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Spatial Homogenization of Stochastic Wave Equation with Large Interaction

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Abstract. A dynamical approximation of a stochastic wave equation with large interaction is derived. A random invariant manifold is discussed. By a key linear transformation, the random invariant manifold is shown to be close to the random invariant manifold of a second-order stochastic ordinary differential equation.

1 Introduction

Let D be an open bounded regular domain in $\mathbb{R}^n$ with $0 \le n \le 3$ and denote by Γ the boundary of D. We consider the following singularly perturbed stochastic wave equation under the homogeneous Neumann boundary condition with a large $k > 0$:

$$u^k_{tt} + \nu u^k_t = k\Delta u^k + f(u^k) + \dot{W}, \quad x \in D, \qquad \frac{\partial u^k}{\partial n}\Big|_{\partial D} = 0. \tag{1.1}$$

This system describes the motion of the particles in the continuum in a stochastic force field $\dot{W}$, where k represents the strength of near neighbor, particle-particle, quasi-elastic interaction forces. If the interaction is strong enough, namely, k is large enough, the behavior of the system has a spatial homogenization phenomenon. First, let us look at a spatial homogeneous system

$$u_{tt} + \nu u_t = \overline{f}(u) + \dot{\overline{W}}, \tag{1.2}$$

where $\overline{f}$ is the spatial homogenization of f, which is to be determined later, and

$$\overline{W}(t) = \frac{1}{|D|}\int_D W(t,x)\,dx.$$

For a large k, we show that the random dynamics of the stochastic problem (1.1) is approximated by that of the spatial homogeneous system (1.2).

Random invariant manifolds are very important in modelling random dynamics of a stochastic system [17], especially infinite dimensional systems (see *e.g.*, [1, 16]). Duan et al. [5, 6] generalized deterministic methods to construct a random invariant manifold for stochastic partial differential equations with multiplicative noise. Lu and

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Schmalfuss [8] and Liu [7] constructed random invariant manifolds for a stochastic wave equation. There has been considerable attention dedicated to the approximation of the stochastic wave equation on finite time intervals [3, 4] and for the long time behavior in an almost sure sense [10, 18]. Lv et al. [9, 11] explored the approximation of the random inertial manifold of singularly perturbed stochastic wave equation.

Spatial homogeneity of the asymptotic behavior has also been paid much attention in the study of infinite-dimensional dynamics systems. Carvalho and Hale [2] studied the spatial homogeneity when the diffusion coefficients are large. Qin [15] discussed the spatial homogeneity and invariant manifolds for the damped hyperbolic equations.

In this paper, we are concerned with the random dynamics of the stochastic problem (1.1) approximated by using the spatial homogeneous system (1.2) when k is sufficiently large, that implies the spatial homogeneity of the stochastic system. To show this, after presenting preliminaries and main results in Section 2, we consider an associated linear system in Section 3, whose solutions can be expressed explicitly. In Section 4, we introduce a linear transformation that is a critical step in our discussions. In Section 5, we study the nonlinear problem (1.1) by constructing a random invariant manifold which is asymptotically complete. By virtue of an estimate of the random invariant manifold and the linear transformation, we obtain the limit of the random invariant manifold.

2 Preliminaries and Main Results

Recall the abstract linear operator $A = -\Delta$ on D with the zero Neumann boundary condition. Then there are $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_i \le \cdots$ and functions $e_i(x)$ $(i = 1, 2, \dots)$ such that $Ae_i = \lambda_i e_i$.

Define $L^2(D)$ as all square integrable functions on D and

$$H^1_n(D) = \Big\{ u \in L^2(D) : \nabla u \in L^2(D) \text{ and } \frac{\partial u}{\partial n}\Big|_{\partial D} = 0 \Big\}.$$

Rewrite the stochastic nonlinear wave equation as the following second-order stochastic evolutionary problem

$$\begin{aligned} u^k_{tt} + \nu u^k_t &= -kAu^k + f(u^k) + \dot{W}, \\ u^k(0) = u_0, &\quad u^k_t(0) = u_1, \end{aligned} \tag{2.1}$$

with $(u_0, u_1) \in H^1_n(D) \times L^2(D)$, and $\dot{W}$ is the formal derivative of the Wiener process W. For the above problem, we assume that the following conditions (H) hold:

