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AN EXISTENCE THEOREM OF CONSTANT MEAN CURVATURE GRAPHS IN EUCLIDEAN SPACE*

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Abstract. We prove the following result of existence of graphs with constant mean curvature in Euclidean space: given a convex bounded planar domain Ω of area $a(\Omega)$ and a real number H such that $a(\Omega)H^2 < \pi/2$, there exists a graph on Ω with constant mean curvature H and whose boundary is $\partial\Omega$.

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1. Introduction and statement of results. Let Ω be a smooth bounded domain in $\mathbb{R}^2$ and let H be a given non-zero constant. We consider classical solutions $u \in C^2(\Omega) \cap \partial C^0(\overline{\Omega})$ of the constant mean curvature boundary value problem (P_H):

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} + 2H = 0 \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{2}$$

The geometric meaning of (1)–(2) is that the graph of a solution u describes a nonparametric surface of $\mathbb{R}^3$ spanning $\partial\Omega \times \{0\}$ and with constant mean curvature H with respect to the orientation $N = (\nabla u, -1)/\sqrt{1+|\nabla u|^2}$. In variational terms, a constant mean curvature surface is a critical point for area given the constraint of fixed volume. From the physical viewpoint, a soap film in equilibrium between two regions of different gas pressure—no gravity—is modelled mathematically by the fact that the surface it defines has nonzero constant mean curvature and the constant H is the pressure difference across the surface. We refer to [**3**] as a complete guide to quasilinear elliptic equations. A suitable introduction to the properties of the constant mean curvature equation (1) is [**8**]. We will use both references in our proofs.

A few existence results are known of the Dirichlet problem (1)–(2), even if Ω is a convex domain. Serrin established an existence result when the boundary condition (2) is replaced by $u = \phi$, where ϕ is a function defined on $\partial\Omega$ that extends to a C^2 function on $\overline{\Omega}$. In [**10**], he proved that given a constant H, there is solvability of (1) for arbitrary continuous boundary data ϕ if and only if $2|H| \leq \kappa$, where κ denotes the curvature of $\partial\Omega$ as a planar curve. In particular, Ω must be strictly convex. However, when the boundary condition is $u = 0$, it is natural to think that the Serrin's condition on H could be relaxed. Thus, if the curvature κ of $\partial\Omega$ satisfies

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$\kappa > 1/|H|$, the problem (P_H) has a solution. (See, for example, [**4**].) In this case, spherical caps included in a halfsphere are used as barrier surfaces for searching the new graphs.

Montiel and the present author used quarter-cylinders as barriers to obtain estimates of the solutions of (P_H). In this sense, the following result is proved in [**7**].

THEOREM 1. *Let Ω be a convex bounded domain whose boundary $\partial\Omega$ has length L. If H is a given number such that*

$$|H| < \frac{\sqrt{3}\pi}{L}, \tag{3}$$

then (P_H) has a unique solution.

Throughout this paper, for a convex bounded domain we assume that the curvature κ of $\partial\Omega$ satisfies $\kappa \geq 0$. In this result (as well as in [**4**]), C^0 estimates of a solution of the Dirichlet problem are used in order to obtain $C^1(\overline{\Omega})$ estimates. In Theorem 1, we used an isoperimetric inequality together with a height estimate for a compact constant mean curvature surface immersed in $\mathbb{R}^3$ measured from a plane P. This estimate is done in terms of the value of the constant H and the area A of the region of M above the plane P: if h denotes the height of the surface with respect to P, then

$$h \leq \frac{A|H|}{2\pi} \tag{4}$$

and the equality holds if an only if M is a spherical cap. Using again quarter-cylinders as barriers, it is proved in [**5**] that for each convex bounded or unbounded domain Ω included in a strip of width $1/|H|$ there exists a graph on Ω bounded by $\partial\Omega$ and with constant mean curvature H. Recently, the present author has given results on existence for nonconvex domains that satisfy some R-sphere condition on the boundary. See [**6**].

In this paper, we prove the following existence theorem for (P_H).

