences of X_n mod either basic 4-cell collapsed to a point. But the complexes obtained this way are $C(\nu)$ and $C((24n + 2)\nu)$, and every homotopy self-equivalence of these induces the identity in cohomology (this follows from cup product considerations and the fact that the suspensions of ν and $(24n + 2)\nu$ in $\pi_8(S^5)$ are not 2-torsion). Hence a simple diagram chase shows that f also induces the identity in cohomology.

Let M be a closed orientable topological n-manifold, and let $E \rightarrow B$ be an orientable fiber bundle with fiber M. A result of Gottlieb states that $\chi(M)\mu \in H^n(M)$ is in the image of $H^n(E)$, where μ is an orientation class and $\chi(M)$ denotes the Euler characteristic [**3**, § 8]. If $\chi(M) \neq 0$, it is natural to ask whether $|\chi(M)|$ is the lowest positive multiple of μ in the image of $H^n(E)$. It is not difficult to find examples of manifolds for which $|\chi(M)|$ is the lowest positive multiple of μ in the image of $H^n(E)$; any simply connected closed manifold with Euler characteristic ± 1 will suffice. On the other hand, evaluation map considerations show that $n + 1 = \chi(CP^n)$ is the smallest positive multiple which works for CP^n (compare [**4**, Theorem 16]). Thus one might conjecture that the answer to the question is yes for all M. However, this is not the case for $M = S^2 \times \ldots \times S^2 \# CP^n$, since the Euler characteristic of this manifold is $2^n + n - 1$ while the restriction map $H^{2n}(E) \rightarrow H^{2n}(M)$ is surjective by Proposition 5.

Added in proof. Infinitely many inequivalent X_n fibrations over S^8 are distinguishable by the sectioning obstruction in $\pi_7(X_n)$. Certain smoothable closed-up plumbing manifolds also satisfy the conclusion of Proposition 6.

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Purdue University,
West Lafayette, Indiana

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