(H) (a) There is a constant $L_f > 0$ such that $|f(s) - f(t)| \le L_f|s - t|$ for any $s, t \in \mathbb{R}$ and $f(0) = 0$.
(b) The stochastic process W is a Q-Wiener process with $\operatorname{tr} Q < \infty$.
(c) The probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$ is the canonical probability space with the Wiener measure $\mathbb{P}$ [1]. More precisely, W is the identity on Ω, where

$$\Omega = \Big\{ \omega \in C(\mathbb{R}, L^2(D)) : \omega(0) = 0 \Big\}.$$

Using a similar discussion as that in [14], we have the following result regarding the existence of solutions of system (2.1).

Theorem 2.1 *For $k>0$, assume that conditions (H) hold. Then for any $(u_0,u_1)\in H_n^1(D)\times L^2(D)$ and any $T>0$, system (2.1) has a unique solution*

$$(u^k,u_t^k)\in L^2\big(\Omega, C(0,T;H_n^1(D)\times L^2(D))\big).$$

Denote the norm of $L^2(D)$ by $\|\cdot\|$. The following result is regarding an approximation that describes the random dynamics of system (2.1) for the large $k>0$.

Theorem 2.2 *Assume that conditions (H) hold and L_f is sufficiently small. System (2.1) has an invariant manifold $\mathcal{M}^k(\omega)$ that is exponentially attracting. That is, for any solution $(u^k(t),u_t^k(t))$ of system (2.1), there is a solution $(\overline{u}^k(t),\overline{u}_t^k(t))$ lying on $\mathcal{M}^k(\omega)$ such that*

$$\big\|(u^k(t),u_t^k(t))-(\overline{u}^k(t),\overline{u}_t^k(t))\big\|\longrightarrow 0\quad \text{as } t\longrightarrow\infty.$$

Furthermore, for any $t>0$ we have

$$\|(\overline{u}^k(t),\overline{u}_t^k(t))-(\overline{u}(t),\overline{u}_t(t))\|\longrightarrow 0\quad \text{as } k\longrightarrow\infty,$$

where $(\overline{u},\overline{u}_t)$ is the solution of the spatial homogeneous system (1.2) with initial conditions

$$\overline{u}(0)=\int_D\overline{u}^k(0)\,dx\quad \text{and}\quad \overline{u}_t(0)=\int_D\overline{u}_t^k(0)\,dx.$$

3 Linear Problem

Let us consider a linear problem

$$\text{(3.1)}\qquad \begin{gathered}\eta_{tt}^k+\nu\eta_t^k=k\Delta\eta^k+\dot{W},\\ \frac{\partial\eta^k}{\partial n}\Big|_{\partial D}=0,\end{gathered}$$

where $\eta^k=\sum_{i=1}^\infty\eta_i^ke_i$. So we get

$$\ddot{\eta}_i^k+\nu\dot{\eta}_i^k=-k\lambda_i\eta_i^k+\sqrt{q}_i\dot{\beta}_i.$$

For $i\geq 2$, we know that the above equation has a unique stationary solution η_i^* with the distribution

$$\mathcal{N}\Big(0,\frac{q_i}{2\nu k\lambda_i}\Big),$$

which is strong mixing with the exponential rate. Furthermore, one can see that for $i\geq 2$ there holds $\eta_i^*\to 0$ as $k\to\infty$. So the behavior of the higher modes ($i\geq 2$) is simple, and is killed after a long time as $k\to\infty$.

