ON THE EXTENSIONS OF LIE ALGEBRAS

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1. Introduction. In this paper we give some results on the extensions of Lie algebras, with emphasis on the case of prime characteristic, although part of the paper is also of interest at characteristic 0. An *extension* of a Lie algebra L is a pair (E, π), where E is a Lie algebra and π is a homomorphism of E onto L. The *kernel* K of the extension is ker π. The extension is called *central* if $K \subseteq zE$ (the centre of E), *abelian* (*solvable*) if K is abelian (solvable), *split* if there is a homomorphism σ of L into E such that $\pi\sigma = 1_L$, and *trivial* if K is a direct summand of E. All the Lie algebras and representations considered in the paper are assumed to be finite-dimensional.

In § 2 we determine the relationship between Cartan decompositions of L and E, partially generalizing and extending to characteristic p some results of Chevalley (**9**, Chapter VI). In particular, over an infinite field we prove that H is a Cartan subalgebra of L if and only if $H = \pi(C)$ for some Cartan subalgebra C of E; moreover, we prove that such a C is unique up to conjugation if K is solvable, provided at characteristic p an additional hypothesis is satisfied, e.g., that $[E, K]$ is nilpotent of class less than p. This gives the conjugacy of Cartan subalgebras of a solvable Lie algebra E at characteristic 0, a result of Chevalley (**9**, pp. 221–222), and the same result at characteristic p provided, e.g., that E^ω (the intersection of all terms of the lower central series of E) is nilpotent of class less than p.

In § 3 we use the comparison between the Cartan decompositions of L and E to give a quick determination of the central extensions of the simple Lie algebras at characteristic p of classical type (in the sense of Mills and Seligman (**14**); they are the analogues of the simple algebras over the complex numbers). The extensions will be proved to be trivial unless L is of type A_{n-1} with $p|n$, i.e., $L \cong \mathrm{PSM}(n)$, in which case, all other extensions may be obtained from $\mathrm{SM}(n)$ (the $n \times n$ matrices of trace 0), which itself has only trivial extensions. I obtained this result some years ago for application (where its use is crucial) in joint work with Zassenhaus (see **7**; **5**) on the Lie algebras with a non-degenerate trace form and with a quotient trace form, but gave no proof of it there. The result was obtained independently and in another way by R. Steinberg, whose proof is sketched in (**16**). The special case in which L has non-degenerate Killing form was obtained by Campbell (**8**).

By the classical Levi theorem, all extensions of a semi-simple Lie algebra of

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characteristic 0 split, and, in particular, all central extensions are trivial. Section 4 continues the work of § 3 in an attempt to find sufficient conditions for splitting of a solvable extension of a simple algebra of classical type at characteristic p. By the standard reduction, one can obtain such conditions by just considering abelian extensions, and, indeed, abelian extensions for which the induced representation $\text{ad}_K L$ of L in K is irreducible. Two sufficient conditions for the splitting of such extensions will be given, one involving the weights of $\text{ad}_K L$ and the other its Casimir operator or trace form. Each condition is satisfied in some cases where the other is not, but the problem of finding best possible conditions remains open.

In § 5 we determine the central extensions of another important class of simple Lie algebras of characteristic p, the Albert-Zassenhaus algebras, and use this to answer a question about isomorphisms between these algebras.

2. Cartan subalgebras of extensions. For a representation Δ of a nilpotent Lie algebra H in M we shall use the terminology *primary function*, *primary component*, and *Fitting null and one components*, essentially as given in (**13**, pp. 41–43). In particular, a primary function of Δ is a mapping $\sigma: h \rightarrow \sigma_h(\lambda)$ of H into monic irreducible polynomials for which there exists an $x \neq 0$ in M such that for every h in H there is a t with $\sigma_h(\Delta h)^t x = 0$. If the characteristic roots of every Δh are in the base field F, then one can replace the consideration of primary functions and primary spaces by that of weights and weight spaces. We shall say that a primary function corresponds to the weight 0 if $\sigma_h(\lambda) = \lambda$ for all h in H. If H is a subalgebra of a Lie algebra L, then the Fitting null component of H for the decomposition of L relative to $\text{ad}H$ is the zero-algebra of H in L, and H is a Cartan subalgebra of L if and only if H equals its own zero-algebra in L.

LEMMA 2.1. *Let $\sigma_1, \ldots, \sigma_k$ be a set of primary functions belonging to a representation Δ of a nilpotent Lie algebra H over an infinite field F, and suppose that no σ_i corresponds to the weight* 0. *Then there exists an h in H such that, for $i = 1, \ldots, k$, $\sigma_{ih}(\lambda) \neq \lambda$.*

Proof. Suppose first that F is algebraically closed, and let $\alpha_1, \ldots, \alpha_k$ be the weights corresponding to $\sigma_1, \ldots, \sigma_k$. The hypothesis states that no α_i is 0. If the characteristic is 0, then $\alpha_1, \ldots, \alpha_k$ are linear functionals on H so that there is an h in H such that no $\alpha_i(h)$ is 0, and the conclusion holds. If the characteristic is a prime p, *weights are not necessarily linear.* However, if m is the smallest power of p which is equal to or greater than the nilpotency class of H, then (**17**, p. 96) $\alpha_1^m, \ldots, \alpha_k^m$ are polynomial functions on H, so that there is an h in H such that no $[\alpha_i(h)]^m$ is 0, and again the conclusion holds.

