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APPROXIMATE CONVEXITY IN VECTOR OPTIMISATION

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Approximate convex functions are characterised in terms of Clarke generalised gradient. We apply this characterisation to derive optimality conditions for quasi efficient solutions of nonsmooth vector optimisation problems. Two new classes of generalised approximate convex functions are defined and mixed duality results are obtained.

1. INTRODUCTION

There have been several studies in the past to demonstrate the key role played by 'duality' in Economics and Optimisation Theory. Many dual models have been proposed for the constrained vector optimisation problems and corresponding duality results have been investigated. Among them the two dual models namely Wolfe dual model and Mond-Weir dual model have been widely studied both for smooth as well as nonsmooth vector optimisation problems ([**1, 2, 4, 6, 7, 8, 10**] and references cited therein). Later, combining the two dual models a mixed dual model was proposed and duality results were obtained by Xu [**12**]. In order to have a deeper insight of the mixed dual model Bector, Chandra and Abha ([**1, 2**]) defined the notion of incomplete Lagrange function and observed that the Mond-Weir dual is connected to the incomplete Lagrange function exactly in the same manner as the Wolfe dual is connected to the usual Lagrange function. This inspired them to study mixed duality for various classes of nonlinear scalar-valued programming problems. It is worth to note that the notions of convexity and generalised convexity play a crucial role in establishing the primal-dual relationships. Moreover, advances in nonsmooth analysis and nonsmooth subdifferenital calculus rules led various authors to search for the class of nonconvex functions possessing properties that are similar to convex functions and also satisfy the basic subdifferential calculus rules. In this context, Ngai, Luc and Thera [**9**] defined a new class of approximate convex functions and showed that functions belonging to this class enjoy many of the desired properties.

In this article we intend to use the notion of approximate convexity to develop mixed duality results for nonsmooth vector optimisation problems. The structure of the paper is as follows. In section 2 we present a characterisation of approximate convex function

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in terms of Clarke generalised gradient. This characterisation motivates us to further introduce two new classes of generalised approximate convex functions. In section 3 necessary and sufficient optimality conditions are derived for quasi efficient solution of the nonsmooth vector optimisation problem. Mixed duality results are established under generalised approximate convexity assumptions in section 4. Finally the paper concludes with some observations.

2. Approximate convex functions

The locally Lipschitz condition and Clarke generalised gradient are frequently used as the principal tools in analyzing nonsmooth vector optimisation problems. For the sake of completeness we first recall these two definitions. In what follows we assume that X is a nonempty subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$.

Definition 1: f is locally Lipschitz at $x \in X$ if there exist a positive constant L and a neighbourhood U of x such that $\forall\, x_1, x_2 \in U$

$$\|f(x_1) - f(x_2)\| \leqslant L\|x_1 - x_2\|.$$

Definition 2: ([5]) Let f be locally Lipschitz at $x \in X$. The Clarke generalised directional derivative of f at x in the direction $v \in \mathbb{R}^n$ is given by

$$f^0(x; v) = \lim_{\lambda \downarrow 0} \sup_{y \to x} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

The locally Lipschitz condition on the function guarantees the existence and finiteness of the above limit. Also, as a function of v, $f^0(x; v)$ is subadditive and positively homogenous. These two properties together with the Hahn-Banach Theorem permit the following definition.

Definition 3: ([5]) The Clarke generalised gradient of f at $x \in X$, denoted by $\partial f(x)$, is defined as

$$\partial f(x) = \{\xi \in \mathbb{R}^n : f^0(x; v) \geqslant \xi^t v,\ \forall\ v \in \mathbb{R}^n\}.$$

For instance, the function $f(x) = \|x - x_0\|$ is not differentiable at x_0, its Clarke generalised gradient at x_0 is a closed unit ball $B[0, 1] :\equiv \mathbf{B}$ in $\mathbb{R}^n$.

To start with, we state the relaxed notion of convexity namely approximate convexity that has been used in this article.

