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REPRESENTATION FORMULAS FOR INTEGRABLE AND ENTIRE FUNCTIONS OF EXPONENTIAL TYPE II

CLÉMENT FRAPPIER

1. **Introduction.** We adopt the terminology and notations of [5]. If $f \in B_\tau$ is an entire function of exponential type τ bounded on the real axis then we have the complementary interpolation formulas [1, p. 142–143]

$$\sin\gamma\, f'(t) + \tau\cos\gamma\, f(t) = \tau \sum_{k=-\infty}^{\infty} (-1)^k \frac{\sin^2\gamma}{(k\pi+\gamma)^2} f\left(\frac{k\pi+\gamma}{\tau}+t\right) \tag{1}$$

and

$$\sin\gamma\, f'(t) - \cos\gamma\, \tilde{f}'(t) = 2\tau \sum_{k=-\infty}^{\infty} (-1)^k \frac{\sin^2\left(\frac{k\pi+\gamma}{2}\right)}{(k\pi+\gamma)^2} f\left(\frac{k\pi+\gamma}{\tau}+t\right) \tag{2}$$

where t, γ are reals and

$$\tilde{f}(t) := \frac{it}{\sqrt{2\pi}} \int_{-\tau}^{\tau} \operatorname{sign}(u) e^{iut} \psi(u)\, du \tag{3}$$

is the conjugate function associated to f, which has always a representation of the form [1, p. 138]:

$$f(t) = f(0) + \frac{t}{\sqrt{2\pi}} \int_{-\tau}^{\tau} e^{iut}\psi(u)\, du, \tag{4}$$

with $\psi \in L^2(-\tau,\tau)$. If, in addition, $h_f\left(\frac{\pi}{2}\right) \le 0$, where

$$h_f(\theta) := \overline{\lim_{r\to\infty}} \frac{\ln|f(re^{i\theta})|}{r}$$

is the indicator function of f, then

$$\tilde{f}(t) = \frac{it}{\sqrt{2\pi}} \int_0^{\tau} e^{iut}\psi(u)\, du, \qquad t \in \mathbb{R} \tag{3'}$$

where $\psi \in L^2(o,\tau)$, with (see the second part of the proof of Lemma 1)

$$f(t) = f(0) + \frac{t}{\sqrt{2\pi}} \int_0^{\tau} e^{iut}\psi(u)\, du. \tag{4'}$$

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The assumption $h_f\left(\frac{\pi}{2}\right) \leq 0$ appears naturally in our context since it is realized in particular for those functions $f \in B_n$ of the form $f(z) = P(e^{iz})$, where P is any algebraic polynomial of degree $\leq n$. It follows from (3′), (4′) that

$$\tilde{f}(t) = i(f(t) - f(0)) \quad \text{if } h_f\left(\frac{\pi}{2}\right) \leq 0. \tag{5}$$

In that case formula (2) may be written in the form

$$e^{i\gamma} f'(t) = 2i\tau \sum_{k=-\infty}^{\infty} (-1)^k \frac{\sin^2\left(\frac{k\pi+\gamma}{2}\right)}{(k\pi+\gamma)^2} f\left(\frac{k\pi+\gamma}{\tau} + t\right). \tag{6}$$

Except for $\gamma \equiv \frac{\pi}{2} \pmod{\pi}$, *the example* $f(z) = e^{-i\tau z}$ shows that formula (6) is not true in general without the restriction $h_f\left(\frac{\pi}{2}\right) \leq 0$.

REMARK. It follows from (6) that *the inequality* (take $\gamma = -t\tau$)

$$|f'(t)| \leq \tau \sup_{k \in Z} \left| f\left(\frac{k\pi}{\tau}\right)\right|, \qquad t \in \mathbb{R}, \tag{7}$$

holds whenever $f \in B_\tau$ satisfies $h_f\left(\frac{\pi}{2}\right) \leq 0$. This is a refinement of the famous Bernstein's inequality, namely $|f'(t)| \leq \tau \sup_{-\infty<u<\infty} |f(u)|$, $t \in \mathbb{R}$. The inequality (7) does not hold for arbitrary $f \in B_\tau$ (take $f(z) = \sin \tau z$); however we have [7], for all $f \in B_\tau$,

$$|\tau^2 f(t) + f''(t)| \leq A(\tau) \sup_{k \in Z} \left| f\left(\frac{k\pi}{\tau}\right)\right|, \quad t \in \mathbb{R}, \tag{8}$$

with an explicit constant $A(\tau)$.

It is also known [10, p. 50] that if $f \in B_\tau$ satisfies the condition $h_f\left(\frac{\pi}{2}\right) = 0$ then:

$$\tau f(t) + i f'(t) - i e^{2i\gamma} f'(t) = \tau \sum_{k=-\infty}^{\infty} \frac{\sin^2 \gamma}{(k\pi+\gamma)^2} f\left(\frac{2(k\pi+\gamma)}{\tau} + t\right). \tag{9}$$

(A factor τ is missing in formula (2.2) of the aforementioned paper.)

Applying (9) to the function $g \in B_{2\tau}$, $g(z) := e^{i\tau z} f(z)$, where $f \in B_\tau$, we readily obtain (1).

2. **Statement of Results.** We adopt the following convention: $\sum_{a \leq \nu \leq b} A_\nu := 0$ whenever $a > b$, $a, b \in \mathbb{R}$. The formula (9) is a corollary of the following

THEOREM 1. *Let $f \in B_\tau$ such that $f(x) = O(|x|^{-\varepsilon})$, $\varepsilon > 0$, $x \to \pm\infty$. For all reals $\gamma \not\equiv 0 \pmod{\pi}$ and $\alpha \geq 0$ we have*

(10)

$$\frac{2i\alpha f'(t)}{(1-e^{2i\gamma})} + \frac{2(\alpha-2)\tau f(t)}{(1-e^{2i\gamma})} + \frac{4\tau f(t)}{(1-e^{2i\gamma})^2} + \tau \sum_{k=-\infty}^{\infty} \frac{e^{-\alpha(k\pi+\gamma)i}}{(k\pi+\gamma)^2} f\left(\alpha \frac{(k\pi+\gamma)}{\tau} + t\right)$$
$$= \sum_{1 \leq \nu \leq \alpha} \frac{e^{-2\nu i\gamma}}{\pi} \int_{-\infty}^{\infty} f(\alpha x + t) e^{\alpha i\tau x} \frac{[e^{2(\nu-\alpha)i\tau x} - 1 + 2(\alpha-\nu)i\tau x]}{x^2}\, dx.$$

If, in addition, $h_f\left(\frac{\pi}{2}\right) \le 0$ *then the summation, in the righthand member of (10) is restricted over the integers* ν *such that* $1 \le \nu \le \frac{\alpha}{2}$.