Now suppose that F is an arbitrary infinite field and that Ω is its algebraic closure. Under scalar extension to Ω, for each i, the primary component corresponding to σ_i decomposes into a direct sum of weight spaces of the representation Δ_Ω of H_Ω. Let the corresponding weights be $\alpha_{i1}, \ldots, \alpha_{il}$ $(l = l_i)$.

Then $\lambda - \alpha_{ij}(h)$ divides $\sigma_{ih}(\lambda)$ for all h in H. For some m, the functions $\alpha_{ij}{}^m$ ($i = 1, \ldots, k; j = 1, \ldots, l_i$) are polynomial functions on H_Ω. Since F is infinite, there exists an h in H such that $\Pi_{i,j}\alpha_{ij}{}^m(h) \neq 0$. For such an h, no $\sigma_{ih}(\lambda)$ is λ, and the lemma is proved.

COROLLARY 2.1. *Over an infinite field, any Cartan subalgebra H of a Lie algebra L is the zero-algebra of (the space spanned by) some element of H.*

Proof. Let $\sigma_1, \ldots, \sigma_k$ be the primary functions of $\mathrm{ad}_L H$ which do not correspond to the weight 0, and let h be one of the elements whose existence Lemma 2.1 asserts. Then it is easy to see that H is the zero-algebra of (h).

Another immediate consequence of Lemma 2.1 is the following result. This generalizes the key lemma of (**2**), which Barnes proved by a different approach.

COROLLARY 2.2. *Suppose that H is a Cartan subalgebra of a Lie algebra L over an infinite field, and that H acts diagonally in some scalar extension L_Ω. Then there is an h in H such that the minimal polynomial of* ad *h factors in Ω into the product of distinct linear polynomials and such that H is the zero-algebra of h (h is regular if H has minimal dimension).*

Proof. The element h of the proof of Corollary 2.1 works again.

LEMMA 2.2. *Let L_0 be the zero-algebra of some element x of a Lie algebra L over an infinite field. Then L_0 contains a Cartan subalgebra of L.*

Proof. The classical proof that the zero-algebra of a regular element of L is a Cartan subalgebra (see, for example, **13**, p. 59) actually shows that if L_0 is not nilpotent, then there is an element y in L_0 such that the zero-algebra of y is properly contained in L_0. The lemma follows by induction on the dimension of L_0.

THEOREM 2.1. *Let (E, π) be an extension of a Lie algebra L over a field F. If C is a Cartan subalgebra of E, then $\pi(C)$ is a Cartan subalgebra of L. Conversely, if H is a Cartan subalgebra of L and F is infinite, then there exists a Cartan subalgebra C of L such that $\pi(C) = H$.*

Proof. First, suppose that C is a Cartan subalgebra of E, and let K denote the kernel of π. If F is infinite, by Corollary 2.1 there is an x in C of which C is the zero-algebra. Suppose such an x is chosen and that y in E is such that $(\mathrm{ad}\, \pi(x))^m \pi(y) = 0$ for some m. Then $(\mathrm{ad}\, x)^m y \in K$. Write $(\mathrm{ad}\, x)^m y = z_0 + z_1$, where z_0 and z_1 belong to the Fitting null and one components K_0 and K_1, respectively, of the restriction of ad x to K. Since ad x is non-singular on K_1, there is a z_1' in K_1 such that $z_1 = (\mathrm{ad}\, x)^m z_1'$. Writing $y' = y - z_1'$, we have that $(\mathrm{ad}\, x)^m y' = z_0 \in C$. Hence, for some n, $(\mathrm{ad}\, x)^n y' = 0$, $y' \in C$, and $\pi(y) = \pi(y') \in \pi(C)$. Therefore, $\pi(C)$ contains the zero-algebra of $\pi(x)$. Also, $\pi(C)$ is nilpotent since C is, and, therefore, $\pi(C)$ is Cartan subalgebra of L. If F is finite, let Ω be an infinite extension field. Then $\pi(C)_\Omega = \pi_\Omega(C_\Omega)$ is a Cartan subalgebra of L_Ω. The normalizer $N_L\pi(C) \subseteq \pi(C)_\Omega \cap L = \pi(C)$, so that again $\pi(C)$ is a Cartan subalgebra of L.

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Conversely, suppose that H is a Cartan subalgebra of L, and that F is infinite. Take an h in L of which H is the zero-algebra, take an x in E such that $\pi(x) = h$, and let E_0 be the zero-algebra of x in E. By Lemma 2.2, there is a Cartan subalgebra C of E contained in E_0. Hence, for every y in C there is an m such that $(\text{ad } x)^m y = 0$, and $(\text{ad } h)^m \pi(y) = 0$, so that $\pi(C) \subseteq H$. But by the first part of the theorem, $\pi(C)$ is a Cartan subalgebra of L. It follows that $\pi(C) = H$, and the theorem is proved.†

LEMMA 2.3. *If L is of characteristic p and N is a nilpotent ideal of class less than p, then* $\exp(\text{ad } z)$ *is an automorphism of L for every z in N.*

Proof. Write $D = \text{ad } z$. Then $D^p = 0$, and if $i + j \geqq p$ and $x, y \in L$, then $(xD^i)(yD^j) = 0$, so that

$$\sum_{i=0}^{p-1} \frac{xD^i}{i!} \sum_{j=0}^{p-1} \frac{yD^j}{j!} = \sum_{m=0}^{p-1} \sum_{i=0}^{m} \frac{(xD^i)(yD^{m-i})}{i!(m-i)!} = \sum_{m=0}^{p-1} (xy) \frac{D^m}{m!}.$$

Thus, in this case, we do better than the general result that exp (ad x) *is an automorphism if* $(\text{ad } x)^{[(p+1)/2]} = 0$.