THEOREM 2. *Let Ω be a convex bounded domain. Let H be a real number such that*

$$a(\Omega)H^2 < \frac{\pi}{2}, \tag{5}$$

where $a(\Omega)$ denotes the area of Ω. Then (1)–(2) has a unique solution.

In a recent paper and with different techniques, Montiel has proved the solvability of (P_H) if $a(\Omega)H^2 < \alpha^2\pi$, where $\alpha = (\sqrt{5}-1)/2$ is the *golden ratio*. Then $\alpha^2 \approx 0.3819$. Thus our estimate in Theorem 2 improves Montiel's result. Note that the constant $1/2$ in (5) is not optimal, as it occurs when Ω is a round disc: there is a family of spherical caps that are graphs on Ω with mean curvature varying in the interval $[0, \sqrt{\pi/a(\Omega)})$. From this example, one is led to the conjecture that it suffices that $a(\Omega)H^2 < \pi$, which is optimal in the case of a circle.

On the other hand, the classical isoperimetric inequality in the plane states that $L^2 \geq 4\pi a(\Omega)$. This inequality links Theorems 1 and 2 as follows. For the Dirichlet problem (P_H), there exists a positive value $H_{\max}$ such that (P_H) has a unique solution if $|H| \leq H_{\max}$ and (P_H) has no solution if $|H| > H_{\max}$. See [**8**]. Condition (3) implies that $H_{\max} \geq \sqrt{3}\pi/L$, but (5) gives a better estimate $H_{\max} \geq \sqrt{\pi}/\sqrt{2a(\Omega)}$. As we have already stated, the examples of spherical caps that are graphs on round discs make us think that $H_{\max} = \sqrt{\pi}/\sqrt{a(\Omega)}$.

Remark. It is worthwhile to point out that in the setting of the existence problem for *parametric surfaces* with prescribed mean curvature and given boundary curve, there are some results of the same nature as Theorem 2. A theorem formulated by Wente [**12**] and sharpened by Steffen [**11**] proves the existence of a disc immersed in $\mathbb{R}^3$ of constant mean curvature H and spanning a closed Jordan curve Γ provided that $a_\Gamma H^2 < 2\pi/3$, where a_Γ is the least spanning area of Γ.

2. Proof of Theorem 2. In the proof of Theorem 2, we shall need two previous results. The first one relates the algebraic volume of a constant mean curvature surface with a certain L^1-norm defined in terms of the coordinates of its Gauss map. The second result is concerned with an estimate of the height of a graph with constant mean curvature in terms of the area of the planar domain in which the graph is defined.

First, recall some facts on solutions of (1). Firstly, a symmetry property holds for solutions of (P_H): if u is a solution of (P_H), then $-u$ solves the problem (P_{-H}). On the other hand, the maximum principle (or the comparison principle with horizontal planes) ensures that either $u \geq 0$ and $H \geq 0$ or $u \leq 0$ and $H \leq 0$. Also, a monotonicity principle holds for the family of Dirichlet problems (P_H) and that it is again a consequence of the comparison principle [**3**, Theorem 10.1]: if $H' < H$, then $u_{H'} < u_H$ on Ω. Finally, the uniqueness of a given solution u of (P_H) follows from the comparison principle.

Consider an immersion $\phi : M \to \mathbb{R}^3$ from an oriented compact surface M with non-empty boundary ∂M into Euclidean space. The algebraic volume V is defined as

$$V = -\frac{1}{3}\int_M \langle N, \phi\rangle dM,$$

where N stands for the Gauss map for the immersion ϕ compatible with the orientation of M. The starting point is the following result that was stated without proof in [**7**, p. 597]. For the sake of completeness we present a proof of it.

Proposition 1. *Let $\phi : M \to \mathbb{R}^3$ be an immersion of constant mean curvature H and with boundary included in a plane P. Then*

$$2HV = \int_M |\nabla\langle\phi, \vec{a}\rangle|^2 dM, \tag{6}$$

where $\vec{a}$ is a unit vector orthogonal to P.