See 5.1.5 for the limiting case $\gamma \equiv 0 \pmod{\pi}$.

The summation over ν, in (10), is interpreted as being equal to zero if $\alpha \le 1$; we obtain (1) with $\alpha = 1$. If $h_f\left(\frac{\pi}{2}\right) \le 0$ then the corresponding summation is zero for $0 < \alpha < 2$ and we can also see that (9) is a consequence of the particular case $\alpha = 2$. The distance between two interpolation points, in the summation of the lefthand member of (10), is equal to $\frac{\alpha\pi}{\tau}$; it can be made arbitrarily large but, in order to compensate, we need a lot of integrals in the righthand member. A similar circumstance happens in a paper of Olivier and Rahman [9] where it is proved that the quadrature formula

$$(11)\qquad \int_{-\infty}^{\infty} f(x)\,dx = \frac{(m+1)\pi}{\tau} \sum_{\substack{\mu=0\\ \mu \text{ even}}}^{m-1} \left(\frac{m+1}{2\tau}\right)^{\mu} a_{\mu,m-1} \sum_{\nu=-\infty}^{\infty} f^{(\mu)}\left(\frac{(m+1)\pi\nu}{\tau}\right)$$

holds, in particular, for entire functions of order 1, type τ, belonging to $L^1(-\infty,\infty)$. Here $m \ge 1$ is an odd integer and $\mu!\,a_{\mu,m-1} = \psi^{(\mu)}(0)$ where

$$\psi(z) = \prod_{1 \le \mu \le \frac{m-1}{2}} \left(1 + \frac{z^2}{\mu^2}\right).$$

In (11) the distance between two interpolation points is $\frac{(m+1)\pi}{\tau}$; it can be made arbitrarily large but, in order to compensate, we need a lot of summations in the righthand member.

We observe also that the integrand, in (10), is equal to

$$f(\alpha x + t)\, e^{(2\nu-\alpha)i\tau x} \frac{d}{dx}\left(\frac{e^{-2(\nu-\alpha)i\tau x} - 1}{x}\right);$$

integrating by parts we see immediately that the righthand member of formula (10) is equal to

$$\sum_{1 \le \nu \le \alpha} \frac{e^{-2\nu i\gamma}}{\pi} \int_{-\infty}^{\infty} (\alpha f'(\alpha x + t) + (2\nu - \alpha) i\tau f(\alpha x + t)) \left(\frac{e^{(2\nu-\alpha)i\tau x} - e^{\alpha i\tau x}}{x}\right) dx.$$

Multiplying both members of formula (10) by $(1 - e^{2i\gamma})^2$ and letting $\gamma \longrightarrow 0$ give only a trivial result. A related result is given in that case by the

THEOREM 2. *Let* $f \in B_\tau$. *For all real* t *we have*

$$(12)\qquad \frac{1}{\pi} \int_{-\infty}^{\infty} f\left(\frac{2x}{\tau} + t\right) \left(\frac{\sin x}{x}\right)^2 dx - \frac{5}{6} f(t) - \frac{1}{\tau^2} f''(t) = \frac{1}{2\pi^2} \sum_{\substack{\nu=-\infty\\ \nu \ne 0}}^{\infty} \frac{f\left(\frac{2\pi\nu}{\tau} + t\right)}{\nu^2}.$$

The formula (12) is an extension to entire functions of exponential type of a trigonometric formula (see Lemma 3, below) involving the Fejer's means, $\sigma_n(s;\theta) :=$

$\sum_{j=-n}^{n}\left(1-\frac{|j|}{n}\right)b_j e^{ij\theta}$, associated to a trigonometric polynomial $s(\theta) := \sum_{j=-n}^{n} b_j e^{ij\theta}$. Like Theorem 1 it will be proved with the method of approximation described in [5] (see also sections 5.1.5 and 5.2). In order to do that we shall need a particular case of a result given in [4], namely

$$(13)\qquad \frac{3\tau^2}{\pi^2}\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty}\frac{1}{\nu^2}f\left(\frac{2\pi\nu}{\tau}\right) = \tau^2 f(0) + 6i\tau f'(0) - 6f''(0), \quad f\in B_\tau, h_f\left(\frac{\pi}{2}\right)\leq 0.$$

We take the opportunity to present here a generalisation of the result in question. It is readily seen that (13) is the case $\sigma = \tau$, $r = 2$ of the

THEOREM 3. *Let $f\in B_\tau$ such that $h_f\left(\frac{\pi}{2}\right)\leq 0$. Suppose that $\sigma\leq\tau$ and $0\leq x\leq 1-\frac{\tau}{\sigma}$. We have, for r=2,3,4,…,*

$$(14)\qquad -r!\left(\frac{\sigma}{2\pi}\right)^r\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty}\frac{e^{2\pi i\nu x}}{\nu^r}f\left(\frac{2\pi\nu}{\sigma}\right) = \sum_{k=0}^{r}\binom{r}{k}B_k(x)(i\sigma)^k f^{(r-k)}(0).$$