For any Lie algebra L, L^ω denotes the intersection of all terms of the lower central series of L.

THEOREM 2.2. *Suppose that (E, π) is an extension of L with solvable kernel K, over an infinite field. Let H be a Cartan subalgebra of L, and write $N = (\pi^{-1}(H))^\omega$. In case the characteristic is $p > 0$ suppose, in addition, either that N is contained in a nilpotent ideal of class less than p or that* $(\text{ad } n)^{[(p+1)/2]} = 0$ *for all n in N. Then a Cartan subalgebra C of E such that $\pi(C) = H$ is unique up to a conjugation of E of the form* $\prod_{i=0}^{d-1} \exp(\text{ad } n_i)$, *where $n_i \in N^{(i)}$ and d is the derived length of N.*

Proof. Let C_1 be another Cartan subalgebra of E such that $\pi(C_1) = H$. Since C and C_1 are also Cartan subalgebras of $\pi^{-1}(H)$, we see that, without loss of generality, we can assume that $L = H$ (at characteristic 0, ad n_i is nilpotent since $N \subseteq [E, K]$ is nilpotent). The proof is by induction on d. If $N = 0$, then $\pi^{-1}(H)$ is nilpotent and thus, $C = C_1 = \pi^{-1}(H)$. Now suppose that $d > 0$ and that the result is true for $d - 1$, and use bars to denote objects modulo $N^{(d-1)}$. Then $\bar{C}$ and $\bar{C}_1$ are Cartan subalgebras of $\bar{E}$ which are conjugate by an automorphism of the given form for elements $\bar{n}_i$ $(i = 0, \ldots, d-2)$. With n_i in the coset $\bar{n}_i$, $\exp(\text{ad } n_i)$ is an automorphism of E. Hence, it suffices

†*Added in proof.* There is an overlap between the material of § 2 and results of D. W. Barnes (*On Cartan subalgebras of Lie algebras*, Math. Z. *101* (1967), 350–355). His paper contains another proof of Theorem 2.1. His Theorem 4 states that if L is solvable of characteristic p and if L' satisfies the $(p-1)$st Engel condition, then all Cartan subalgebras of L are conjugate, but he mistakenly assumes that exp d is an automorphism if d is a derivation with $d^p = 0$, instead of $d^{[(p+1)/2]} = 0$. He thus has only proved the result with $[(p+1)/2] - 1$ instead of $p - 1$, and it is then a special case of Corollary 2.3 below. (The two papers were done independently, and Barnes submitted his five weeks earlier.)

to assume that $\bar{C}_1 = \bar{C}$ and to find n_{d-1}. In particular, it suffices to find a conjugacy of the form $\exp(\text{ad } k)$, $k \in K$, assuming that K is abelian, which we now do. Since K is abelian, $\text{ad}_K C$ induces a representation Δ of H in K. Write K_0 and K_1 for the Fitting null and one components of Δ. By Lemma 2.1, there is an x in H such that K_0 is the Fitting null component of Δx. Take y in C and y_1 in C_1 such that $\pi(y) = \pi(y_1) = x$. If B is the zero-algebra of y, then $C \subseteq B$, $\pi(B) = H$, and $B \cap K = K_0 = C \cap K$. Hence, $B = C$, and, similarly, C_1 is the zero-algebra of y_1. By adding an element of K_0 to y, we may assume that $y_1 - y \in K_1$. Since ad y is non-singular on K_1, there is a k in K_1 such that $[k, y] = y_1 - y$. Since $(\text{ad } k)^2 = 0$, $(\exp(\text{ad } k))y = y + [k, y] = y_1$. Hence, $\exp(\text{ad } k)C = C_1$, and the theorem follows.

COROLLARY 2.3. *Suppose that E is a solvable Lie algebra. In case the characteristic is $p > 0$, suppose, in addition, that the base field is infinite and either that E^ω is nilpotent of class less than p or that* $(\text{ad } x)^{[(p+1)/2]} = 0$ *for all x in E^ω. Then all Cartan subalgebras of E are conjugate.**

Proof. We can apply the theorem with $K = E^\omega$.

We note that when the extension is split, say $E = S + K$, it is not necessarily true that a Cartan subalgebra of S is contained in one of E, even if K is solvable, as is shown by the example $(x) + (y, z)$, where $[x, y] = [y, z] = z$ and $[x, z] = 0$.

We next compare the root space (or more generally, primary space) decompositions of L and its extension.

LEMMA 2.4. *Suppose that (E, π) is an extension of L with kernel K, and that C is a Cartan subalgebra of E. If σ is a primary function of* $\text{ad}_E C$ *with primary space E_σ, and if $\pi(E_\sigma) \neq 0$, then $\sigma_x(\lambda) = \sigma_{x+y}(\lambda)$ for all x in C and y in $C \cap K$, and the function $\sigma' \colon \pi(C) \to F[\lambda]$ defined by setting $\sigma'_{\pi(x)}(\lambda) = \sigma_x(\lambda)$ is a primary function on $\pi(C)$. Moreover, $\pi(E_\sigma) = L_{\sigma'}$, and every primary function of* $\text{ad}_L \pi(C)$ *is obtained in this way.*