Definition 4: ([9]) f is said to be an approximate convex function at $x_0 \in X$ if $\forall\, c > 0\ \exists\, \delta > 0$ (δ depends on c and x_0) such that

$$f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda) f(y) + c\lambda(1 - \lambda)\|x - y\| \quad \forall\ x, y \in B(x_0, \delta) \cap X, \forall\ \lambda \in (0, 1). \tag{1}$$

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Functions belonging to this class possess many interesting properties similar to that of convex functions. The main feature of this class is that it includes the classes of convex functions, weakly convex functions, strongly convex functions of order m, $m \geqslant 1$, and strictly continuously differentiable functions. It is important to note that a lower semicontinuous approximate convex function at x_0 is locally Lipschitz at x_0. For additional details we refer the readers to [**9**]. Furthermore, there exist real-valued functions which are approximate convex but not necessarily convex. For example, consider the function $f(x) = x^3 - x^2$, $x \in \mathbb{R}$. f is approximate convex at $x_0 = 0$ as $\forall c > 0\ \exists 0 < \delta < \left(-1 + \sqrt{(1 + 1.5c)}\right)/3$ such that (1) holds but f is neither convex nor concave at x_0.

Below we present a characterisation of approximate convex function in terms of the Clarke generalised gradient.

Theorem 1. *If f is a lower semi continuous approximate convex function at $x_0 \in X$ then $\forall c > 0\ \exists \delta > 0$ such that*

$$f(y) \geqslant f(x_0) + \xi^t(y - x_0) - c\|y - x_0\|, \ \forall\, y \in B(x_0, \delta) \cap X, \ \forall \xi \in \partial f(x_0). \tag{2}$$

Proof: Since f is a lower semicontinuous approximate convex function at x_0 hence it is a locally Lipschitz function at x_0. Moreover, $\forall c > 0\ \exists \delta > 0$ such that (1) holds. Let $y \in B(x_0, \delta) \cap X$. Choosing $h > 0$ sufficiently small so that $x_0 + h$, $y + h \in B(x_0, \delta)$. The Clarke generalised directional derivative of f at x_0 in the direction of $(y - x_0)$ is

$$\begin{aligned}
f^0(x_0; y - x_0) &= \limsup_{\substack{h \to 0 \\ t \to 0+}} \frac{f((x_0 + h) + t(y - x_0)) - f(x_0 + h)}{t} \\
&= \limsup_{\substack{h \to 0 \\ t \to 0+}} \frac{f(t(y + h) + (1 - t)(x_0 + h)) - f(x_0 + h)}{t} \\
&\leqslant \limsup_{\substack{h \to 0 \\ t \to 0+}} \left(f(y + h) - f(x_0 + h) + c(1 - t)\|y - x_0\|\right) \\
&\leqslant \limsup_{\substack{h \to 0 \\ t \to 0+}} \left(f(y) - f(x_0) + L|h| + L|h| + c(1 - t)\|y - x_0\|\right) \\
&\leqslant f(y) - f(x_0) + c\|y - x_0\|.
\end{aligned}$$

Thus $f(y) \geqslant f(x_0) + \xi^t(y - x_0) - c\|y - x_0\|$, $\forall\, \xi \in \partial f(x_0)$. ∎

Inspired by the previous result we introduce two useful classes of generalised approximate convex functions for using them in deriving the duality results in the later section.

Definition 5: f is said to be an approximate quasiconvex function at $x_0 \in X$ if $\forall c > 0\ \exists \delta > 0$ such that whenever $y \in B(x_0, \delta) \cap X$ and $f(y) \leqslant f(x_0)$ then

$$\xi^t(y - x_0) - c\|y - x_0\| \leqslant 0, \quad \forall\, \xi \in \partial f(x_0). \tag{3}$$

Remark 1. (i) Approximate convex function at x_0 is also an approximate quasiconvex function at x_0. The converse need not follow. For example, consider the function

$$f(x) = \begin{cases} x^2 & -2\pi \leqslant x < 0 \\ -\sin x & 0 \leqslant x < 2\pi \end{cases}$$

f is approximate quasiconvex but not approximate convex at $x_0 = 0$ as for $c \in (0, 1)$ we can not find a $\delta > 0$ such that (2) holds.

(ii) There is no relationship between approximate quasiconvexity and quasiconvexity (see reference [**3**, pp. 163]) of a function at x_0. For example, $f(x) = \sin x$, $x \in [-2\pi, 2\pi]$, is approximate quasiconvex with $0 < \delta < \pi$ but not quasiconvex at $x_0 = 0$.

Furthermore, it can be proved that

$$f(x) = \begin{cases} 0 & x \leqslant 0 \\ -x & 0 \leqslant x \leqslant 1 \\ -1 & x \geqslant 1\,. \end{cases}$$

is quasiconvex but not approximate quasiconvex at $x_0 = 0$ since for any $c \in (0, 1)$ $\nexists\, \delta > 0$ such that (3) holds.