We have also the

THEOREM 3′. *Let $f\in B_\tau$. Suppose that $\sigma\geq 2\tau$ and $0\leq x\leq 1-\frac{2\tau}{\sigma}$. We have, for $r = 2,3,4,\ldots$*

$$(15)\qquad -r!\left(\frac{\sigma}{2\pi}\right)^r\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty}\frac{e^{2\pi i\nu(x+\frac{\tau}{\sigma})}}{\nu^r}f\left(\frac{2\pi\nu}{\sigma}\right) = \sum_{k=0}^{r}\binom{r}{k}B_k\left(x+\frac{\tau}{\sigma}\right)(i\sigma)^k f^{(r-k)}(0).$$

In (14), (15) we have $B_k(z) := \sum_{j=0}^{k}\binom{k}{j}B_j\, z^{k-j}$ where B_j is the j^{th} Bernoulli number defined by the generating function $\frac{z}{e^z-1} = \sum_{j=0}^{\infty}\frac{B_j}{j!}z^j$. Of course (14) and (15) are valid under a less restrictive hypothesis of the form $f(x) = O(|x|^{r-1-\varepsilon})$.

3. **Some Lemmas.** In order to prove the second statement of Theorem 1 we need the

LEMMA 1. *If $F\in B_\tau$ is integrable then for every $\delta\notin(-\tau,\tau)$ we have*

$$(16)\qquad \int_{-\infty}^{\infty}F(x)e^{i\delta x}\,dx = 0.$$

If in addition $h_f\left(\frac{\pi}{2}\right)\leq 0$ then (16) holds for $\delta\notin(-\tau,0)$.

PROOF. The first part of Lemma 1 is known: the Fourier transform of an integrable and entire function of exponential type τ is a continuous function equal to zero outside $[-\tau,\tau]$ (see [8, p. 109, Theorem 3.1.3]).

The second part is also essentially known but an adaptation of a standard proof of the classical Paley-Wiener theorem (e.g. the first proof in [3, p. 105]) is necessary. We need to observe that if, in addition, $h_f\left(\frac{\pi}{2}\right)\leq 0$ then [3, Theorem 6.2.4]

$$|F(x+iy)|\leq \sup_{-\infty<u<\infty}|F(u)|,\qquad -\infty<x<\infty,\ y\geq 0$$

(instead of $|F(x+iy)| \leq e^{\tau|y|}\sup_{-\infty<u<\infty}|F(u)|$). The result follows since $B_\tau \cap L^1(-\infty,\infty) \subseteq B_\tau \cap L^2(-\infty,\infty)$ (see [8, p. 126, Theorem 3.3.5]).

The next two lemmas contain the appropriate formulas on trigonometric polynomials that we shall need for the proofs of Theorems 1 and 2.

LEMMA 2. *Let* $t(\theta) := \sum_{j=-n}^{n} c_j e^{ij\theta}$ *be a trigonometric polynomial of degree* $\leq n$, $n \geq 2$. *For all reals* θ *and* $\gamma \not\equiv 0 \pmod{2\pi}$ *we have*

$$
\begin{aligned}
&c_n e^{in\theta} + \sum_{0\leq s\leq \frac{n+m-1}{n-m}} e^{-(s+1)i\gamma} \sum_{j=-n}^{(s+1)m-sn-1} ((s+1)m - sn - 1 - j)\, c_j e^{ij\theta} \\
&\quad = -\frac{it'(\theta)}{(1-e^{i\gamma})} - \frac{(m-1)t(\theta)}{(1-e^{i\gamma})} - \frac{(n-m)t(\theta)}{(1-e^{i\gamma})^2} \\
&\qquad - \frac{e^{-i\gamma}}{4(n-m)} \sum_{k=1}^{n-m} \frac{e^{-\frac{(m-1)}{(n-m)}(2k\pi+\gamma)i}}{\sin^2\left(\frac{2k\pi+\gamma}{2(n-m)}\right)}\, t\left(\theta + \frac{2k\pi+\gamma}{n-m}\right),
\end{aligned}
\tag{17}
$$

where $m < n$ *is an integer.*

PROOF. Let us consider the integral

$$I_\rho(\theta) := \frac{1}{2\pi i}\oint_{|\zeta|=\rho} \frac{t(-i\ln\zeta)\, d\zeta}{(\zeta - e^{i\theta})^2 \zeta^{m-1}(\zeta^{n-m} - e^{i(n-m)\theta + i\gamma})}.$$

We have

$$\lim_{\rho\to\infty} I_\rho(\theta) = c_n.$$

On the other hand, using the residue theorem (with $\rho > 1$),

$$
\begin{aligned}
I_\rho(\theta) &= \operatorname{Res}(\zeta = e^{i\theta}) + \sum_{k=1}^{n-m} \operatorname{Res}\left(\zeta_k = e^{i\left(\theta + \frac{2k\pi+\gamma}{n-m}\right)}\right) + \operatorname{Res}(\zeta = 0) \\
&= -i\frac{e^{-in\theta}t'(\theta)}{(1-e^{i\gamma})} - \frac{(m-1)e^{-in\theta}t(\theta)}{(1-e^{i\gamma})} - \frac{(n-m)e^{-in\theta}t(\theta)}{(1-e^{i\gamma})^2} \\
&\quad - \frac{e^{-in\theta - i\gamma}}{4(n-m)} \sum_{k=1}^{n-m} \frac{e^{-\frac{(m-1)(2k\pi+\gamma)i}{(n-m)}}}{\sin^2\left(\frac{2k\pi+\gamma}{2(n-m)}\right)}\, t\left(\theta + \frac{2k\pi+\gamma}{n-m}\right) + \operatorname{Res}(\zeta = 0).
\end{aligned}
$$