Proof. Let σ be a primary function of $\text{ad}_E C$ and suppose that $b \in E_\sigma$ with $\pi(b) \neq 0$. If $x \in C$ and $y \in C \cap K$, then there is an m such that

$$(\sigma_x(\text{ad } x))^m b = (\sigma_{x+y}(\text{ad}(x + y)))^m b = 0.$$

Hence

$$(\sigma_x(\text{ad } \pi(x)))^m \pi(b) = (\sigma_{x+y}(\text{ad } \pi(x)))^m \pi(b) = 0.$$

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*After the paper was written, G. Seligman kindly sent me a copy of part of the typescript of his forthcoming book on Lie algebras of characteristic p. It contained a proof of the conjugacy of the Cartan subalgebras of solvable Lie algebras E for which E^ω is abelian, over arbitrary fields. The device Seligman uses to reduce the case in which the base field F is finite to that in which F is infinite can also be used to show that our Theorem 2.2 and Corollary 2.3 remain valid when F is finite. Seligman also gives an example of a solvable E, with E^ω not nilpotent, having non-conjugate Cartan subalgebras.

Since $\pi(b) \neq 0$ and $\sigma_x(\lambda)$ and $\sigma_{x+y}(\lambda)$ are irreducible, it follows that $\sigma_x(\lambda) = \sigma_{x+y}(\lambda)$ and that σ' is a primary function of $\mathrm{ad}_L \pi(C)$, with $\pi(b)$ in $L_{\sigma'}$. If τ is another primary function of $\mathrm{ad}_E C$ and $\sigma \neq \tau$, then $\sigma' \neq \tau'$. The fact that $L = \sum_\sigma \pi(E\sigma)$ now implies the remaining statements of the lemma.

COROLLARY 2.4. *Let L be a Lie algebra over an infinite field F, H a Cartan subalgebra, and M an L-module. Suppose that* ad h *and* h_M *have all characteristic roots in F for all h in H. Then any 2-dimensional M-cocycle for L differs by a coboundary from a cocycle f such that if α and β are roots, $x \in L_\alpha$, and $y \in L_\beta$, then $f(x, y) \in M_{\alpha+\beta}$ ($= 0$ if $\alpha + \beta$ is not a weight of M).*

Proof. An element of $H^2(L, M)$ corresponds to an abelian extension (E, π) of L with kernel M. Let C be a Cartan subalgebra of E such that $\pi(C) = H$. Any root γ for H corresponds to a root, also denoted by γ, for C, and $\pi(E_\gamma) = L_\gamma$. Hence, there is a linear mapping τ of L into E such that $\pi\tau = 1_L$ and $\tau(L_\gamma) \subseteq E_\gamma$ for each root γ. With f the corresponding cocycle,

$$f(x, y) = [\tau(x), \tau(y)] - \tau[x, y] \in E_{\alpha+\beta} \cap M = M_{\alpha+\beta}.$$

3. Central extensions of algebras of classical type.

LEMMA 3.1. *Suppose that E is a Lie algebra with no non-trivial central extension, and that π is a homomorphism of E into L. Then for every central extension (E_1, π_1) of L there exists a homomorphism σ of E into E_1 such that $\pi = \pi_1\sigma$.*

Proof. Let K_1 be the kernel of (E_1, π_1). Take a linear mapping τ of L into E_1 such that $\pi_1(\tau(x)) = x$, $x \in L$. Then $[\tau(x), \tau(y)] = \tau[x, y] + g_1(x, y)$ $(x, y \in L)$ for a 2-cocycle $g_1 \in Z^2(L, K_1)$. Define $g: E \times E \to K_1$ by setting $g(z, w) = g_1(\pi(z), \pi(w))$. Then $g \in Z^2(E, K_1)$, where K_1 is a trivial module for E. Hence, there exists an h in $C^1(E, K_1)$ such that $g = \delta h$, that is, $g_1(\pi(z), \pi(w)) = h[z, w]$. Set $\sigma(z) = \tau(\pi(z)) + h(z)$ $(z \in E)$. Then σ is linear from E into E_1,

$$[\sigma(z), \sigma(w)] = [\tau(\pi(z)) + h(z), \tau(\pi(w)) + h(w)] = \tau[\pi(z), \pi(w)] + g_1[\pi(z), \pi(w)] = \sigma[z, w] \quad (z, w \in E),$$

and $\pi_1\sigma = \pi$, and the proof is complete.

The Lie algebras of classical type over a field F of characteristic $p > 3$ are the algebras satisfying the axioms of Mills and Seligman (**14**). The simple algebras were shown in (**14**) to be the analogues of the complex simple Lie algebras, L_C, that is, the algebras over F obtained by reducing modulo p the structure constants of a Chevalley basis of L_C; in addition, if L_C is of type A_{n-1}, where $p|n$, one must divide the resulting algebra (a copy of $\mathrm{SM}(n, F)$, the $n \times n$ matrices of trace 0) by its one-dimensional centre (the scalar matrices) to obtain an algebra isomorphic to $\mathrm{PSM}(n, F)$. Algebras isomorphic to $\mathrm{PSM}(n, F)$ $(p|n)$ are said to be of *type* PA.

THEOREM 3.1. *Any central extension* (E, π) *of* L *is trivial if* L *is simple of classical type not* PA *or if* $L = \mathrm{SM}(n, F)$ $(p|n)$. *If* $L = \mathrm{PSM}(n, F)$, *either the central extension is trivial, or there is an isomorphism* σ *of* $\mathrm{SM}(n, F)$ *into* E *such that* E *is a direct sum of* $\sigma\mathrm{SM}(n, F)$ *and an ideal contained in the kernel* K, *and such that* $\pi\sigma$ *is the natural mapping of* $\mathrm{SM}(n, F)$ *onto* $\mathrm{PSM}(n, F)$.