For $i=1$, we have a k-independent system, and so denote η_1^k by η_1; that is,

$$\text{(3.2)}\qquad \ddot{\eta}_1+\nu\dot{\eta}_1=\sqrt{q_1}\dot{\beta}_1.$$

To describe the dynamics of the above system, we change it into an equivalent first-order system

$$\dot{\eta}_1 = \delta_1, \quad \dot{\delta}_1 = -\nu\delta_1 + \sqrt{q_1}\dot{\beta}_1.$$

The linear part has eigenvalues 0 and $-\nu$ with two eigenvectors

$$\overline{e}_1 = \frac{1}{\nu}\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \overline{e}_2 = \frac{1}{\nu}\begin{pmatrix}1\\-\nu\end{pmatrix},$$

respectively. Recall the inner product as [12]:

$$\langle U_1,\ U_2\rangle = 2\Big[\frac{\nu^2}{4}u_1u_2 + \Big(\frac{\nu}{2}u_1 + v_1\Big)\Big(\frac{\nu}{2}u_2 + v_2\Big)\Big],$$

where $U_1 = (u_1, v_1)^T$ and $U_2 = (u_2, v_2)^T$. Using this inner product, the components of (η_1, δ_1) along the eigenvectors $\overline{e}_1$ and $\overline{e}_2$ respectively, are

$$\dot{x}_1 = \sqrt{q_1}\dot{\beta}_1, \quad \dot{y}_1 = -\nu y_1 - \sqrt{q_1}\dot{\beta}_1,$$

with $x_1 = \langle(\eta_1, \delta_1), \overline{e}_1\rangle$ and $y_1 = \langle(\eta_1, \delta_1), \overline{e}_2\rangle$. Thus, we decouple η_1 and δ_1 into x_1 and y_1. The dynamics of this system can be described by one-dimensional Wiener process

$$x_1(t) = x_1(0) + \sqrt{q}_1\beta_1(t)$$

and the one-dimensional Gaussian process

$$y_1(t) = e^{-\nu t}y_1(0) - \sqrt{q_1}\int_0^t e^{-\nu(t-s)}d\beta_1(s).$$

Notice that the dynamics of (x_1, y_1) are independent of k. Then the dynamics of the linear problem (3.1) as $k \to \infty$, are described by those of equation (3.2), namely, the case of equation (1.2) with $f(u) = 0$.

4 A Linear Transformation

We present an explicit representation of a linear transformation that plays an important role in the proof of our main result in the next section.

For $i \geq 2$ and the fixed k, we change the second-order system

$$\ddot{\eta}_i + \nu\dot{\eta}_i = -k\lambda_i\eta_i + \sqrt{q_i}\dot{\beta}_i.$$

into an equivalent first-order system

$$\dot{\eta}_i = \delta_i, \quad \dot{\delta}_i = -k\lambda_i\eta_i - \nu\delta_i + \sqrt{q_i}\dot{\beta}_i.$$

A simple calculation gives the eigenvalues of the linear part as

$$\mu_i^{\pm} = -\frac{\nu}{2} \pm \sqrt{\frac{\nu^2}{4} - k\lambda_i},$$

with the two corresponding eigenvectors

$$e_i^{\pm} = \begin{bmatrix}1\\ \mu_i^{\pm}\end{bmatrix}, \quad i \geq 2.$$

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Use the inner product defined as [12]:

$$\langle U_1, U_2\rangle = 2\Big[k\lambda_i u_1\overline{u}_2 + \Big(\frac{\nu^2}{4} - 2\lambda_2\Big) u_1\overline{u}_2 + \Big(\frac{\nu}{2}u_1 + v_1\Big)\Big(\frac{\nu}{2}\overline{u}_2 + \overline{v}_2\Big)\Big],$$

where $U_1 = (u_1, v_1)^T$ and $U_2 = (u_2, v_2)^T$. We set

$$\overline{e}_i^+ = \frac{1}{2\sqrt{k\lambda_i - \lambda_2}}\begin{bmatrix}1\\ \mu_i^+\end{bmatrix} \quad \text{and} \quad \overline{e}_i^- = \frac{1}{2\sqrt{k\lambda_i - \lambda_2}}\begin{bmatrix}1\\ \mu_i^-\end{bmatrix}.$$