To compute the residue at $\zeta = 0$ we observe that, in a neighborhood of the origin,

$$t(-i\ln\zeta) = \sum_{j=-n}^{n} c_j\zeta^j, \quad \frac{1}{\zeta^{n-m} - e^{i(n-m)\theta+i\gamma}} = -\sum_{s=0}^{\infty} \frac{\zeta^{(n-m)s}}{e^{i(s+1)((n-m)\theta+\gamma)}},$$

and

$$\frac{1}{(\zeta - e^{i\theta})^2} = \sum_{r=1}^{\infty} \frac{r\zeta^{r-1}}{e^{i(r+1)\theta}},$$

whence

$$\frac{t(-i\ln\zeta)}{(\zeta-e^{i\theta})^2\zeta^{m-1}(\zeta^{n-m}-e^{i(n-m)\theta+i\gamma})}$$
$$=-\sum_{j=-n}^{n}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{rc_j\zeta^{j+r-m+(n-m)s}}{e^{i\theta(r+1+(s+1)(n-m))}e^{i\gamma(s+1)}}$$
$$=\cdots-\frac{1}{\zeta}\sum_{0\le s\le\frac{n+m-1}{n-m}}e^{-in\theta-i\gamma(s+1)}\sum_{j=-n}^{(s+1)m-sn-1}((s+1)m-sn-1-j)c_je^{ij\theta}$$
$$+\ldots,\quad m<n.$$

Thus,

$$\operatorname{Res}(\zeta=0)=-\sum_{0\le s\le\frac{n+m-1}{n-m}}e^{-in\theta-i\gamma(s+1)}\sum_{j=-n}^{(s+1)m-sn-1}((s+1)m-sn-1-j)c_je^{ij\theta}$$

and we readily obtain (17). ∎

The formula (12) will be obtained by comparing two representations of the Fejer's means associated to a trigonometric polynomial $t(\theta):=\sum_{j=-n}^{n}c_je^{ij\theta}$. One of them is the classical representation of De la Vallée-Poussin:

$$\sigma_n(t;\theta)=\frac{1}{\pi}\int_{-\infty}^{\infty}t\left(\frac{2x}{n}+\theta\right)\left(\frac{\sin x}{x}\right)^2dx. \tag{18}$$

The other is stated in the

LEMMA 3. [4, Theorem 2]. *If $t(\theta):=\sum_{j=-n}^{n}c_je^{ij\theta}$ is a trigonometric polynomial of degree $\le n$ then, for all real θ,*

$$\sigma_n(t;\theta)-\frac{1}{6}\left(5+\frac{1}{n^2}\right)t(\theta)-\frac{1}{n^2}t''(\theta)=\frac{1}{2n^2}\sum_{k=1}^{n-1}\frac{t\left(\theta+\frac{2k\pi}{n}\right)}{\sin^2\left(\frac{k\pi}{n}\right)},\ n\ge 2. \tag{19}$$

4. **Proofs of the Theorems.** Given $f\in B_\tau$, the functions $f_h(x):=\sum_{k=-\infty}^{\infty}\varphi(hx+k)f\left(x+\frac{k}{h}\right)$, $h>0$, where $\varphi(x)=\left(\frac{\sin\pi x}{\pi x}\right)^2$, are trigonometric polynomials with period $1/h$ and degree $\le N:=1+\left[\frac{\tau}{2\pi h}\right]$. These functions have Fourier coefficients

$$c_j(h)=h\int_{-\infty}^{\infty}\varphi(hx)f(x)e^{-2\pi ijhx}\,dx \tag{20}$$

so that

$$f_h(x)=\sum_{j=-N}^{N}c_j(h)e^{2\pi ijhx}. \tag{21}$$

We may assume that $\sup_{-\infty<t<\infty}|f(t)|\le 1$; we have then

$$|f_h(x)|\le 1,\ -\infty<x<\infty. \tag{22}$$

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Also,

$$|f_h(x) - f(x)| \leq 2(1 - \varphi(hx)), \ -\infty < x < \infty, \tag{23}$$

from which the uniform convergence on every bounded set of the real axis follows. These observations are proved in [6] with some obvious modifications.

PROOF OF THEOREM 1. We apply (17) to the trigonometric polynomial $f_h\left(\frac{\theta}{2\pi h}\right)$. We take $\theta = 0$ (the general case in (10) is obtained after an obvious translation), $n = N$ and $m = \frac{p}{q}N$ where p and q are integers such that $\frac{p}{q} < 1$ and $h \equiv \frac{\tau}{2\pi(S-1)}$, $S \equiv 0 \pmod{2q}$, $S \longrightarrow \infty$. This readily gives us the formula

$$T_1(h) = T_2(h), \tag{24}$$

where

$$\begin{aligned} T_1(h) &:= 2\pi h C_N(h) + \frac{if_h'(0)}{(1-e^{i\gamma})} + \frac{2\pi h\left(\frac{p}{q}N - 1\right)}{(1-e^{i\gamma})}f_h(0) + \frac{2\pi h\left(1-\frac{p}{q}\right)N}{\left(1-e^{i\gamma}\right)^2}f_h(0) \\ &\quad + \frac{2\pi h N e^{-i\gamma}}{4\left(1-\frac{p}{q}\right)} \sum_{k=1}^{n-m} \frac{e^{-\frac{(m-1)}{(n-m)}(2k\pi+\gamma)i}}{N^2 \sin^2\left(\frac{2k\pi+\gamma}{2(n-m)}\right)} f_h\left(\frac{2k\pi+\gamma}{2\pi h(n-m)}\right) \end{aligned} \tag{25}$$

and

$$T_2(h) := -2\pi h \sum_{0 \leq \nu \leq \frac{n+m-1}{n-m}} e^{-(\nu+1)i\gamma} \sum_{j=-N}^{(\nu+1)m-\nu N-1} ((\nu+1)m - \nu N - 1 - j)c_j(h). \tag{26}$$

Proceeding as in [5], we obtain

$$\begin{aligned} \lim_{h\to 0} T_1(h) &= \frac{if'(0)}{(1-e^{i\gamma})} + \frac{\frac{p}{q}\tau f(0)}{(1-e^{i\gamma})} + \frac{\left(1-\frac{p}{q}\right)\tau f(0)}{\left(1-e^{i\gamma}\right)^2} \\ &\quad + \left(1-\frac{p}{q}\right)\tau e^{-i\gamma} \sum_{k=-\infty}^{\infty} \frac{e^{-\frac{p/q(2k\pi+\gamma)i}{(1-p/q)}}}{(2k\pi+\gamma)^2} f\left(\frac{2k\pi+\gamma}{(1-p/q)\tau}\right). \end{aligned} \tag{27}$$