Proof. Let N be the Gauss map of the immersion. We define on M two 1-forms α and β by

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$$\alpha_p(v) = \langle (d\phi)_p(v), \vec{a}\rangle \langle \phi(p), \vec{a}\rangle,$$

$$\beta_p(v) = \langle (d\phi)_p(v) \wedge N(p), \phi(p) \wedge \vec{a}\rangle \langle \phi(p), \vec{a}\rangle,$$

for each $p \in M$ and $v \in T_pM$. Compute their codifferentials $\delta\alpha$ and $\delta\beta$. Let $p \in M$ and let $\{e_1, e_2\}$ be an orthonormal basis of the tangent plane T_pM. Then

$$\delta\alpha(p) = \sum_{i=1}^{2} \sigma(e_i, e_i)\langle N(p), \vec{a}\rangle + \sum_{i=1}^{2} \langle \vec{a}, e_i\rangle^2,$$
$$\delta\beta(p) = \sum_{i=1}^{2} \langle e_i \wedge N(p), e_i \wedge \vec{a}\rangle \langle \phi(p), \vec{a}\rangle + \sum_{i=1}^{2} \langle e_i \wedge N(p), \phi(p) \wedge \vec{a}\rangle \langle e_i, \vec{a}\rangle,$$

where σ stands for the second fundamental form of the immersion ϕ. Then

$$\delta\alpha = |\nabla\langle \phi, \vec{a}\rangle|^2 + 2H\langle \phi, \vec{a}\rangle \langle N, \vec{a}\rangle,$$

$$\delta\beta = 3\langle \phi, \vec{a}\rangle \langle N, \vec{a}\rangle - \langle N, \phi\rangle.$$

Let us integrate these two inequalities over M. Because the mean curvature H is constant and since α and β vanish on the boundary ∂M, we get the desired identity (6). $\triangle$

As a consequence of Proposition 1 and the height estimate (4), we obtain the following result.

THEOREM 3. *Let u be a solution of the Dirichlet problem for the prescribed constant mean curvature (P_H). Let $h = \sup_\Omega |u|$. Then*

$$h \leq \frac{a(\Omega)|H|}{2(\pi - a(\Omega)H^2)}, \tag{7}$$

where $a(\Omega)$ denotes the area of the domain Ω.

Proof. Let $\vec{a} = (0, 0, 1)$. Without loss of generality we assume that $u \geq 0$. Since the orientation N of $M = \text{graph}(u)$ is chosen pointing downwards, the mean curvature H is positive. A straightforward computation yields $|\nabla\langle x, \vec{a}\rangle|^2 = 1 - \langle N, \vec{a}\rangle^2$, where x denotes a point of M. Let A be the area of M. Then (6) gives

$$2H \int_\Omega u \; dx \; dy = A - \int_M \langle N, \vec{a}\rangle^2 dM.$$

Using (4), we obtain

$$\begin{aligned} \frac{2\pi h}{H} \leq A \leq 2H \int_\Omega u \; dx \; dy + \int_M \langle N, \vec{a}\rangle^2 dM \leq 2ha(\Omega)H + \int_M |\langle N, \vec{a}\rangle| \\ = 2h \; a(\Omega)H + \int_\Omega 1 \; dx \; dy = (2Hh + 1)a(\Omega), \end{aligned}$$

yielding the desired inequality (7). $\triangle$

We are now prepared to prove Theorem 2. The reasoning below follows the work in [**7**]. We give a more detailed exposition. We shall apply the method of continuity to solve the Dirichlet problem (1)–(2). We refer the reader to the discussion in [**3**] for a modern treatment of the theory of the Dirichlet problem for the prescribed mean curvature equation. As usual, the proof is based on the establishment of global $C^{1,\alpha}(\overline{\Omega})$ a priori estimates for prospective solutions of (P_H). Let c be a positive number with $a(\Omega)c^2 < \pi/2$. Consider the set $\mathcal{S}$ defined as

$$\mathcal{S} = \{H \in [0, c];\ \text{there exists a solution } u_H \text{ of } (P_H)\}.$$

Since $u_0 = 0$ solves the minimal case, the set $\mathcal{S}$ is not empty.

