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ENDOMORPHISMS OF THE QUASI-INJECTIVE HULL OF A MODULE

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R is a ring and M is a right R-module for which $R^{l}=\{m \in M \mid mR=0\}$ is the zero submodule. Let $\hat{M}$ and $\bar{M}$ be the injective hull and the quasi-injective hull of M respectively. Then $\bar{M}=KM$ where $K=\mathrm{Hom}_R(\hat{M}, \hat{M})$ [1]. The ring $D=\mathrm{Hom}_R(\bar{M}, \bar{M})$ plays an important role, in many cases, in the studying of R especially when D is a division ring. For $x \in M$, we denote the annihilator of x in R by $x^{\gamma}=\{r \in R \mid xr=0\}$, whereas $x^{\gamma l}=\{m \in M \mid mx^{\gamma}=0\}$. If N is a submodule of M and $x \in M$, $x^{-1}(N)$ is the right ideal in R consisting of elements r in R where $xr \in N$.

LEMMA. *D is a division ring, if and only if, $k \in K$ either $kM=0$ or k is one-to-one.*

Proof. $k \in K$, Let T_k be the kernel of k. If D is a division ring and $kM \neq 0$. Since $k\bar{M} \subset \bar{M}$. k is one-to-one on $\bar{M}$. $T_k=0$ follows $\hat{M}$ is an essential extension of $\bar{M}$.

Suppose for each $k \in K$ either $kM=0$ or $T_k=0$. Let $d \in D$ and $d \neq 0$. There exist $k \in K$ and $m \in M$ such that $d(km)=(dk)m \neq 0$. Let $\bar{d}$ be an extension of d in K. $\bar{d}kM \neq 0$. Hence $\bar{d}k$ is one-to-one on $\hat{M}$. Since $kM \cap M \neq 0$. It implies $\bar{d}M \neq 0$ and $T_{\bar{d}}=0$. Consequently d is one-to-one on $\bar{M}$. D is a division ring follows the definition of $\bar{M}$.

THEOREM. *D. is a division ring, if and only if,*

(1) *every nonzero submodule of M is large,*

(2) *x, y in M, $x^{\gamma} > y^{\gamma}$ (properly) then $x=0$.*

Proof. If D is a division ring then condition (1) must be satisfied. Otherwise D would have nonzero element with nonzero kernel. Suppose x, y in M such that $x^{\gamma} > y^{\gamma}$. Then the mapping $f: yR \to xR$ where $f(yr)=xr$ can be extended to an element in D. Since the kernel of f is nonzero, f must be identically zero. Hence $x=0$.

Suppose M satisfies conditions (1) and (2). Let $k \in K$ and $kM \neq 0$. There exist n, n' in M such that $kn=n' \neq 0$. $(n')^{\gamma}=n^{-1}(T_k) \supset n^{\gamma}$. If $T_k \neq 0$, then there exists $r \in R$ such that $nr \in T_k$ and $nr \neq 0$. This means $(n')^{\gamma} > n^{\gamma}$ and $n'=0$. Contradiction. Hence $T_k=0$. By the lemma, D is a division ring.

COROLLARY. *If D is a division ring. Then for any $x \in M$ and any nonzero submodule N of M, if $x^{-1}(N) \supset y^{\gamma}$, $y \neq 0$, then $x^{-1}(N) > y^{\gamma}$.*

Proof. If $x=0$ then there is nothing to prove. So we assume $x \neq 0$. If $Dx=Dy$

then $x^\gamma = y^\gamma$. But $xR \cap N \neq 0$. $x^{-1}(N) > x^\gamma = y^\gamma$. If $Dx \neq Dy$ then x and y are linearly independent. By Theorem 2.3 [1], there exists $r \in R$ such that $xr = 0$ and $yr \neq 0$. Again $x^{-1}(N) > y^\gamma$.

Reference

1. R. E. Johnson and E. T. Wong, *Quasi-injective modules and irreducible rings*, J. London Math. Soc. **36** (1961), 260–268.

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