DEFINITION 6: f is said to be an approximate pseudoconvex function at $x_0 \in X$ if $\forall\, c > 0 \,\exists\, \delta > 0$ such that whenever

$$y \in B(x_0, \delta) \cap X \text{ and } \xi^t(y - x_0) + c\|y - x_0\| \geqslant 0 \text{ for some } \xi \in \partial f(x_0)$$

then

$$f(y) + 2c\|y - x_0\| \geqslant f(x_0).$$

REMARK 2. (i) An approximate convex function at x_0 is an approximate pseudoconvex function at x_0 but the converse in general does not hold. For example, consider the function

$$f(x) = \begin{cases} x^3 + x & x \geqslant 0 \\ 2x & x < 0 \end{cases}$$

At $x_0 = 0$ f is approximate pseudoconvex but not approximate convex as for $c \in (0, 1)$ and for y sufficiently small condition (2) fails to hold.

(ii) It is important to observe that there is no relationship between an approximate pseudoconvex function and a pseudoconvex function (see reference [**3**, pp. 163]), that is, a function can be approximate pseudoconvex but not pseudoconvex or pseudoconvex but not approximate pseudoconvex at some point x_0. These facts are illustrated below.

Let

$$f(x) = \begin{cases} x+1 & x \leqslant -1 \\ 0 & -1 < x < 1 \\ x-1 & x \geqslant 1. \end{cases}$$

Then f is approximate pseudoconvex but not pseudoconvex at $x_0 = 0$.

Define

$$f(x) = \begin{cases} x^2 + 4x & x \leqslant 0 \\ x & 0 < x < 1 \\ x^2 & x \geqslant 1. \end{cases}$$

It can easily be proved that f is pseudoconvex but not approximate pseudoconvex at $x_0 = 0$.

(iii) An approximate pseudoconvex function at x_0 is in general not approximate quasiconvex at x_0. The following function justifies this assertion at $x_0 = 0$.

$$f(x) = \begin{cases} -(x)^{3/2} & x \geqslant 0 \\ e^x & x < 0. \end{cases}$$

However it can easily be shown that a continuous approximate pseudoconvex function at x_0 is approximate quasiconvex at x_0.

(iv) Furthermore, an approximate quasiconvex function at x_0 is not necessarily an approximate pseudoconvex function at x_0. For instance,

$$f(x) = \begin{cases} x^2 & x \leqslant 0 \\ -x^2 - 2x & x > 0. \end{cases}$$

is approximate quasiconvex but not approximate pseudoconvex at $x_0 = 0$.

3. Optimality Conditions

Using the notion of approximate convexity we derive the necessary and sufficient optimality conditions for the quasi efficient solution of the following vector optimisation problem (VP).

$$\text{(VP)} \qquad \min\ f(x) = \big(f_1(x), \dots, f_p(x)\big)$$
$$\text{subject to } \ g_j(x) \leqslant 0\,,\ j = 1, \dots, m,$$

where f_i, $g_j : X \to \mathbb{R}$, $i = 1, \dots, p$, $j = 1, \dots, m$. Let $S_P = \{x \in X \mid g_j(x) \leqslant 0, j = 1, \dots, m\}$ be the feasible set of (VP).

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DEFINITION 7: $x_0 \in S_P$ is said to be a local efficient solution of (VP) if there exists a neighbourhood U of x_0 such that for any $x \in S_P \cap U$ the following can not hold

$$\begin{aligned} f_i(x) &\leqq f_i(x_0), \quad \forall\, i = 1, \dots, p, \\ f_r(x) &< f_r(x_0), \quad \text{for some r.} \end{aligned}$$

DEFINITION 8: $x_0 \in S_P$ is said to be a local quasi efficient solution (or quasi efficient solution) of (VP) if there exist $\alpha \in \operatorname{int}(R^p_+)$ and a neighbourhood U of x_0 such that for any $x \in S_P \cap U$ (or $x \in S_P$) the following can not hold

$$\begin{aligned} f_i(x) &\leqq f_i(x_0) - \alpha_i \|x - x_0\|, \quad \forall\, i = 1, \dots, p \\ f_r(x) &< f_r(x_0) - \alpha_r \|x - x_0\|, \quad \text{for some r.} \end{aligned}$$

