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A finite set covering theorem II

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Let n, s, t be integers with $s > t > 1$ and $n > (t+2)2^{s-t-1}$. We prove that if n subsets of a set S with s elements have intersection I and union J then some t of them have intersection I and union J. The result is best possible.

1. Introduction

Small letters denote non-negative integers and large letters denote sets. Also $[i, j]$ denotes the set $\{i, i+1, i+2, \ldots, j\}$. We assume $s > t > 1$ and let $S = [1, s]$. If a family M of subsets of S has union S we say it *covers* S. If it covers S and has empty intersection 0 we say it *laces* S. We say we *invert* an element k of S in M when we adjoin k to all sets in M not possessing k and delete k from all sets in M possessing k. Clearly inversion will not affect lacing. An important family of subsets of S is

$$E = \{X;\ X = P \cup Q,\ P \subset [1, t+1],\ |P| \leq 1,\ Q \subset S \backslash P\} .$$

The number e of sets in E is

$$e = e(s, t) = (t+2)2^{s-t-1} ,$$

and these e sets lace S. Our result is the

THEOREM. *Let* n, s, t *be integers with* $s > t > 1$ *and let* $N = \{X_1, X_2, \ldots, X_n\}$ *be* n *different subsets* X_i *of* $S = [1, s]$. *Firstly suppose* N *covers* S *but no* t *sets* X_i *of* N *cover* S. *Then*

(*i*) $n \leq e$, *and*

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(ii) *if* $3 \le t$ *and* $n = e$ *we can obtain* N *from* E *by permuting the elements of* S .

Secondly suppose N *laces* S *but no* t *sets* X_i *of* N *lace* S . *Then*

(iii) $n \le e$, *and*

(iv) *if* $3 \le t$ *and* $n = e$ *we can obtain* N *from* E *by permuting and inverting elements of* S .

When $t = 2$ the value e can be attained in many ways beside E , for instance

$$F = \{X;\ X = P \cup Q,\ P = 1 \text{ or } 2 \text{ or } 3 \text{ or } [1, 3],\ Q \subset [4, s]\} .$$

In an earlier paper [*1*] we proved parts *(i)* and *(ii)* of the theorem and we will use them in proving the remainder. When the characterization of the extreme case is not required, the theorem takes the pleasing form presented in the abstract at the beginning of this paper.

2. Preliminary results

Let M be a family of subsets of S and for each $k \in S$ put

$$A_k(M) = \{X;\ X \cup k \in M,\ X \backslash k \in M\} ,$$

$$B_k(M) = \{X;\ X \in M,\ k \in X,\ X \backslash k \notin M\} ,$$

$$C_k(M) = \{X;\ X \in M,\ k \notin X,\ X \cup k \notin M\} ,$$

$$B'_k(M) = \{X \backslash k;\ X \in B_k(M)\} .$$

For example $C_1(E) \neq 0$ but $B_1(E) = B_s(E) = C_s(E) = 0$. Then

$$M = A_k(M) \cup B_k(M) \cup C_k(M)$$

is a partition of M , and

$$\Delta = \Delta(M, k) = A_k(M) \cup B'_k(M) \cup C_k(M)$$

defines and partitions a family Δ of subsets of S . Clearly

(1) $$A_k(M) = A_k\big(\Delta(M, k)\big) ,$$

(2) $$B_k\big(\Delta(M, k)\big) = 0 ,$$

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(3) $$|B_k(M)| + |C_k(M)| = |C_k(\Delta(M, k))| ,$$

(4) $$|M| = |\Delta| .$$

We will need

LEMMA 1. *If* $j, k \in S$ *and* M *is a family of subsets of* S *and* $\Delta_k = \Delta(M, k)$ *then*

$$\left.\begin{aligned} |A_j(M)| &\le |A_j(\Delta_k)| \\ |B_j(M)| &\ge |B_j(\Delta_k)| \\ |C_j(M)| &\ge |C_j(\Delta_k)| \end{aligned}\right\} \text{ if } j \ne k .$$

Proof. To simplify notation suppose $k = 1$ and $j = 2$. Then for each fixed subset R of $[3, s]$ consider 16 cases as set out in Table 1.