Proof. Since $\pi(E^2) = \pi(E)$, by throwing away a direct summand of E contained in K we may assume that E is perfect. In proving an extension to be trivial we may also assume that F is infinite. If L is simple of classical type other than PA, take a Cartan subalgebra H of E such that $\pi(H)$ is a standard Cartan subalgebra of L. Then $K \subset H$; by Lemma 2.4, the roots of E with respect to H correspond to those of L with respect to $\pi(H)$, and for each non-zero root α, E_α is 1-dimensional, spanned, say, by e_α. Write $[e_\alpha e_{-\alpha}] = h_\alpha$. Then H is spanned by K together with all h_α, since their images span $\pi(H)$. For any h in H,

$$[[e_\alpha e_{-\alpha}]h] = [[e_\alpha h]e_{-\alpha}] + [e_\alpha[e_{-\alpha}h]] = (\alpha(h) - \alpha(h))[e_\alpha, e_{-\alpha}] = 0.$$

Hence, H is abelian and the elements h_α span H. If $\beta_1, \ldots, \beta_r$ is a fundamental system of roots for L (so that $\dim H \geqq r$ because of the type of L) and if α is a non-zero root, then α (or $-\alpha$) is a sum of a sequence of the β_i's such that each partial sum is also a root. Hence

$$[e_\alpha e_{-\alpha}] \in F[\ldots[e_{\beta_{i_1}} e_{\beta_{i_2}}]\ldots e_{\beta_{i_j}}][e_{-\beta_{i_1}}\ldots e_{-\beta_{i_j}}] \in (h_{\beta_1}, \ldots, h_{\beta_r})$$

by the Jacobi identity. Therefore, H is r-dimensional, and $K = 0$.

Similarly, if $L = \mathrm{SM}(n, F)$, where $p|n$, then we lift the $(n-1)$-dimensional Cartan subalgebra of diagonal matrices of trace 0 to a Cartan subalgebra H of E. As in the preceding case, there are $n - 1$ roots of L such that the corresponding h's span H, so that again $K = 0$.

Finally, the result for $\mathrm{PSM}(n, F)$ follows from that for $\mathrm{SM}(n, F)$ by Lemma 3.1.

One could also prove this last case by using a fundamental system with respect to the usual Cartan subalgebra of L, and defining, as before, elements $h_1, \ldots, h_{n-1}$ which span H, If these are dependent, then $K = 0$, and if they are independent, then one has an obvious mapping of $\mathrm{SM}(n, F)$ onto E which is an isomorphism.

COROLLARY 3.1. *Suppose that* L *is semi-simple of characteristic not* 2, 3. *If* L *has a representation with non-degenerate trace form, then every central extension of* L *is trivial.*

Proof. Scalar extension preserves non-degeneracy of the trace form as well as the property of being centreless or, equivalently (because of the form), perfect. Over an algebraically closed field, by (**7**), the algebra is a direct sum of simple algebras of classical type not PA, and the result follows from the theorem.

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4. Sufficient conditions for extensions to split. In this section we shall assume, when discussing roots and weights, that all relevant transformations representing elements of a Cartan subalgebra have their characteristic roots in the base field. Let (E, π) be an extension of L with kernel K. If C is a Cartan subalgebra of E and if $[C \cap K, K] = 0$, then a representation $\Delta = \mathrm{ad}_K\pi(C)$ of $\pi(C)$ in K is defined by setting $\Delta(\pi(x)) = \mathrm{ad}_K x$ for all x in C. By Lemma 2.4, each root α of L (for $\pi(C)$) has a unique corresponding root of E (for C), which we also denote by α, where $\alpha(\pi(c)) = \alpha(c)$ and $\pi(E_\alpha) = L_\alpha$.

THEOREM 4.1. *Suppose that C is a Cartan subalgebra of E and that*

$$[C \cap K, K] = 0.$$

If no weight of $\mathrm{ad}_K\pi(C)$ *is a root of L (for $\pi(C)$), and if* $[E_\alpha, E_\beta] = 0$ *whenever α and β are roots of L such that $\alpha + \beta$ is not a root of L, then the extension splits.*

Proof. Let S be the sum of the root spaces E_α of E such that $\pi(E_\alpha) \neq 0$. For each such α, the corresponding α for $\pi(C)$ is not a weight of $\mathrm{ad}_K\pi(C)$, and hence $E_\alpha \cap K = 0$. Therefore, the restriction of π to S is a one-to-one mapping onto L. The final condition of the hypotheses guarantees that S is a subalgebra. Hence, S is the required Levi factor.

When K is abelian, which we henceforth assume, we obtain an induced representation $\mathrm{ad}_K L$.