By the above inner product, considering the components of (η_i, δ_i) along the eigenvectors $\overline{e}_i^+$ and $\overline{e}_i^-$ respectively, gives

$$\begin{pmatrix}x_i\\ y_i\end{pmatrix} = \frac{1}{\sqrt{k\lambda_i - \lambda_2}}\begin{bmatrix} k\lambda_i + \frac{\nu^2}{4} - 2\lambda_2 - \sqrt{\frac{\nu^2}{4} - k\lambda_i} & -\sqrt{\frac{\nu^2}{4} - k\lambda_i}\\ k\lambda_i + \frac{\nu^2}{4} - 2\lambda_2 + \sqrt{\frac{\nu^2}{4} - k\lambda_i} & \sqrt{\frac{\nu^2}{4} - k\lambda_i}\end{bmatrix}\begin{pmatrix}\eta_i\\ \delta_i\end{pmatrix}.$$

Thus, we further get

$$\begin{pmatrix}\eta_i\\ \delta_i\end{pmatrix} = \frac{\sqrt{k\lambda_i - \lambda_2}}{2(k\lambda_i + \frac{\nu^2}{4} - 2\lambda_2)}\begin{bmatrix} 1 & 1\\ -1 - \frac{k\lambda_i + \frac{\nu^2}{4} - 2\lambda_2}{\sqrt{\frac{\nu^2}{4} - k\lambda_i}} & 1 + \frac{k\lambda_i + \frac{\nu^2}{4} - 2\lambda_2}{\sqrt{\frac{\nu^2}{4} - k\lambda_i}}\end{bmatrix}\begin{pmatrix}x_i\\ y_i\end{pmatrix};$$

that is,

$$\eta_i = \frac{\sqrt{k\lambda_i - \lambda_2}}{2(k\lambda_i + \frac{\nu^2}{4} - 2\lambda_2)}(x_i + y_i), \qquad i \geq 2. \tag{4.1}$$

5 Nonlinear System

Now we get back to the nonlinear problem (1.1). We restrict our attention to the limit of the dynamics of u^k as $k \to \infty$. The expected limit equation becomes

$$u_{tt} + \nu u_t = Pf(u) + P\dot{W},$$

where P is the projection from H to $H_1 = \operatorname{span}\{e_1\}$. Denote by $Q = Id_H - P$ and split u^k as

$$u^k = Pu^k + Qu^k := u^{1,k} + u^{2,k}.$$

Then the problem (1.1) can be re-expressed as

$$\begin{aligned} u_{tt}^{1,k} + \nu u_t^{1,k} &= Pf(u^{1,k} + u^{2,k}) + \dot{W}_1,\\ u_{tt}^{2,k} + \nu u_t^{2,k} &= k\Delta u^{2,k} + Qf(u^{1,k} + u^{2,k}) + \dot{W}_2,\\ \frac{\partial u^{2,k}}{\partial n}\Big|_{\partial D} &= 0, \end{aligned}$$

where $W_1 = PW$ and $W_2 = QW$.

5.1 Random Invariant Manifold

In this subsection, we construct a random inertial invariant manifold for the problem (1.1) with the fixed $k > 0$.

The construction of random inertial manifold has been discussed by a number of works [7,8]. To make this paper self-contained and state our discussions in a straightforward way, we give a brief introduction of this construction. For the detailed analysis, we refer the reader to [7,8].

We use the transformation $\widetilde{u}^k = u^k - \eta^k$, where $\eta^k = \eta^{1,k} + \eta^{2,k}$ with $\eta^{1,k} = \eta_1$ satisfies equation (3.2) with $\eta_1(0) = 0$, and $\eta_t^{1,k}$ and $(\eta^{2,k}, \eta_t^{2,k})$ are stationary such that (η^k, η_t^k) satisfies system (3.1). Make the change of variables

$$\widetilde{u}_t^k = \widetilde{v}^k \quad \text{and} \quad \widetilde{U}^k = (\widetilde{u}^k, \widetilde{v}^k).$$

By the definition of $\widetilde{U}^k$, we get a random differential equation

$$\widetilde{U}_t^k(t,\omega) = C\widetilde{U}^k(t,\omega) + F\big(\widetilde{U}^k(t,\omega), \theta_t\omega\big), \tag{5.1}$$

where

$$C = \begin{bmatrix} 0 & 1 \\ -kA & -\nu \end{bmatrix} \quad \text{and} \quad F(\widetilde{U}^k, \omega) = \begin{bmatrix} 0 \\ f(\widetilde{u}^k + \eta^k) \end{bmatrix}.$$