It has been assumed here that $0 < \gamma < 2\pi$, an unnecessary condition since $T_1(h)$ is a periodic function of γ with period 2π. In the following we shall also assume that f is integrable. If it is not the case but f satisfies a condition of the form

$$f(x) = O(|x|^{-\varepsilon}), \ \varepsilon > 0, \ x \longrightarrow \pm\infty, \tag{28}$$

then the functions $g_\delta(z) := \frac{\sin \delta z}{\delta z} f(z)$ are elements of $B_{\tau+\delta}$ ($\delta > 0$) belonging to $L^1(-\infty, \infty)$. An appropriate limiting process (not difficult to justify) then gives us the result under the less restrictive hypothesis (28).

Let us change now j to $(\nu+1)m - \nu N - 1 - j$ in (26). Using (20) and the basic formula

$$\sum_{j=1}^{M-1} jz^j = \frac{(M-1)z^{M+1} - Mz^M + z}{(z-1)^2} \tag{29}$$

we see that

$$T_2(h) = -2\pi h^2 \sum_{0\le\nu\le\frac{n+m-1}{n-m}} e^{-(\nu+1)i\gamma} \int_{-\infty}^{\infty} \varphi(hx)f(x)k_{\nu,h}(x)\,dx, \tag{30}$$

where

$$k_{\nu,h}(x) := \left[\frac{((\nu+1)m-(\nu-1)N-1)e^{2\pi ihx(N+2)}}{\left(e^{2\pi ihx}-1\right)^2} \right. \\ \left. \frac{-((\nu+1)m-(\nu-1)N)e^{2\pi ihx(N+1)}+e^{2\pi ihx(\nu N-(\nu+1)m+2)}}{\left(e^{2\pi ihx}-1\right)^2}\right]. \tag{31}$$

Since $h^2|\varphi(hx)f(x)k_{\nu,h}(x)| \le c(\tau)|f(x)|$, $-\infty < x < \infty$, we may invoke the dominated convergence theorem to obtain

$$\lim_{h\to 0} T_2(h) = \sum_{0\le\nu\le\frac{q+p}{q-p}} \frac{e^{-(\nu+1)i\gamma}}{2\pi} \int_{-\infty}^{\infty} f(x)k_\nu(x)\,dx, \tag{32}$$

where

$$k_\nu(x) := \left[\frac{((\nu+1)\frac{p}{q}-\nu+1)i\tau xe^{i\tau x}-e^{i\tau x}+e^{i\tau x(\nu-(\nu+1)p/q)}}{x^2}\right] \tag{33}$$

Using (24), (27) and (32) we obtain a formula which is, up to a few changes of variables, equivalent to formula (10) whenever α is a positive rational number. The result is extended to real and positive values of α with an argument similar to that used in [5].

It remains to examine formula (10) whenever the additional hypothesis $h_f\left(\frac{\pi}{2}\right) \le 0$ is imposed. The integrand, in (10), namely

$$F(z) := f(\alpha z+t)\frac{\left[e^{(2\nu-\alpha)i\tau z}-e^{\alpha i\tau z}+(2\alpha-2\nu)i\tau ze^{\alpha i\tau z}\right]}{z^2} \tag{34}$$

is an entire function of exponential type. If $h_f\left(\frac{\pi}{2}\right) \le 0$ then we shall have also $h_F\left(\frac{\pi}{2}\right) \le 0$ whenever the second factor in (34) satisfies the same condition. But that is of course realized if $s\nu - \alpha \ge 0$ i.e. $\nu \ge \frac{\alpha}{2}$. For these values of ν the Lemma 1 (with $\delta = 0$) shows that the integral is zero in (10). This completes the proof of Theorem 1 since formula (10) is seen to be equivalent to a known identity in the case $\alpha = 0$. ∎

PROOF OF THEOREM 2. It suffices to prove (12) for $t = 0$. We apply (19), with $\theta = 0$, to the trigonometric polynomial $t_h(\theta) = f_h\left(\frac{\theta}{2\pi h}\right)$ where N is chosen such that $N \equiv 0 \pmod 2$. This gives us

$$\sigma_N(t_h;0) - \frac{1}{6}\left(5+\frac{1}{N^2}\right)f_h(0) - \frac{1}{(2\pi hN)^2}f_h''(0) = \frac{1}{2N^2}\sum_{k=1}^{N-1}\frac{f_h\left(\frac{2\pi k}{2\pi hN}\right)}{\sin^2\left(\frac{k\pi}{N}\right)}. \tag{35}$$

Using the representation (18) we obtain

$$\sigma_N(t_h;0)=\frac{1}{\pi}\int_{-\infty}^{\infty}f_h\left(\frac{2x}{2\pi hN}\right)\left(\frac{\sin x}{x}\right)^2 dx, \tag{36}$$

with

$$\left|f_h\left(\frac{2x}{2\pi hN}\right)-f\left(\frac{2x}{\tau}\right)\right|\le\left|f_h\left(\frac{2x}{2\pi hN}\right)-f_h\left(\frac{2x}{\tau}\right)\right|+\left|f_h\left(\frac{2x}{\tau}\right)-f\left(\frac{2x}{\tau}\right)\right|.$$