COROLLARY 4.1. *Suppose that H is a Cartan subalgebra of L and that the kernel K of an extension is abelian. If no weight of* $\mathrm{ad}_K H$ *is a sum of two roots (including* 0) *of L (for H), then the extension splits.*

COROLLARY 4.2. *Suppose that K is abelian and L is of classical type, of characteristic p. Let Σ and P be the sets of roots of L and weights of* $\mathrm{ad}_K L$, *respectively (for some Cartan subalgebra of L). If* (1) $0 \notin \mathrm{P}$ *and* (2) $\Sigma \cup \mathrm{P}$ *contains no circular string $\beta, \beta + \alpha, \ldots, \beta + (p-1)\alpha$, where α and β are roots, then (E, π) splits.*

Proof. If $\alpha \in \Sigma \cap \mathrm{P}$, consider the representation of the 3-dimensional simple algebra generated by L_α and $L_{-\alpha}$ on $K_\alpha + \ldots + K_{(p-1)\alpha}$. It follows from properties of representations of the 3-dimensional algebra that $i\alpha \in \mathrm{P}$, $i = 1, \ldots, p-1$, a contradiction. Similarly, if $\beta, \alpha \in \Sigma$ it can be seen that $\beta + \alpha \notin \mathrm{P}$. The result now follows from Corollary 4.1.

We now consider the case in which $\mathrm{ad}_K L$ is irreducible. For L simple of classical type, the restricted irreducible representations were classified by Curtis (**10**) by their maximal weights. For each such representation, the irreducible representation of the corresponding simple Lie algebra over the complex numbers with corresponding maximal weight is called the associated representation (**11**). A reduction modulo p of the associated representation gives a representation of L having the given irreducible representation as a constituent.

A result like that of Corollary 4.2 may be stated in terms of conditions on the associated representations of the irreducible constituents of $\mathrm{ad}_K L$ (assuming that these constituents are all restricted). Indeed, (E, π) will split if for each of these associated representations *the maximal weight*: (1′) *is not a sum of fundamental roots and* (2′) *is not too big* (with respect to p); e.g., if L is of type A_2 and the maximal weight is (a_1, a_2) (written with respect to a given fundamental system), we may take for (2′) the condition $a_1 + a_2 < p - 1$. Condition (2′) assures that no weight for L is associated with two distinct weights of the associated representation, and then, that conditions (1) and (2) of Corollary 4.2 are satisfied, (1) by (1′) and a result of Freudenthal (**12**).

We next consider what can be salvaged at characteristic p from the characteristic 0 proof of the Whitehead-Levi theorem. Suppose that B is a non-degenerate invariant bilinear form on a Lie algebra L, that $\{u_i\}$ and $\{u^i\}$ are bases of L dual with respect to B, and that Δ is a representation of L. We call $\sum_i \Delta(u_i)\Delta(u^i)$ the Casimir operator of Δ with respect to B, and denote it by $\Gamma(B, \Delta)$. It is in fact independent of the choice of dual bases, and commutes with all $\Delta(x)$. If Δ is absolutely irreducible, then $\Gamma(B, \Delta)$ is a scalar transformation which we denote by $c(B, \Delta)I$.

THEOREM 4.2. *Let (E, π) be an extension of L with abelian kernel K. Suppose, for each irreducible constituent Δ of $\mathrm{ad}_K L$, that L has a non-degenerate invariant bilinear form B such that $\Gamma(B, \Delta) \neq 0$. Then (E, π) splits.*

Proof. It is enough to consider the case in which $\mathrm{ad}_K L$ is irreducible. Since $\Gamma(B, \Delta)$ commutes with all $\Delta(x)$, it is non-singular. The standard characteristic 0 proof (**13**, p. 90) for the case in which the usual Casimir operator (for the Killing form) is non-singular may be seen to remain valid in the present case.

COROLLARY 4.3. *If each irreducible constituent Δ of $\mathrm{ad}_K L$ has non-degenerate trace form, and, at characteristic p, if $p \nmid \dim L$, then (E, π) splits.*

Proof. If we take B to be the trace form of Δ, then $\mathrm{tr}\, \Gamma(B, \Delta) = (\dim L)1$, whence the result.

We remark that if F is algebraically closed, L simple, and Δ irreducible, then $B_\Delta = t(B, \Delta)B$, where B_Δ denotes the trace form of Δ, B is the given form, and $t(B, \Delta) \in F$, and

$$\mathrm{tr}\, \Gamma(B, \Delta) = (\text{degree } \Delta)c(B, \Delta) = (\dim L)t(B, \Delta).$$

Hence, *if B_Δ is non-degenerate and $p \nmid \dim L$, then $\Gamma(B, \Delta) \neq 0$ and $p \nmid$* (degree Δ), *and conversely*. If L is of classical type and Δ is restricted, then $c(B, \Delta)$ may be computed from the maximal weight of Δ as at characteristic 0 (**13**, p. 247).

Now suppose that L is the simple 3-dimensional algebra ($p > 2$), with basis e, f, h, where $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. For the trace form B of the representation of degree 2, $B(h, h) = 2$ and $B(e, f) = 1$. Hence h, e, f and $h/2, f, e$ are dual bases of L for this form, and if an irreducible representation

Δ has maximal weight α with $\alpha(h) = m$, then $\Gamma(B, \Delta) = (m(m+2)/2)I$. It follows that an abelian extension of L with restricted irreducible $\mathrm{ad}_K L$ splits unless $\mathrm{ad}_K L$ is the representation with degree $p - 1$. On the other hand, let V be the irreducible L-module of degree $p - 1$ and let v and w be maximal and minimal vectors of V and a and b scalars not both 0. Let $E = L + V$ (direct sum as vector spaces) and let the multiplication of basis elements be that of the split abelian extension except that $[h, e] = -[e, h] = 2e + aw$ and $[h, f] = -[f, h] = -2f + bv$. This gives an extension of L which does not split since (h) is a Cartan subalgebra but does not act diagonally.