Now we show that $\mathcal{S}$ is open. This is accomplished by using the Implicit Function Theorem for Banach spaces. Let $\phi : \overline{\Omega} \to \mathbb{R}^3$ be an isometric immersion, where N denotes a unit normal vector field along ϕ in $\mathbb{R}^3$. For each $u \in C_0^{2,\alpha}(\overline{\Omega})$, the maps $\phi_t : \Omega \to \mathbb{R}^3$ defined as $\phi_t(p) = \phi(p) + tu(p)N(p)$, $(p \in \overline{\Omega})$, are immersions for t near zero. Consider on $\overline{\Omega}$ the metric induced by ϕ_t and let H be the mean curvature. The linearized operator (up to a factor) $L : C_0^{2,\alpha}(\overline{\Omega}) \to C^{\alpha}(\overline{\Omega})$, defined by

$$L(u)(p) = \frac{d}{dt}\bigg|_{t=0} H(\phi_t(p))),$$

turns out to be $L = \Delta + |\sigma|^2$, where Δ denotes the Laplacian operator on $\overline{\Omega}$ with the induced metric from ϕ and σ is its second fundamental form. Here $L(u)$ is a self-adjoint linear elliptic operator. We claim that the kernel of L is trivial. This is proved as follows. Assume that $H \in \mathcal{S}$ and denote $G_H = \text{graph}(u_H)$. Because the mean curvature is constant, the function $\langle N, \vec{a}\rangle$ defined on G_H satisfies

$$\Delta\langle N, \vec{a}\rangle = -|\sigma|^2\langle N, \vec{a}\rangle, \tag{8}$$

so that

$$L\langle N, \vec{a}\rangle = 0 \quad \text{and} \quad \langle N, \vec{a}\rangle < 0.$$

Hence, if $v \in C^{2,\alpha}(\overline{\Omega})$ satisfies $L(u)v = 0$ and $v = 0$ on $\partial\Omega$, then $v = 0$. (See, for example, [**2**, Theorem 1].) Then L is a Fredholm operator of index zero. Hence we use the Riesz spectral theory of compact operators to assert that the Fredholm alternative applies and the invertibility of (1)–(2) is assured. (See [**1**].) The Implicit Function Theorem in Banach spaces guarantees an interval of solutions of (P_H) around the value H.

Finally, to prove that $\mathcal{S}$ is a closed set, the Schauder approach reduces the question to establishing apriori $C^{1,\alpha}(\Omega)$ bounds for any solution u_H with $0 \le H \le c$ [**3**, Theorem 13.8]. In our situation, it suffices to prove that there is a fixed constant M independent of H such that

$$|u_H|_{C^1(\overline{\Omega})} = \sup_{\Omega}|u_H| + \sup_{\Omega}|\nabla u_H| < M$$

holds for any $u_H \in C_0^{2,\alpha}(\overline{\Omega})$ and $H \in [0, c]$.

The apriori C^0 bounds for u_H is obtained as follows. The monotonicity principle ensures that $0 \le u_H \le u_c$, for $0 \le H \le c$. On the other hand, the hypothesis on $a(\Omega)$ and inequality (7) imply that $u_c < 1/(2c)$ and, consequently, $0 \le u_H < 1/(2c)$.

Now, we seek apriori estimates for $|\nabla u_H|$. By the expression of N in terms of ∇u_H we obtain

$$\langle N, \vec{a}\rangle = -\frac{1}{\sqrt{1+|\nabla u_H|^2}}.$$

Then we have apriori estimates of $|\nabla u_H|$ provided that $\langle N, \vec{a}\rangle$ remains bounded away from zero. But equation (8) tells us that $\Delta\langle N, \vec{a}\rangle \geq 0$ and so the maximum $\langle N, \vec{a}\rangle$ on $\overline{\Omega}$ is attained at some boundary point. This proves the well-known maximum principle $\sup_\Omega |\nabla u_H| = \sup_{\partial\Omega} |\nabla u_H|$ for the constant mean curvature equation (1). The above bound $1/(2c)$ on the height of our graphs provides barriers which serve to estimate $|\nabla u_H|$ on $\partial\Omega$. The reasoning that follows is based on the use of appropriate pieces of quarter-cylinders as barriers. (See **[5]**, **[7]** and **[9]** for examples in the same context.)