REMARK 3. An immediate consequence of the above definitions is that a local efficient solution is local quasi efficient solution of (VP). The converse relation is in general not true. To illustrate these facts we first consider the vector optimisation problem

$$\begin{aligned} &\min \ f(x) = (e^{x^2} + 1, -x^3 + x) \\ &\text{subject to } x \geqslant 0. \end{aligned}$$

$x_0 = 0$ is an efficient solution as well as quasi efficient solution for $\alpha = (1,1)^t$. However, if we take another vector optimisation problem

$$\begin{aligned} &\min \ f(x) = (\ln(x+1) - x,\, x^3 - x) \\ &\text{subject to } x \geqslant 0, \end{aligned}$$

then $x_0 = 0$ is a local quasi efficient solution for $\alpha = (1,1)^t$ but not a local efficient solution.

The next two theorems provide the necessary and sufficient conditions for (VP) to possess a quasi efficient solution.

THEOREM 2. (Necessary Optimality Conditions) *Suppose x_0 is a quasi efficient solution of* (VP) *and the functions f_i, $i = 1, \dots, p$, and g_j, $j = 1, \dots, m$ are locally Lipschitz at x_0. Then there exist $\alpha \in \operatorname{int}(R^p_+)$, $\lambda \in \mathbb{R}^p_+$ and $\mu \in \mathbb{R}^m_+$ such that*

$$0 \in \sum_{i=1}^{p} \lambda_i \partial f_i(x_0) + \sum_{j=1}^{m} \mu_j \partial g_j(x_0) + \sum_{i=1}^{p} \lambda_i \alpha_i \mathbf{B} \tag{4}$$

$$\mu_j g_j(x_0) = 0\,. \tag{5}$$

PROOF: It follows from the quasi efficiency of x_0 that there exists $\alpha \in \operatorname{int}(\mathbb{R}^p_+)$ such that the following system has no solution $x \in X$

$$\begin{aligned} f_i(x) &\leqq f_i(x_0) - \alpha_i \|x - x_0\|, \quad i = 1, \dots, p, \\ f_r(x) &< f_r(x_0) - \alpha_r \|x - x_0\|, \quad \text{for some } r, \\ g_j(x) &\leqslant 0, \qquad\qquad\qquad j = 1, \dots, m. \end{aligned}$$

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Consequently x_0 is an efficient solution of the auxiliary vector optimisation problem (VP')
involving locally Lipschitz functions.

$$\text{(VP}'\text{)} \qquad \begin{aligned} &\min\big(f_1(x) + \alpha_1\|x - x_0\|, \ldots, f_p(x) + \alpha_p\|x - x_0\|\big) \\ &\text{subject to } \; g_j(x) \leqslant 0, \quad j = 1, \ldots, m. \end{aligned}$$

Applying Fritz–John necessary optimality conditions on (VP$'$) we get the existence of scalars λ_i, $i = 1, \ldots, p$ and μ_j, $j = 1, \ldots, m$ such that

$$\text{(6)} \qquad \begin{gathered} 0 \in \sum_{i=1}^{p} \big(\lambda_i \partial(f_i + \alpha_i\|x - x_0\|)\big)(x_0) + \sum_{j=1}^{m} \mu_j \partial g_j(x_0) \\ \mu_j g_j(x_0) = 0, \quad j = 1, \ldots, m \\ \lambda_i \geqq 0, \;\; \mu_j \geqq 0, \;\; i = 1, \ldots, p, \; j = 1, \ldots, m, \; (\lambda, \mu) \neq 0. \end{gathered}$$

(6) can be rewritten as

$$0 \in \sum_{i=1}^{p} \lambda_i \partial f_i(x_0) + \sum_{j=1}^{m} \mu_j \partial g_j(x_0) + \sum_{i=1}^{p} \lambda_i \alpha_i \mathbb{B}.$$

Hence the result. $\square$

REMARK 4. The necessary optimality conditions developed above are of Fritz–John Type. Under appropriate constraint qualifications or regularity conditions on the functions we can easily derive the KKT type necessary optimality conditions. In that case we can take $\lambda^t e = 1$. One such constraint qualification is Mangasarian Fromovitz Constraint Qualification which states that

$$0 \in \sum_{j \in I(x_0)} \mu_j \partial g_j(x_0) \;\Rightarrow \mu_j = 0, \; \forall j \in I(x_0); \quad I(x_0) = \big\{ j \mid g_j(x_0) = 0 \big\}.$$