We have seen that for L simple of type A_1 there are irreducible restricted representations for which the hypotheses of Corollary 4.1 are not satisfied but which do have non-zero Casimir operator. For L simple of type A_2 there are restricted irreducible Δ for which $\Gamma(B, \Delta) = 0$ for all B but which do satisfy the hypotheses of Corollary 4.1 (and 4.2), e.g., if Δ has maximal weight (2, 3) and $p = 17$.

5. Central extensions of algebras of quasi-classical type. Let F be a field of characteristic p, G a finite additive subgroup of F, f a biadditive mapping of $G \times G$ into F, and L an algebra with a basis $\{u_\alpha | \alpha \in G\}$ indexed by G and multiplication given by

$$[u_\alpha, u_\beta] = \{\alpha - \beta + f(\alpha, \beta)\} u_{\alpha+\beta}.$$

If $f = 0$, then L is a Lie algebra called a *Zassenhaus algebra* (**17**). If $f \neq 0$, then L is a Lie algebra if and only if there exists an additive mapping l of G into F such that

$$f(\alpha, \beta) = \alpha l(\beta) - \beta l(\alpha) \qquad (\alpha, \beta \in G); \tag{5.1}$$

such a Lie algebra is called an *Albert algebra* (**1**). In addition to being the only known simple algebras of rank one (over a perfect field) other than A_1, these algebras gain importance from their role in the theory of Lie algebras of *quasi-classical type* (a perfect centreless Lie algebra is said to be of quasi-classical type if for each non-zero root ρ, the root spaces L_ρ and $L_{-\rho}$ generate the 3-dimensional simple Lie algebra). We proved in (**6**) that any Lie algebra of quasi-classical type over a perfect field is a direct sum of simple algebras which are either of classical type or Zassenhaus or Albert algebras.

Ree (**15**) proved the remarkable fact that all Zassenhaus algebras of the same dimension over an algebraically closed field F are isomorphic. However, the question of isomorphism between Albert and Zassenhaus algebras of the same dimension over F has been open. It is known (**3**) that they have isomorphic algebras of outer derivations. We show here that no Albert algebra is isomorphic to a Zassenhaus algebra, if $p > 3$, by examining their central extensions.

THEOREM 5.1. *If $p > 3$ and L is a Zassenhaus algebra over F, then $H^2(L, F)$ is 1-dimensional, while if L is an Albert algebra over F, then $H^2(L, F) = 0$.*

Proof. If L is a Zassenhaus or Albert algebra over F, then each Fu_α is a root space for the Cartan subalgebra Fu_0. Let g be a 2-cocycle on L with respect to the trivial module F. By Corollary 2.4, we may assume that $g(u_\alpha, u_\beta) = 0$ unless $\alpha = -\beta \neq 0$. For $\alpha \in G$, write $g(u_\alpha, u_{-\alpha}) = c_\alpha$. Then the condition $\delta g(u_\alpha, u_\beta, u_\gamma) = 0$ is automatically satisfied unless $\alpha + \beta + \gamma = 0$, in which case,

$$(5.2)\quad \{\alpha - \beta + f(\alpha, \beta)\}c_{\alpha+\beta} + \{-\alpha - 2\beta - f(\alpha, \beta)\}c_\alpha + \{2\alpha + \beta - f(\alpha, \beta)\}c_\beta = 0.$$

If $f = 0$ and if $c_\alpha = \alpha^3 - \alpha$ for each α, then (5.2) is satisfied, as is shown by a straightforward computation. Hence, in the Zassenhaus case, the 2-cochain k with $k(u_\alpha, u_{-\alpha}) = \alpha^3 - \alpha$ and $k(u_\alpha, u_\beta) = 0$ if $\alpha + \beta \neq 0$ $(\alpha, \beta \in G)$, is a cocycle. This cocycle is not a coboundary, since, if $\delta h = k$, then $\delta h(u_\alpha, u_{-\alpha}) = 2\alpha h(u_0) = \alpha^3 - \alpha$ for all α, a contradiction. Returning to the original cocycle g, for a given non-zero α we may assume that $c_\alpha = 0$ by subtracting δh from g, where h is the 1-cochain with $h(u_0) = c_\alpha/2\alpha$, $h(u_\beta) = 0$ $(\beta \neq 0)$. Then by (5.2), $(i - 1)\alpha c_{(i+1)\alpha} = (i + 2)\alpha c_{i\alpha}$, and by induction, $c_{i\alpha} = (1/6)(i + 1)i(i - 1)c_{2\alpha}$.

By multiplying each basis element of L by α^{-1} we may assume that $\alpha = 1$. Then, if $f = 0$, we subtract $(c_{2\alpha}/6)k$ from g to obtain a cocycle with $c_{i\alpha} = 0$ for all i. It is sufficient to show that all c_β vanish for this cocycle. If β is not of the form $i\alpha$, then application of (5.2) to the pairs β, 2α; $\beta + \alpha$, α; and β, α yields

$$c_{\beta+2\alpha} = \frac{\beta + 4\alpha}{\beta - 2\alpha} c_\beta; \qquad c_{\beta+2\alpha} = \frac{\beta + 3\alpha}{\beta} c_{\beta+\alpha} = \frac{(\beta + 3\alpha)(\beta + 2\alpha)}{\beta(\beta - \alpha)} c_\beta.$$

Equating the expressions for $c_{\beta+2\alpha}$ and expanding, we obtain $12\alpha^3 c_\beta = 0$, and $c_\beta = 0$. This completes the proof for the Zassenhaus case.