Let $E = H_0^1(D) \times L^2(D)$. Set

$$E_{11} = \operatorname{span}\left\{\begin{bmatrix} e_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e_1 \end{bmatrix}\right\} \quad \text{and} \quad E_{22} = \operatorname{span}\left\{\begin{bmatrix} e_i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e_i \end{bmatrix} : i = 2, 3, \dots\right\}.$$

Then $E = E_{11} \oplus E_{22}$, where E_{11} is orthogonal to E_{22} by the orthogonality of $\{e_i\}$, and $\dim E_{11} = 2$. Moreover, both E_{11} and E_{22} are invariant subspaces of the operator C.

Since the eigenvalues of A are $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n \le \cdots$ with corresponding eigenvectors e_i $(i = 1, 2, \dots)$ by restricting C to E_{11}, the eigenvalues of $C|_{E_{11}}$ are 0 and $-\nu$ with the associated eigenvectors

$$e_1^+ = \frac{1}{\nu}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_1^- = \frac{1}{\nu}\begin{pmatrix} 1 \\ -\nu \end{pmatrix},$$

respectively.

By restricting C to E_{22}, the eigenvalues of $C|_{E_{22}}$ are

$$\mu_i^\pm = -\frac{\nu}{2} \pm \sqrt{\frac{\nu^2}{4} - k\lambda_i}, \quad i = 2, 3, \dots,$$

with the corresponding eigenvectors

$$e_i^\pm = \begin{bmatrix} e_i \\ \mu_i^\pm e_i \end{bmatrix}, \quad i = 2, 3, \dots.$$

Let

$$E_1 = \operatorname{span}\{e_1^+\} \quad \text{and} \quad E_{-1} = \operatorname{span}\{e_1^-\}.$$

So we see that $E_{11} = E_1 \oplus E_{-1}$, and E_1 and E_{-1} are invariant subspaces of the operator C. Let P_1 and P_{-1} be the corresponding spectral projections [13], and let P_{22} be the unique orthogonal projection onto E_{22}. Then there exists a decomposition $E = E_1 \oplus E_{-1} \oplus E_{22}$ with projections P_1, P_{-1}, and P_{22}, respectively. Note that E_1 is not orthogonal to E_{-1}.

To overcome this obstacle, we may use an equivalent inner product on E, as defined for deterministic wave equations ([12]), to ensure that E_1 is orthogonal to E_{-1}.

Let $U_i = (u_i, v_i)$ $(i = 1, 2)$ be two elements of E_{11} or E_{22}. Assume that $\nu^2 > 8\lambda_2$. We define the new inner products on E_{11} and E_{22} by

$$\langle U_1, U_2\rangle_{E_{11}} = 2\Big[\frac{\nu^2}{4}u_1u_2 + \Big(\frac{\nu}{2}u_1 + v_1\Big)\Big(\frac{\nu}{2}u_2 + v_2\Big)\Big],$$
$$\langle U_1, U_2\rangle_{E_{22}} = 2\Big[\langle kAu_1, u_2\rangle + \Big(\frac{\nu^2}{4} - 2\lambda_2\Big)\langle u_1, u_2\rangle + \Big\langle \frac{\nu}{2}u_1 + v_1, \frac{\nu}{2}u_2 + v_2\Big\rangle\Big],$$

respectively, where $\langle\cdot,\cdot\rangle$ is the usual inner product of $L^2(D)$. Define a new inner product on E by

$$\langle U, V\rangle_E = \langle U_{11}, V_{11}\rangle_{E_{11}} + \langle U_{22}, V_{22}\rangle_{E_{22}},$$

where $U = U_{11}+U_{22}$ and $V = V_{11}+V_{22}$ with $U_{ii}, V_{ii} \in E_{ii}$ $(i = 1, 2)$. The corresponding norm is denoted by $\|\cdot\|_E$.

Since $\nu^2 > 8\lambda_2$, one can see that $\langle\cdot,\cdot\rangle_{E_{11}}$ is equivalent to the usual inner product on E_{11}, and $\langle\cdot,\cdot\rangle_{E_{22}}$ is equivalent to the usual inner product on E_{22}. Hence, the new inner product $\langle\cdot,\cdot\rangle_E$ is equivalent to the usual inner product on E.