Another weakened form of constraint qualification called basic regularity condition is given as follows

$$\begin{aligned} 0 \in \sum_{i=1, i \neq r}^{p} \lambda_i \partial f_i(x_0) + \sum_{j \in I(x_0)} \mu_j \partial g_j(x_0) + \sum_{i=1, i \neq r}^{p} \lambda_i \alpha_i \mathbb{B}, \; \text{for some r} \\ \Rightarrow \lambda_i = 0, \; \forall i = 1, \ldots, p, \; i \neq r, \; \mu_j = 0, \; \forall j \in I(x_0); \; I(x_0) = \big\{ j \mid g_j(x_0) = 0 \big\}. \end{aligned}$$

THEOREM 3. (Sufficient Optimality Conditions) *Let conditions* (4) *and* (5) *be satisfied at* $x_0 \in X$ *along with* $\lambda > 0$ *and* $\lambda^t e = 1$. *Suppose* f_i, $i = 1, \ldots, p$, *and* g_j, $j = 1, \ldots, m$ *are approximate convex functions at* x_0. *Then* x_0 *is a local quasi efficient solution of* (VP).

PROOF: From hypothesis it follows that for some $\xi_i \in \partial f_i(x_0)$, $\beta_j \in \partial g_j(x_0)$, $b \in \mathbb{B}$

$$0 = \sum_{i=1}^{p} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \beta_j + \sum_{i=1}^{p} \lambda_i \alpha_i b, \tag{7}$$

$$\mu_j g_j(x_0) = 0, \quad j = 1, \dots, m. \tag{8}$$

In lieu of Theorem 1 we have that $\forall\, c > 0\ \exists\, \delta > 0$ such that $\forall\, x \in B(x_0, \delta)$

$$f_i(x) - f_i(x_0) \geqslant \langle \xi_i, x - x_0 \rangle - c\|x - x_0\|,\ \forall\, \xi_i \in \partial f_i(x_0)$$
$$g_j(x) - g_j(x_0) \geqslant \langle \beta_j, x - x_0 \rangle - c\|x - x_0\|,\ \forall\, \beta_j \in \partial g_j(x_0).$$

The above two inequalities along with $\lambda > 0$, $\mu \geqq 0$, relations (7) and (8) yield

$$\lambda^t f(x) - \lambda^t f(x_0) + \mu^t g(x) \geqslant -\sum_{i=1}^{p} \lambda_i \alpha_i \langle b, x - x_0 \rangle - \gamma \|x - x_0\|,$$
$$\gamma = c(1 + \mu^t e) > 0.$$
$$\geqslant -\sum_{i=1}^{p} \lambda_i \alpha_i \|x - x_0\| - \gamma \|x - x_0\|$$
$$= -\eta \|x - x_0\|, \quad \eta = \sum_{i=1}^{p} \lambda_i \alpha_i + \gamma > 0.$$

Let $x \in S_P \cap B(x_0, \delta)$. Then

$$\lambda^t (f(x) - f(x_0) + \eta e \|x - x_0\|) \geqslant 0.$$

Thus $\forall\, c > 0\ \exists\, \delta > 0$ and $\eta_c = \lambda^t \alpha + c(1 + \mu^t e) > 0$ such that $\forall\, x \in S_P \cap B(x_0, \delta)$

$$f_i(x) \leqq f_i(x_0) - \eta_c \|x - x_0\|, \quad \forall i = 1, \dots, p$$
$$f_r(x) < f_r(x_0) - \eta_c \|x - x_0\|, \quad \text{for some r}$$

is not possible, thereby implying that x_0 is a local quasi efficient solution of (VP). ∎

4. Duality

The present section is devoted to develop the duality relationship between (VP) and its mixed dual under generalised approximate convexity assumptions.

Let the index set $M = \{1, \dots, m\}$ be partitioned into two disjoint subsets K and J such that $M = K \cup J$. The mixed dual for (VP) is given by

(VD)

$$\max \quad f(u) + \mu_J g_J(u) e$$
$$\text{subject to} \quad \mu_k g_k(u) \geqslant 0, \quad k \in K$$
$$0 \in \partial \lambda^t f(u) + \partial \mu^t g(u) + \langle \lambda, \alpha \rangle \mathbf{B}$$
$$\lambda \geqslant 0,\ \lambda^t e = 1,\ \mu \geqq 0,\ \alpha > 0,$$
$$e = (1, \dots, 1)^t \in \mathbb{R}^p.$$

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Let

$$S_D = \{(u, \lambda, \mu, \alpha) \mid \mu_k g_k(u) \geqslant 0,\ k \in K,\ 0 \in \partial \lambda^t f(u) + \partial \mu^t g(u) + \langle \lambda, \alpha \rangle \mathbb{B},\ \lambda \geqslant 0,\ \lambda^t e = 1,\ \mu \geqq 0,\ \alpha > 0\}$$

denotes the feasible set of (VD). Recall that S_P was the feasible set of (VP).