Next, suppose that $f \neq 0$, and take α, β such that $f(\alpha, \beta) \neq 0$, where, as before, we may suppose that $c_\alpha = 0$. We claim that $c_\beta = 0$. Applying (5.2) successively to the pairs $\beta + i\alpha$, α $(i = 0, 1, \ldots, p - 2)$ we obtain

$$c_{\beta+(p-1)\alpha} = \frac{\{\beta + p\alpha + f(\beta, \alpha)\} \ldots \{\beta + 2\alpha + f(\beta, \alpha)\}}{\{\beta + (p - 3)\alpha + f(\beta, \alpha)\} \ldots \{\beta - \alpha + f(\beta, \alpha)\}} c_\beta$$
$$= \frac{\{\beta - 2\alpha + f(\beta, \alpha)\}}{\{\beta + \alpha + f(\beta, \alpha)\}} c_\beta$$

if $\beta + i\alpha + f(\beta, \alpha) \neq 0$ $(i = -1, 0, \ldots, p - 3)$. But if $\beta + i\alpha + f(\beta, \alpha) = 0$, then (5.2) for $\beta + (i + 1)\alpha$, α yields $c_{\beta+(i+1)\alpha} = 0$, and hence $c_{\beta+i\alpha} = \ldots = c_\beta = 0$. We apply (5.2) for β, $-\alpha$ and obtain

$$\{\beta + \alpha - f(\beta, \alpha)\}c_{\beta-\alpha} = \{\beta - 2\alpha - f(\beta, \alpha)\}c_\beta.$$

Thus, we have two equations for $c_{\beta-\alpha}$ and c_β. The determinant of the coefficients is $6\alpha f(\beta, \alpha) \neq 0$, and therefore $c_\beta = 0$. Suppose that $\gamma \in G$ and $\gamma \neq 0$. If $l(\alpha) = 0$, then, by (5.1), if $l(\gamma) \neq 0$, then $f(\alpha, \gamma) \neq 0$, and hence $c_\gamma = 0$, while if $l(\gamma) = 0$, then $f(\beta, \gamma) \neq 0$ and again $c_\gamma = 0$. Hence, we may assume that $l(\alpha) \neq 0$ and by symmetry, also that $l(\beta) \neq 0$. By (5.1), $\alpha/\beta \neq l(\alpha)/l(\beta)$; if $f(\alpha, \gamma) = 0$, then $\gamma/\alpha = l(\gamma)/l(\alpha)$ and $\gamma/\beta \neq l(\gamma)/l(\beta)$ so that $f(\beta, \gamma) \neq 0$. Hence, in every case, $c_\gamma = 0$, that is, $g = 0$, and the proof is complete.

COROLLARY 5.1. *No Albert algebra with $p > 3$ is isomorphic to a Zassenhaus algebra.*

References

1. A. A. Albert and M. S. Frank, *Simple Lie algebras of characteristic p*, Univ. e Politec. Torino Rend. Sem. Mat. *14* (1954–55), 117–139.
2. R. T. Barnes, *On splitting fields for certain Lie algebras of prime characteristic*, Proc. Amer. Math. Soc. *17* (1966), 930–935.
3. R. E. Block, *On torsion-free abelian groups and Lie algebras*, Proc. Amer. Math. Soc. *9* (1958), 613–620.
4. ——— *Trace forms on Lie algebras*, Can. J. Math. *14* (1962), 553–564.
5. ——— *The Lie algebras with a quotient trace form*, Illinois J. Math. *9* (1965), 277–285.
6. ——— *On the Mills-Seligman axioms for Lie algebras of classical type*, Trans. Amer. Math. Soc. *121* (1966), 378–392.
7. R. E. Block and H. Zassenhaus, *The Lie algebras with a non-degenerate trace form*, Illinois J. Math. *8* (1964), 543–549.
8. H. E. Campbell, *On the Casimir operator*, Pacific J. Math. *7* (1957), 1325–1331.
9. C. Chevalley, *Théorie des groupes de Lie*, Tome III, *Théorèmes généraux sur les algèbres de Lie*, Actualités Sci. Indust., No. 1226 (Hermann, Paris, 1955).
10. C. W. Curtis, *Representations of Lie algebras of classical type with applications to linear groups*, J. Math. Mech. *9* (1960), 307–326.
11. ——— *On the dimensions of the irreducible modules of Lie algebras of classical type*, Trans. Amer. Math. Soc. *96* (1960), 135–142.
12. H. Freudenthal, *The existence of a vector of weight* 0 *in irreducible Lie groups without centre*, Proc. Amer. Math. Soc. *7* (1956), 175–176.
13. N. Jacobson, *Lie algebras* (Interscience, New York, 1962).
14. W. H. Mills and G. B. Seligman, *Lie algebras of classical type*, J. Math. Mech. *6* (1957), 519–548.
15. R. Ree, *On generalized Witt algebras*, Trans. Amer. Math. Soc. *83* (1956), 510–546.
16. R. Steinberg, *Générateurs, relations et revêtements des groupes algébriques*, Colloq. Théorie des Groupes Algébriques (Bruxelles, 1962), pp. 113–127, Librairie Universitaire, Louvain (Gauthier-Villars, Paris, 1962).
17. H. Zassenhaus, *Ueber Lie'sche Ringe mit Primzahlcharakteristik*, Abh. Math. Sem. Univ. Hamburg *13* (1939), 1–100.

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