Since

$$\begin{aligned}\left|f_h\left(\frac{2x}{2\pi hN}\right)-f_h\left(\frac{2x}{\tau}\right)\right|&=\left|\int_{\frac{2x}{\tau}}^{\frac{2x}{2\pi hN}}f_h'(u)du\right|\le\left|\frac{2x}{2\pi hN}-\frac{2x}{\tau}\right|\max_{0\le u\le 1/h}|f_h'(u)|\\&\le\left|\frac{2x}{2\pi hN}-\frac{2x}{\tau}\right|\cdot 2\pi hN,\end{aligned}$$

by Berstein's inequality for trigonometric polynomials, and

$$\left|f_h\left(\frac{2x}{\tau}\right)-f\left(\frac{2x}{\tau}\right)\right|\le 2\left(1-\varphi\left(\frac{2hx}{\tau}\right)\right),$$

by (23), we see that $\lim_{h\to 0}f_h\left(\frac{2x}{2\pi hN}\right)=f\left(\frac{2x}{\tau}\right)$.
Thus,

$$\begin{aligned}\lim_{h\to 0}\sigma_N(t_h;0)&=\frac{1}{\pi}\int_{-\infty}^{\infty}\lim_{h\to 0}f_h\left(\frac{2x}{2\pi hN}\right)\left(\frac{\sin x}{x}\right)^2 dx\\&=\frac{1}{\pi}\int_{-\infty}^{\infty}f\left(\frac{2x}{\tau}\right)\left(\frac{\sin x}{x}\right)^2 dx.\end{aligned}\tag{37}$$

On the other hand,

$$\frac{1}{N^2}\sum_{k=1}^{N-1}\frac{f_h\left(\frac{2\pi k}{2\pi hN}\right)}{\sin^2\left(\frac{k\pi}{N}\right)}=\sum_{k=1}^{\frac{N}{2}-1}\frac{f_h\left(\frac{2\pi k}{2\pi hN}\right)}{N^2\sin^2\left(\frac{k\pi}{N}\right)}+\sum_{k=-\frac{N}{2}}^{-1}\frac{f_h\left(\frac{2\pi k}{2\pi hN}\right)}{N^2\sin^2\left(\frac{k\pi}{N}\right)},$$

with

$$\left|\frac{f_h\left(\frac{2\pi k}{2\pi hN}\right)}{N^2\sin^2\left(\frac{k\pi}{N}\right)}\right|\le\frac{1}{4k^2},\qquad 0<|k|\le\frac{N}{2}.$$

Here again

$$\lim_{h\to 0}f_h\left(\frac{2\pi k}{2\pi hN}\right)=f\left(\frac{2k\pi}{\tau}\right)$$

so that

$$\lim_{h\to 0}\frac{1}{N^2}\sum_{k=1}^{N-1}\frac{f_h\left(\frac{2\pi k}{2\pi hN}\right)}{\sin^2\left(\frac{k\pi}{N}\right)}=\frac{1}{\pi^2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{f\left(\frac{2k\pi}{\tau}\right)}{k^2}. \tag{38}$$

The result follows from (35), (37) and (38). ∎

The Theorem 3 may be proved by applying the residue theorem to the integral

$$\oint_{C_{N,R}} \frac{e^{x\zeta} f\left(\frac{\zeta}{i\sigma}\right) d\zeta}{\left(e^{\zeta}-1\right)\zeta^{r}}, \qquad N \longrightarrow \infty,\ R \longrightarrow \infty,$$

where $C_{N,R}$ is the boundary of the rectangle

$$\{z = x + iy : \ |x| \leq R,\ |y| \leq (2N+1)\pi\}.$$

Since it is known to be true for $\sigma = \tau$ we shall give here a simpler proof.

PROOF OF THEOREM 3. The following formula is proved in [4, Theorem 1]: let $F \in B_\sigma$ such that $h_F\left(\frac{\pi}{2}\right) \leq 0$; for all integers $r \geq 2$ we have

$$r!\left(\frac{\sigma}{2\pi}\right)^{r} \sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty} \frac{F\left(\frac{2\pi\nu}{\sigma}\right)}{\nu^{r}} = -\sum_{k=0}^{r}\binom{r}{k} B_k (i\sigma)^k F^{(r-k)}(0). \tag{39}$$

Now,

$$\sum_{k=0}^{r}\binom{r}{k} B_k(x)(i\sigma)^k f^{(r-k)}(0) = \sum_{k=0}^{r}\sum_{j=0}^{k}\binom{r}{k}\binom{k}{j} B_j x^{k-j}(i\sigma)^k f^{(r-k)}(0).$$

We rearrange the order of summation, change j to $j+k$ and use the relation $\binom{r}{j+k}\binom{j+k}{k} = \binom{r}{k}\binom{r-k}{j}$ to obtain

$$\begin{aligned}\sum_{k=0}^{r}\binom{r}{k} B_k(x)(i\sigma)^k f^{(r-k)}(0) &= \sum_{k=0}^{r}\binom{r}{k} B_k(i\sigma)^k \sum_{j=0}^{r-k}\binom{r-k}{j}(i\sigma x)^j f^{(r-k-j)}(0)\\ &= \sum_{k=0}^{r}\binom{r}{k} B_k(i\sigma)^k\left(e^{i\sigma x w} f(w)\right)^{(r-k)}(w=0),\end{aligned}$$

by Leibnitz's formula. The function $F(w) := e^{i\sigma x w} f(w)$ is, for $x \geq 0$, an element of $B_{\tau+\sigma x}$ and $h_F\left(\frac{\pi}{2}\right) = h_f\left(\frac{\pi}{2}\right) - \sigma x \leq 0$. If $\tau + \sigma x \leq \sigma$, i.e. $x \leq 1 - \frac{\tau}{\sigma}$, then F belongs to B_σ; thus, applying (39), we obtain

$$\sum_{k=0}^{r}\binom{r}{k} B_k(x)(i\sigma)^k f^{(r-k)}(0) = -r!\left(\frac{\sigma}{2\pi}\right)^{r}\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty}\frac{F\left(\frac{2\pi\nu}{\sigma}\right)}{\nu^{r}},$$

which is the desired result. ∎

PROOF OF THEOREM 3′. Let us apply (14) to the function $g \in B_{2\tau}, g(z) := e^{i\tau z} f(z)$, which satisfies $h_g\left(\frac{\pi}{2}\right) = h_f\left(\frac{\pi}{2}\right) - \tau \leq 0$. We obtain, with the help of Leibnitz's formula,

$$-r!\left(\frac{\sigma}{2\pi}\right)^{r}\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty}\frac{e^{2\pi i x\nu}}{\nu^{r}} g\left(\frac{2\pi\nu}{\sigma}\right) = \sum_{k=0}^{r}\sum_{j=0}^{r}\binom{r}{k}\binom{r-k}{j} B_k(x)(i\sigma)^k(i\tau)^{r-k-j} f^{(j)}(0).$$