Family M				Family $\Delta_1 = \Delta(M, 1)$			
R	$1 \cup R$	$2 \cup R$	$1 \cup 2 \cup R$	R	$1 \cup R$	$2 \cup R$	$1 \cup 2 \cup R$
0	0	0	0	no change			
0	0	0	B	0	0	B	0
0	0	B	0	no change			
0	0	B	B	no change			
0	C	0	0	C	0	0	0
0	A	0	A	A	0	A	0
0	C	B	0	A	0	A	0
0	A	B	A	A	0	A	B
C	0	0	0	no change			
C	0	0	B	A	0	A	0
A	0	A	0	no change			
A	0	A	B	no change			
C	C	0	0	no change			
C	A	0	A	A	C	A	0
A	C	A	0	no change			
A	A	A	A	no change			

Table 1

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We explain the table by discussing the third line up. This line corresponds to the case $R \in M$, $1 \cup R \in M$, $2 \cup R \notin M$, $1 \cup 2 \cup R \in M$, so that R is in $C_2(M)$ while $1 \cup R$ and $1 \cup 2 \cup R$ are in $A_2(M)$. These facts are indicated by the letters C, A, 0, A in the left hand columns headed R, $1 \cup R$, $2 \cup R$, $1 \cup 2 \cup R$ respectively. The sets R, $1 \cup R$ are in $A_1(M)$ and so are unaltered when changing from M to $\Delta_1 = \Delta(M, 1)$. However the set $1 \cup 2 \cup R$ is in $B_1(M)$ and so is changed to $2 \cup R$. Thus in Δ_1 we get R and $2 \cup R$ in $A_2(\Delta_1)$ and $1 \cup R$ in $C_2(\Delta_1)$, as indicated by the letters A, C, A, 0 in the right hand columns.

The table is not difficult to check. The inequalities of the lemma hold for each line of the table, and the result follows.

LEMMA 2. *Let M be a family of subsets of S which lace S, and such that no t of the subsets lace S. Then if $\Delta = \Delta(M, 1)$ can be obtained from E by permuting and inverting elements of S, so can M.*

Proof. If $C_1(M) = 0$ then Δ is obtained from M by inverting element 1, so M can be obtained from E by permuting and inverting. Now suppose $C_1(M) \neq 0$. Then $C_1(\Delta) \neq 0$ by (3), and therefore, by suitably permuting and inverting the elements $2, 3, \ldots, s$ in M and Δ simultaneously, we can make $\Delta = E$. If $B_1(M) \neq 0$ there must be a set X in M containing two elements of $[1, t+1]$ and hence t sets in M lacing S, a contradiction. Therefore $B_1(M) = 0$ and $M = \Delta$. This completes the proof.

3. Proof of parts (iii) and (iv) of the theorem

The case $t = 2$ is trivial because we can't have a set and its complement among the X_i. For $t > 2$ we use induction. We assume the theorem true for $s-1$, $t-1$ and deduce its validity for s, t. Also we suppose that n is as large as possible.

We define a sequence $N_0, N_1, \ldots, N_s$ of families of subsets of S by $N_0 = N$ and

$$N_k = \Delta(N_{k-1}, k) \quad \text{for} \quad k = 1, 2, \ldots, s .$$

We have $|N_k| = |N_0| = n$ by (4). Also we notice from (2) that

$B_k(N_k) = 0$. So if t sets of N_k were to lace S at least one of them would possess the element k and so lie in $A_k(N_k)$. Then we would easily get t sets of N_{k-1} lacing S . Thus by induction we know that for $0 \le k \le s$ no t sets of N_k lace S .

Next we claim that if $1 \le j \le s$ and $0 \le k \le s$ then $A_j(N_k) \ne 0$. For otherwise, by (1) and Lemma 1, we would have $A_j(N) = 0$. Then by renumbering the elements of S we could have $j = 1$ so $A_1(N) = 0$. Since N laces S this would imply $B_1(N) \ne 0$ and $C_1(N) \ne 0$. Consider the family N' of subsets of $[2, s]$ defined by

$$N' = \{X \backslash 1,\ X \in N\} .$$

There are as many sets in N' as in N and they lace $[2, s]$. Moreover no $t - 1$ of them lace $[2, s]$, for if they did, because $B_1(N) \ne 0$ and $C_1(N) \ne 0$, we would immediately get t sets of N lacing S . Thus by our induction hypothesis

$$|N| = |N'| \le e(s-1, t-1) < e(s, t) = |E| ,$$

contradicting our assumption that n is maximal. This proves that no $A_j(N_k)$ is 0 , and hence that each of $N_0, N_1, \ldots, N_s$ lace S .

Consider now N_s . By (2) and Lemma 1 we have $B_j(N_s) = 0$ for $1 \le j \le s$. Further no t sets $Y_1, \ldots, Y_t$ of N_s can cover S . For suppose they did and let $h \in S$. If h is in every Y_i then every Y_i is in $A_h(N_s)$, and we replace Y_1 by $Y_1 \backslash h$. Repeating this for every $h \in S$ would produce t sets of N_s which lace S , which we just proved was impossible. Thus part *(i)* of the theorem applies to N_s and we have $n = |N_s| \le e$ proving *(iii)*.

Finally suppose $|N_s| = e$ and $t > 2$. Then part *(ii)* of the theorem says that N_s can be obtained from E by permuting elements of S . In turn Lemma 2 says that N_{s-1} can be obtained from E by permuting and inverting elements of S and, by repeated application of the lemma, so

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can $N_0 = N$. This proves *(iv)*.

Reference

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