Theorem 4. (Weak Duality) *Let $(u, \lambda, \mu, \alpha) \in S_D$ and suppose $\sum_{k \in K} \mu_k g_k(\cdot)$ is approximately quasiconvex and $\left(\sum_{i=1}^{p} \lambda_i f_i + \sum_{j \in J} \mu_j g_j\right)(\cdot)$ is approximately pseudoconvex at u. Then $\forall \gamma > 2\lambda^t \alpha\ \ \exists\, \delta_\gamma > 0$ such that the following does not hold*

$$f_i(x) < f_i(u) + \mu_J g_J(u) - \gamma \|x - u\|,\ \forall i = 1, \ldots, p$$

where $x = u + td$, $d \in \mathbb{R}^n$, $0 < t < \delta_\gamma$, is such that $x \in S_P$.

Proof: From the feasibility condition of (VD)

$$0 = \sum_{i=1}^{p} \xi_i + \sum_{j=1}^{m} \beta_j + \sum_{i=1}^{p} \lambda_i \alpha_i b \tag{9}$$

for some $\xi_i \in \partial \lambda_i f_i(u),\ i = 1, \ldots, p,\ \beta_j \in \partial \mu_j g_j(u),\ j = 1, \ldots, m$ and $b \in \mathbb{B}$.

Let $x \in S_P$. Then as $\mu_k \geqq 0,\ k \in K$

$$\sum_{k \in K} \mu_k g_k(x) \leqslant \sum_{k \in K} \mu_k g_k(u)\,. \tag{10}$$

Using the approximate quasi convexity of $\sum_{k \in K} \mu_k g_k(\cdot)$ at u, $\forall c > 0\ \exists \delta > 0$ such that whenever $x \in B(u, \delta) \cap S_P$ and (10) holds then

$$\sum_{k \in K} \beta_k (x - u) - c\|x - u\| \leqslant 0\,.$$

Without loss of generality we can assume that $\|d\| = 1$. Choose $x = u + td$, $0 < t < \delta$, with $x \in S_P$. The above arguments along with (9) yields

$$\left(\sum_{i=1}^{p} \xi_i + \sum_{j \in J} \beta_j + \sum_{i=1}^{p} \lambda_i \alpha_i b\right)(x - u) + c\|x - u\| \geqslant 0.$$

$$\left(\sum_{i=1}^{p} \xi_i + \sum_{j \in J} \beta_j\right)(x - u) + c'\|x - u\| \geqslant 0, \quad c' = \sum_{i=1}^{p} \lambda_i \alpha_i + c > 0. \tag{11}$$

Using approximate pseudoconvexity of $\left(\sum_{i=1}^{p} \lambda_i f_i + \sum_{j \in J} \mu_j g_j\right)(\cdot)$ at u we get the existence of $\delta' > 0$ such that whenever $x \in B(u, \delta')$ and (11) holds then

$$\sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{j \in J} \mu_j g_j(x) + 2c'\|x - u\| \geqslant \sum_{i=1}^{p} \lambda_i f_i(u) + \sum_{j \in J} \mu_j g_j(u).$$

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Set $\gamma = 2c'$ and $\delta_\gamma = \min\{\delta, \delta'\}$. So for $x = u + td$, $0 < t < \delta_\gamma$

$$\lambda^t(f(x) - f(u) - \mu_J g_J(u)e + \gamma e\|x - u\|) \geqslant 0,$$

implying that

$$f_i(x) < f_i(u) + \mu_J g_J(u) - \gamma\|x - u\|, \ \forall\, i = 1, \ldots, p$$

is not possible. Hence the result. ∎

DEFINITION 9: $(u_0, \lambda_0, \mu_0, \alpha_0) \in S_D$ is said to be a local weak quasi efficient solution of (VD) if there exist $\eta \in \operatorname{int}(\mathbb{R}^p_+)$ and a neighbourhood U of $(u_0, \lambda_0, \mu_0, \alpha_0)$ such that for any $(u, \lambda, \mu, \alpha) \in S_D \cap U$ the following can not hold

$$f_i(u_0) + \mu_{0J} g_J(u_0) + \eta_i\|u - u_0\| < f_i(u) + \mu_J g_J(u), \ \forall\, i = 1, \ldots, p.$$

We next prove a very important result namely the strong duality theorem. In fact this result demonstrates the importance of duality in optimisation theory.