We rearrange the order of summation $\left(\sum_{k=0}^{r}\sum_{j=0}^{r-k}a_{j,k}=\sum_{j=0}^{r}\sum_{k=0}^{r-j}a_{j,k}\right)$ and use the relation $\binom{r}{k}\binom{r-k}{j}=\binom{r}{j}\binom{r-j}{k}$ to obtain

$$-r!\left(\frac{\sigma}{2\pi}\right)^{r}\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty}\frac{e^{2\pi i x\nu}}{\nu^{r}}g\left(\frac{2\pi\nu}{\sigma}\right)=\sum_{j=0}^{r}\sum_{k=0}^{r-j}\binom{r}{j}\binom{r-j}{k}B_k(x)(i\sigma)^k(i\tau)^{r-k-j}f^{(j)}(0)$$
$$=\sum_{j=0}^{r}\binom{r}{j}B_{r-j}\left(x+\frac{\tau}{\sigma}\right)(i\sigma)^{r-j}f^{(j)}(0),$$

where the last step uses the addition formula (see for example [2, p. 275]): $B_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k(x)y^{n-k}$. This is equivalent to formula (15). ∎

5. **Other Observations and Results.**

5.1. Some consequences of Theorem 1.

5.1.1. There is a result, similar to Theorem 1, valid for negative values of α. In order to obtain it we need only to change, in formula (10), k to $-k$, γ to $-\gamma$ and α to $-\alpha$.

5.1.2. It is possible to evaluate in closed form the summation over ν in formula (10) (the summation under the integral sign is essentially a geometric progression) but the resulting formula does not take an elegant form. However, in the case $\gamma=\frac{\pi}{2}$ we have $e^{-2\nu i\gamma}=(-1)^{\nu}$; if we suppose furthermore that $[\alpha]$ is an even number then $\sum_{1\leq\nu\leq[\alpha]}(-1)^{\nu}=0$. In that case other simplifications occur and we are led to the

COROLLARY 1. *Let $f\in B_\tau$ such that $f(x)=O\left(|x|^{-\varepsilon}\right)$, $\varepsilon>0, x\longrightarrow\pm\infty$. For all $\alpha\geq 0$ such that $[\alpha]\equiv 0 \pmod 2$ we have*

$$\alpha i f'(t)+(\alpha-1)\tau f(t)+\frac{4\tau}{\pi^2}\sum_{k=-\infty}^{\infty}\frac{e^{\frac{-(2k+1)\pi i\alpha}{2}}}{(2k+1)^2}f\left(\frac{(2k+1)}{2\tau}\pi\alpha+t\right)$$
$$=\frac{1}{\pi}\int_{-\infty}^{\infty}f(\alpha x+t)\frac{\left[e^{(2-\alpha)i\tau x}\left(e^{2i\tau x[\alpha]}-1\right)-[\alpha]i\tau x e^{\alpha i\tau x}\left(e^{2i\tau x}+1\right)\right]}{x^2\left(e^{2i\tau x}+1\right)}\,dx. \tag{40}$$

5.1.3 A special case of particular interest is obtained by letting $\alpha=\tau$ in Theorem 1.

5.1.4 Under suitable conditions we can derive, with respect to α, both members of formula (10). In order to apply the dominated convergence theorem we restrict ourselves to an interval $(m-1,m)$ where m is a positive integer such that $m-1<\alpha<m$. Deriving two times lead us (taking $t=0$ and using Lemma 1) to the integral

$$\int_{-\infty}^{\infty}\left(f''(\alpha x)-2i\tau f'(\alpha x)-\tau^2 f(\alpha x)\right)e^{(2\nu-\alpha)i\tau x}\,dx.$$

Integrating by parts, we obtain a result which is seen to be valid, by continuity, at the extremities of the interval $(m-1,m)$. Precisely, we have the

COROLLARY 2. *Let $f \in B_\tau$ such that $f(x) = O(|x|^{-\delta})$, $\delta > 1$, $x \longrightarrow \pm\infty$. For all $\alpha > 0$ we have*

$$\alpha^2 \sum_{k=-\infty}^{\infty} \left(\tau^2 f\left(\frac{\alpha k\pi}{\tau}\right) + 2i\tau f'\left(\frac{\alpha k\pi}{\tau}\right) - f''\left(\frac{\alpha k\pi}{\tau}\right)\right) e^{-\alpha k\pi i} = \sum_{1\le\nu\le[\alpha]} \frac{4\tau^3}{\pi}\nu^2 \int_{-\infty}^{\infty} f(\alpha x)e^{(2\nu-\alpha)i\tau x}\,dx. \tag{41}$$

Suppose that $\alpha \geq 1$ so that the function $f \in B_\tau$ can be seen as an element of $B_{\tau\alpha}$. We can therefore change τ to $\tau\alpha$ in (41). Using Lemma 1 we see that the integrals are zero whenever $|2\nu - \alpha| \geq 1$; if $\frac{(\alpha+1)}{2}$ is not an integer we remain with only one value of ν, namely $\nu = \left[\frac{\alpha+1}{2}\right]$. Replacing α by $(2\alpha - 1)$ we obtain the

COROLLARY 2′. *Under the same hypothesis as in Corollary 2, except that $\alpha \geq 1$, we have*

$$\sum_{k=-\infty}^{\infty} (-1)^k e^{-2\alpha k\pi i}\left(\tau^2(2\alpha-1)^2 f\left(\frac{k\pi}{\tau}\right) + 2i\tau(2\alpha-1)f'\left(\frac{k\pi}{\tau}\right) - f''\left(\frac{k\pi}{\tau}\right)\right) = \frac{4\tau^3}{\pi}[\alpha]^2\int_{-\infty}^{\infty} f(x)e^{(1-2\{\alpha\})i\tau x}\,dx, \tag{41′}$$

where $\{\alpha\} := \alpha - [\alpha]$ is the fractional part of α.