In terms of this new inner product, by the orthogonality of $\sin kx$, a straightforward calculation shows that

$$E_{-1} \perp E_{22}, \quad E_1 \perp E_{22}, \quad E_1 \perp E_{-1}.$$

Let $E_2 = E_{-1} \oplus E_{22}$; then $E_1 \perp E_2$. Furthermore, for $U = (0, v) \in E$, we have

$$\|U\|_E = \sqrt{2}\|v\|_{L^2(D)}, \tag{5.2}$$

and for any $U = (u, v) \in E$, we have

$$\|U\|_E \geq \sqrt{2}\rho\|u\|_{L^2(D)} \tag{5.3}$$

with $\rho = \min\{\nu/2, \sqrt{\nu^2/4 + (k-2)\lambda_2}\}$.

Let C_1, C_2, C_{-1}, C_{22} denote $C|_{E_1}, C|_{E_2}, C|_{E_{-1}}$, and $C|_{E_{22}}$, respectively. By a discussion similar to Mora's bounds [12], we have

$$\|e^{C_1 t}\| \leq 1 \qquad \text{for } t \leq 0,$$
$$\|e^{C_{-1} t}\| \leq e^{-\nu t} \qquad \text{for } t \geq 0, \tag{5.4}$$
$$\|e^{C_{22} t}\| \leq e^{\operatorname{Re}\mu_2^+ t} \qquad \text{for } t \geq 0. \tag{5.5}$$

From (5.4) and (5.5), we have

$$\|e^{C_2 t}\| \leq e^{\operatorname{Re}\mu_2^+ t} \quad \text{for } t \geq 0.$$

For the nonlinearity F, in terms of the new norm, by (5.2) and (5.3) we have

$$\begin{aligned}\|F(\widetilde{U}_1, \omega) - F(\overline{U}_2, \omega)\|_E &= \sqrt{2}\|f(\widetilde{u}_1 + \eta) - f(\widetilde{u}_2 + \eta)\|_{L^2(D)} \\ &\leq \sqrt{2}L_f\|\widetilde{u}_1 - \widetilde{u}_2\|_{L^2(D)} \leq \frac{L_f}{\rho}\|\widetilde{U}_1 - \widetilde{U}_2\|_E.\end{aligned}$$

So F is Lipschitz with respect to $\widetilde{U}$ and the Lipschitz constant $L_F = L_f/\rho$ is independent of k when $k \geq 2$.

Note that by choosing $\alpha = 0$, $\beta = \operatorname{Re}\mu_2^+$, and $\zeta = \operatorname{Re}\mu_2^+/2$, using a similar discussion to that of Liu [7] and of Lu and Schmalfuss [8] leads to the following theorem.

Theorem 5.1 *There is a one-dimensional inertial manifold $\overline{\mathcal{M}}^k(\omega)$ for equation (5.1), which is represented by*

$$\overline{\mathcal{M}}^k(\omega) = \left\{ (\xi, h^k(\xi,\omega)) : \xi \in E_1 \right\}$$

with $h^k\colon E_1 \to E_2$ is a Lipschitz continuous mapping given by

$$h^k(\xi,\omega) = \int_{-\infty}^{0} (P_{-1} + P_{22}) e^{-Cs} F(\overline{U}(s),\omega)\,ds.$$

Moreover, if $f \in C^1(L^2(D), L^2(D))$, then the random invariant manifold is C^1, which implies that $h \in C^1(E_1, E_2)$. Here $\overline{U}$ is the unique solution of

$$\overline{U}(t) = e^{Ct}\xi + \int_0^t P_1 e^{C(t-s)} F(\overline{U}(s),\omega)\,ds + \int_{-\infty}^{t} (P_{-1} + P_{22}) e^{C(t-s)} F(\overline{U}(s),\omega)\,ds$$

in the space

$$C_\zeta^- = \left\{ u \in C((-\infty,0]; H) : \sup_{t\le 0} e^{-\zeta t}\|u(t)\| < \infty \right\}$$

with the norm

$$|u|_{C_\zeta^-} = \sup_{t\le 0} e^{-\zeta t}\|u(t)\|.$$

Let $\widetilde{u} = u - \eta_1$, and η_1 satisfies (3.2). We have

$$\widetilde{u}_{tt} + \nu \widetilde{u}_t = Pf(\widetilde{u} + \eta_1). \tag{5.6}$$

Hence, we have the following result. Notice that in this case, $E_2 = E_{-1}$ holds.