THEOREM 5. (Strong Duality) *Suppose x_0 is a quasi efficient solution of* (VP) *and an appropriate constraint qualification (like Mangasarian Fromovitz Constraint Qualification) or regularity condition (like basic regularity condition) is satisfied at x_0. Then there exist $\alpha_0 \in \operatorname{int}\mathbb{R}^p_+$, $\lambda_0 \in \mathbb{R}^p_+$, $\mu_0 \in \mathbb{R}^m_+$ such that $(x_0, \lambda_0, \mu_0, \alpha_0)$ is feasible of* (VD). *Further if the conditions of weak duality hold with $\gamma > 2\max\limits_{1 \leqq i \leqq p}\{\alpha_i\}$ then $(x_0, \lambda_0, \mu_0, \alpha_0)$ is a local weak quasi efficient solution of* (VD) *and the objective values of* (VP) *and* (VD) *are equal.*

PROOF: On account of Theorem 2 and Remark 4 there exist $\alpha \in \operatorname{int}\mathbb{R}^p_+$, $\lambda_0 \in \mathbb{R}^p_+$, $\lambda_0^t e = 1$, $\mu_0 \in \mathbb{R}^m_+$ such that $(x_0, \lambda_0, \mu_0, \alpha_0) \in S_D$. Moreover the objective values of (VP) and (VD) are equal to $f(x_0)$. Invoking the weak duality between (VP) and (VD) we have for every $\gamma > 2\sum\limits_{i=1}^{p} \lambda_i\alpha_i$ there exists $\delta_\gamma > 0$ such that for any $u \in B(x_0, \delta_\gamma)$, $x_0 = u + td$, $0 < t < \delta_\gamma$, $d \in \mathbb{R}^n$, $\|d\| = 1$,

$$f_i(x_0) = f_i(u + td) < f_i(u) + \mu_J g_J(u) - \gamma\|u - x_0\|, \ \forall\, i = 1 \ldots, p$$

does not hold, implying that for any $u \in B(x_0, \delta_\gamma)$

$$f_i(x_0) + \mu_J g_J(x_0) + \gamma\|u - x_0\| < f_i(u) + \mu_J g_J(u), \ \forall\, i = 1 \ldots, p$$

does not hold. Consequently $(x_0, \lambda_0, \mu_0, \alpha_0)$ is a local weak quasi efficient solution of (VD). ∎

REMARK 5. In Theorem 5 *and thereby in* Theorem 4 if $\lambda_0 > 0$ then we can show that $(x_0, \lambda_0, \mu_0, \alpha_0)$ is a local quasi efficient solution of (VD).

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5. Concluding Remarks

Optimality conditions and duality results for vector optimisation problems have been derived in the past under ρ convexity and generalised ρ convexity conditions (see ref [**3, 4, 6, 10, 11**]). In all these studies the parameter ρ is assumed to be a fixed real number. Our aim in the present paper is to study the consequences of the variations in ρ. The concepts of approximate convexity and its two variants namely approximate quasiconvexity and approximate pseudoconvexity introduced in the paper serve this purpose. However these concepts are defined in a local sense, that is, for any given positive constant c we can find a neighbourhood of a point under consideration depending on c such that the corresponding inequalities are satisfied in that neighbourhood. It is precisely because of this that the optimality conditions and the duality results obtained here are local in nature. Moreover all the results are derived for a more general solution concept namely quasi efficient solution. Such solution concept for vector optimisation problem exists in literature but not much work has been reported on it.

Furthermore, while approximate convexity is defined by Ngai, Luc and Thera [**9**] for the extended real-valued functions on a real Banach space, in this article, we have restricted ourselves to the real-valued functions defined on a finite dimension subset X of $\mathbb{R}^n$. This is done for the sake of conceptual simplicity. One can attempt to extend the results of this article to the functions over a real Banach space. Detailed study about the two new families of functions and their applications in optimisation can also be investigated in the future. It will be worth exploring some properties characterising these classes of functions.

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