We need to observe here that formula (41′) is also valid whenever α is an integer. In that case, the integral is zero by Lemma 1 and that the series are also zero is a consequence of the quadrature formula (11) (with $m = 1$).

Suppose, in addition, that $h_f\left(\frac{\pi}{2}\right) \leq 0$. The formula (see [5, Corollary 1])

$$\frac{2\pi}{\tau}\sum_{k=-\infty}^{\infty} e^{-\alpha k\pi i}\sin^2\left(\frac{k\pi}{2}\right) f\left(\frac{k\pi}{\tau}\right) = \int_{-\infty}^{\infty} f(x)e^{-\alpha i\tau x}\,dx,\quad 0 \leq \alpha \leq 1,$$

in conjunction with (41′), gives us the following result: if $h_f\left(\frac{\pi}{2}\right) \leq 0$ and $\frac{1}{2} \leq \{\alpha\} < 1$, $\alpha \geq \frac{3}{2}$, then

$$\sum_{k=-\infty}^{\infty} (-1)^k e^{-2\alpha k\pi i}\left(\tau^2(2\alpha-1)^2 f\left(\frac{k\pi}{\tau}\right) + 2i\tau(2\alpha-1)f'\left(\frac{k\pi}{\tau}\right) - f''\left(\frac{k\pi}{\tau}\right)\right) = 8\tau^2[\alpha]^2 \sum_{k=-\infty}^{\infty}(-1)^k e^{-2\alpha k\pi i}\sin^2\left(\frac{k\pi}{2}\right) f\left(\frac{k\pi}{\tau}\right). \tag{41″}$$

5.1.5 Let us put in evidence the term corresponding to $k = 0$ in formula (10). Evaluating the limit as $\gamma \longrightarrow 0$ we see that the expression beside the series becomes

$$-\frac{1}{2\tau}\left[\left(\frac{2}{3} - 2\alpha + \alpha^2\right)\tau^2 f(t) - 2i\tau\alpha(1-\alpha)f'(t) - \alpha^2 f''(t)\right].$$

The resulting formula, namely

$$\alpha^2 f''(t) + 2i\tau\alpha(1-\alpha)f'(t) - \left(\frac{2}{3} - 2\alpha + \alpha^2\right)\tau^2 f(t) + \frac{2\tau^2}{\pi^2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \frac{e^{-\alpha k\pi i}}{k^2} f\left(\frac{\alpha k\pi}{\tau} + t\right)$$
$$= \frac{2\tau}{\pi}\sum_{1\leq\nu\leq\alpha}\int_{-\infty}^{\infty} f(\alpha x + t)e^{\alpha i\tau x}\frac{[e^{2(\nu-\alpha)i\tau x} - 1 + 2(\alpha-\nu)i\tau x]}{x^2}\,dx,$$

is known for $\alpha = 1$. The case $\alpha = 2$ leads us to Theorem 2; however some work, including the use of formula (12) of [5], is necessary.

5.2. A third proof of Theorem 2. A strong result (see [3, Theorem 6.8.11]) says that an entire function $f(z)$ is of exponential type τ and belongs $L^1(-\infty,\infty)$ if and only if

$$f(z) = \int_{-\tau}^{\tau} e^{izu}\phi(u)\,du, \tag{42}$$

where $\phi(\tau) = \phi(-\tau) = 0$ and the function obtained by extending $\phi(u)$ to be 0 outside $(-\tau,\tau)$ has an absolutely convergent Fourier series on the interval $(-\tau-\varepsilon,\tau+\varepsilon)$, $\varepsilon > 0$. Assuming that f is integrable we see, in view of (42), that it is sufficient to establish (12) for functions of the form $f(z) = e^{izu}$, $-\tau \leq u \leq \tau$. Formula (12) is, for these functions, equivalent to the identity

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\cos(2\lambda x)\left(\frac{\sin x}{x}\right)^2 dx - \frac{5}{6} + \lambda^2 = \frac{1}{2\pi^2}\sum_{\substack{\nu=-\infty\\\nu\neq 0}}^{\infty}\frac{\cos(2\pi\nu\lambda)}{\nu^2},\ 0\leq\lambda\leq 1, \tag{43}$$

which follows from

$$\frac{1}{\pi^2}\sum_{\nu=1}^{\infty}\frac{\cos(2\pi\nu\lambda)}{\nu^2} = \frac{1}{6} - \lambda + \lambda^2, \qquad 0\leq\lambda\leq 1, \tag{44}$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\cos(2\lambda x)\left(\frac{\sin x}{x}\right)^2 dx = 1-\lambda, \qquad 0\leq\lambda\leq 1. \tag{45}$$

If f is not integrable but satisfies a condition of the form $f(x) = O\left(|x|^{1-\varepsilon}\right)$, $\varepsilon > 0$, $x \to \pm\infty$, then we may apply the result to $F_\delta(z) := \left(\frac{\sin\delta z}{\delta z}\right)^2 f(z)$, $\delta \to 0$. ∎

5.3. A second proof of Theorem 3′. While proving the Theorem 3(section 4) we observe that $h_F\left(\frac{\pi}{2}\right) \leq \tau - \sigma x$. But $\tau - \sigma x \leq 0$ if $x \geq \frac{\tau}{\sigma}$ and $\frac{\tau}{\sigma} \leq 1 - \frac{\tau}{\sigma}$ for $\sigma \geq 2\tau$. Thus, for $\sigma \geq 2\tau$, the restriction $h_f\left(\frac{\pi}{2}\right) \leq 0$ is not necessary if $\frac{\tau}{\sigma} \leq x \leq 1 - \frac{\tau}{\sigma}$. The relation (15) follows if we change x to $x + \frac{\tau}{\sigma}$.

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