Theorem 5.2 *There is a one-dimensional inertial manifold $\overline{\mathcal{M}}(\omega)$ for equation (5.6) that is represented by $\overline{\mathcal{M}}(\omega) = \{(\xi, h(\xi,\omega)) : \xi \in E_1\}$ with $h\colon E_1 \to E_2$ being a Lipschitz continuous mapping.*

5.2 Limit of $\overline{\mathcal{M}}^k(\omega)$

Let $\overline{U}$ be the unique fixed point of the nonlinear map $\mathcal{J}_\xi\colon C_\zeta^- \to C_\zeta^-$:

$$\mathcal{J}_\xi(\overline{U}) = e^{Ct}\xi + \int_0^t P_1 e^{C(t-s)} F(\overline{U}(s),\omega)\,ds + \int_{-\infty}^{t} (P_{-1} + P_{22}) e^{C(t-s)} F(\overline{U}(s),\omega)\,ds.$$

In view of the Lipschitz property of F, one can see that $\mathcal{J}_\xi$ is Lipschitz with $\operatorname{Lip}(\mathcal{J}_\xi) < 1$, and

$$\|\overline{U}\|_{C_\eta^-} \le \|\mathcal{J}_\xi(\overline{U}) - \mathcal{J}_\xi(0)\|_{C_\zeta^-} + \|\mathcal{J}_\xi(0)\|_{C_\zeta^-} leq \operatorname{Lip}(\mathcal{J}_\xi)\|\overline{U}\|_{C_\zeta^-} + \|\mathcal{J}_\xi(0)\|_{C_\zeta^-}.$$

By the definition of $\mathcal{J}_\xi$ and using the assumption $F(0) = 0$, we have

$$\|\mathcal{J}_\xi(0)\|_{C_\zeta^-} = \|\xi\|.$$

So we have $\|\overline{U}\|_{C_\zeta^-} \le L\|\xi\|$ for some $L > 0$.

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Consider $\overline{U}(t) = (\overline{U}_c(t), \overline{U}_s(t))$ with

$$\|\overline{U}_s(t)\| = \Big\| \int_{-\infty}^{t} (P_{-1} + P_{22}) e^{C(t-s)} F(\overline{U}(s), \omega) ds \Big\|.$$

In view of the Lipschitz property of f, we get

$$\begin{aligned}\|\overline{U}_s(t)\| &\le \sqrt{2} L_f \int_{-\infty}^{t} \|e^{C_2(t-s)}\| \cdot \|\overline{U}(s)\| ds,\\ &\le \sqrt{2} L_f \int_{-\infty}^{t} \|e^{C_2(t-s)}\| e^{\zeta s} e^{-\zeta s} \|\overline{U}(s)\| ds,\\ &\le \sqrt{2} L_f \int_{-\infty}^{t} e^{\mathrm{Re}\, \mu_2^+(t-s)} e^{\zeta s} ds \|\overline{U}\|_{C_\zeta^-}.\end{aligned}$$

Thus, we have

$$\|h(\xi)\| = \|\overline{U}_s(0)\| \le \sqrt{2} L_f \int_{-\infty}^{0} e^{(\zeta - \mathrm{Re}\, \mu_2^+)s} ds \|\overline{U}\|_{C_\zeta^-} \le K\|\xi\|$$

for some constant $K > 0$.