Let ν_H denote the inner conormal of G_H along its boundary. Since $0 \leq u_H$, we have $0 \leq \langle \nu_H, \vec{a}\rangle$. The boundary condition $u_H = 0$ on $\partial\Omega$ yields $\langle N, \vec{a}\rangle^2 + \langle \nu_H, \vec{a}\rangle^2 = 1$. According to the orientation N chosen on G_H, we have

$$\langle \nu_H, \vec{a}\rangle = \frac{|\nabla u_H|}{\sqrt{1+|\nabla u_H|^2}}.$$

As a consequence of the reasoning above, we shall obtain estimates for $|\nabla u_H|$ on Ω if we are able to establish a constant $C'(\Omega, c)$, depending only on Ω and c, such that $\langle \nu_H, \vec{a}\rangle \leq C'(\Omega, c)$. This estimate will be accomplished by the technique of barriers. We define K the quarter-cylinder by

$$K = \{(x, y, z); 0 \leq y \leq \frac{1}{2c}, z = \frac{1}{2c}\sqrt{1-4c^2y^2}\}.$$

The surface K is a graph on the strip $\{0 < y < 1/(2c)\}$ and its mean curvature is c with the downwards orientation. Moreover, $K \subset \{x \in \mathbb{R}^3; \langle x, \vec{a}\rangle \geq 0\}$ and its boundary ∂K is formed by two parallel straight-lines; one of them lies on the (x, y)-plane and the other one lies at height $1/(2c)$ over this plane. Consider $\epsilon > 0$ such $h_c + \epsilon < 1/(2c)$. We move down K an amount ϵ (with respect to the $\vec{a}$-direction), and call $K_\epsilon = K \cap \{x \in \mathbb{R}^3; \langle x, \vec{a}\rangle \geq 0\}$.

Consider the circle of horizontal directions

$$S^1 = \{\vec{v} \in \mathbb{R}^3; |\vec{v}| = 1, \langle \vec{v}, \vec{a}\rangle = 0\}.$$

Let $\vec{v} \in S^1$. After a horizontal translation and a rotation with respect to a vertical axis, we assume that the axis of rotation of K_ϵ is orthogonal to $\vec{v}$, its concave side lies in front of G_H and that K_ϵ does not intersect G_H. If $h_H = \sup_\Omega u_H$ denotes the height of G_H, then

$$h_H \leq h_c < \frac{1}{2c} - \epsilon = \text{ height of } K_\epsilon.$$

Call $C' = \langle \nu_{K_\epsilon}, \vec{a}\rangle$, where ν_{K_ϵ} denotes the inner conormal of K_ϵ along the component boundary that lies in the (x, y)-plane. Notice that C' is a constant and $C' < 1$. Move

K_ϵ towards G_H and parallel to $\vec{a}$ until the first contact point occurs between K_ϵ and G_H. Since the height of G_H is strictly less than the K_ϵ and because the mean curvature H of G_H is strictly less than K_ϵ, the maximum principle ensures that this touching point is some boundary point $p \in \partial\Omega$. At this point p, we have

$$0 \leq \langle \nu_H(p), \vec{a} \rangle < C'. \tag{9}$$

Now, let us repeat the same argument varying $\vec{v}$ on S^1. Because $\partial\Omega$ is a *convex curve*, the successive straight-lines on the (x, y)-plane that are boundary to K_ϵ go touching *each point* of $\partial\Omega$. Hence, inequality (9) holds for each $p \in \partial\Omega$. This gives the uniform bound of $|\nabla u_H|$ along $\partial\Omega$ and it concludes the proof of Theorem 2.

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