To consider estimates of the solution on the random invariant manifold $\overline{\mathcal{M}}^k(\omega)$, we start with the limit of $k \to \infty$. For any $(\overline{u}^k, \overline{v}^k) \in \overline{\mathcal{M}}^k(\omega)$, we rewrite $\overline{u}^k$ as

$$\overline{u}^{2,k} = \sum_{i=2}^{\infty} \overline{u}_i^k e_i.$$

By the transformation (4.1), there exists a constant $c > 0$ such that

$$\|\overline{u}^{2,k}\| \le \frac{c}{\sqrt{k}} \|(\xi, h(\xi))\| \le \frac{c(1+K)}{\sqrt{k}} \|\xi\|. \tag{5.7}$$

Assume that $(\overline{u}, \overline{u}_t) \in \overline{\mathcal{M}}(\omega)$ with the same initial $\xi \in E_1$. We now show that for any $t > 0$ there holds

$$\|\overline{u}^k(t) - \overline{u}(t)\| \longrightarrow 0, \quad \text{as } k \longrightarrow \infty.$$

Let $\overline{U}^{1,k} = \overline{u}^{1,k} - \overline{u}$. So we have

$$\overline{U}_{tt}^{1,k} + \nu \overline{U}_t^{1,k} = Pf(\overline{u}^{1,k} + \overline{u}^{2,k} + \eta^k) - Pf(\overline{u} + \eta_1).$$

Let $\overline{\rho}^{1,k} = \overline{U}_t^{1,k} + \delta \overline{U}^{1,k}$ with $\delta < \nu$. Then we deduce that

$$\begin{aligned}\overline{\rho}_t^{1,k} = -\frac{1}{2}(\nu - \delta)\overline{\rho}^{1,k} - \frac{1}{2}(\nu - \delta)\overline{\rho}^{1,k} + \delta(\nu - \delta)\overline{U}^{1,k}&\\ + Pf(\overline{u}^{1,k} + \overline{u}^{2,k} + \eta^k) - Pf(\overline{u} + \eta_1).&\end{aligned}$$

In view of the definition of $\overline{\rho}^{1,k}$ and the Lipschitz property of f, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big[\|\overline{\rho}^{1,k}\|^2 + \delta(\nu - \delta)\|\overline{U}^{1,k}\|^2\big]\\ &\quad = -\frac{1}{2}(\nu - \delta)\|\overline{\rho}^{1,k}\|^2 - \frac{1}{2}(\nu - \delta)\|\overline{U}_t^{1,k}\|^2 - \frac{\delta^2(\nu - \delta)}{2}\|\overline{U}^{1,k}\|^2\\ &\qquad + \big\langle \delta(\nu - \delta)\overline{U}^{1,k}, \overline{\rho}^{1,k}\big\rangle + \big\langle Pf(\overline{u}^{1,k} + \overline{u}^{2,k} + \eta^k) - Pf(\overline{u} + \eta_1), \overline{\rho}^{1,k}\big\rangle\\ &\quad \le L_\delta\big[\|\overline{\rho}^{1,k}\|^2 + \delta(\nu - \delta)\|\overline{U}^{1,k}\|^2\big] + L_\delta\big(\|\overline{u}^{2,k}\|^2 + \|\eta^k - \eta_1\|^2\big)\end{aligned}$$

for some constant $L_\delta > 0$. By the decaying estimate (5.7) and

$$\|\eta^k - \eta_1\|^2 \longrightarrow 0, \quad \text{a.s. as} \quad k \longrightarrow \infty,$$

for any $t > 0$ it follows from Gronwall's inequality that

$$\|\overline{\rho}^{1,k}(t)\| + \|\overline{U}^{1,k}(t)\| \longrightarrow 0, \quad \text{a.s. as} \quad k \longrightarrow \infty.$$

Denote by $\operatorname{dist}(\cdot,\cdot)$ the Hausdorff semi-distance on $E \times E$ with

$$\operatorname{dist}(A, B) = \sup_{x\in A} \inf_{y\in B} \|x - y\|_E, \quad A, B \subset E.$$

Thus, we obtain the following result.

Theorem 5.3 *For the random invariant manifold $\overline{\mathcal{M}}^k(\omega)$, we have*

$$\lim_{k\to\infty} \operatorname{dist}(\overline{\mathcal{M}}^k, \overline{\mathcal{M}}) = 0.$$

By virtue of Theorems 5.1–5.3, we arrive at Theorem 2.2.

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