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ON THE DISTRIBUTION OF THE SEQUENCE $n^2\theta$ (mod 1)

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1. Introduction. In a paper with the above title H. Heilbronn [**4**] proved that, given any real θ, there exist infinitely many positive integers n such that the distance $||\theta n^2||$ from θn^2 to its nearest integral neighbour satisfies the bound

$$||\theta n^2|| \ll_\epsilon n^{-1/2+\epsilon}.$$

He actually proved a somewhat stronger statement which shows that the integers n occur with some regularity and he suggested that perhaps the exponent $-\frac{1}{2} + \epsilon$ may be replaced by $-1 + \epsilon$. This theorem has attracted considerable attention and spawned a number of generalizations (see [**1**], [**7**] and the references therein), yet no essential improvement has been given for the original problem (but see [**6**], [**8**]).

In this note we draw attention to the connection of this difficult problem to another (also difficult) problem of a different nature and possibly of wider interest. This concerns a twisted incomplete Kloosterman sum about which we make the following

(C) CONJECTURE. *Let a, $q \geqq 1$, $q_1 \geqq 1$, $q_2 \geqq 1$ be integers with $(a, q_2) = 1$ and let $M \geqq 1$, $\epsilon > 0$ be reals. Let $Q = [q_1, q_2]$, the least common multiple, and let χ be a Dirichlet character having conductor q_1 and modulus q. Then if $\bar{m}$ denotes the multiplicative inverse of m modulo q_2, we have*

$$\sum_{\substack{1 \leqq m \leqq M \\ (m,q_2)=1}} \chi(m) e\left(\frac{a\bar{m}}{q_2}\right) \ll_\epsilon (qq_2)^\epsilon (M^{1/2} + Q^{-1/2}M).$$

In the special case $q = 1$, conjecture (C) reduces to the R^* conjecture of Hooley [**5**]. On the other hand, in case $q_2 = 1$, the conjecture is a consequence of the generalized Lindelöf hypothesis for L-functions.

In this paper we shall be concerned with another special case of (C). We take q to be an odd positive integer and $\chi(m)$ to be the Jacobi symbol

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$\left(\frac{m}{q}\right)$. We let a and k be integers with $(a, q) = 1$ and consider the incomplete Salié sum

$$S = \sum_{\substack{1 \leqq m \leqq M \\ (m,q)=1}} \left(\frac{m}{q}\right) e\left(\frac{ak^2\bar{m}}{q}\right).$$

We have no wish to compare the difficulty of this special case of (C) with that of its famous cousins. It is, however, sufficiently strong to enable some conditional progress with Heilbronn's problem. In fact we require for this only a somewhat weakened form of the conjecture. Since, in the case at issue, we have

$$Q \geqq q_2 = q/(k^2, q),$$

conjecture (C) gives at once

(C*) CONJECTURE. *For* $(a, q) = 1$ *we have*

$$\sum_{\substack{1 \leqq m \leqq M \\ (m,q)=1}} \left(\frac{m}{q}\right) e\left(\frac{ak^2\bar{m}}{q}\right) \ll_\epsilon (M^{1/2} + q^{-1/2}M)(k^2, q)^{1/2}q^\epsilon.$$

In this paper we prove

THEOREM. *Assuming conjecture* (C*) *there exist infinitely many positive integers n such that*

$$\|\theta n^2\| \ll_\epsilon n^{-2/3+\epsilon}.$$

The expected exponent $-1 + \epsilon$ can, by similar arguments (which we do not give), be deduced from the assumption of a conjecture which gains also from an average over k. The following conjecture suffices for this.

(C**) CONJECTURE. *Let* $a, q \geqq 2$ *be integers with* $(a, q) = 1$ *and q not a perfect square. Let* $H \geqq 1$, $K \geqq 1$ *be reals. Then*

$$\sum_{\substack{1 \leqq h \leqq H \\ (h,q)=1}} \sum_{0 \leqq k < K} \left(\frac{h}{q}\right) e\left(\frac{a\bar{h}k^2}{q}\right)$$

$$\ll_\epsilon (H^{1/2}K^{1/2} + H^{3/4} + K + q^{-1/2}HK + q^{-1/2}K^2)q^\epsilon.$$

Remark. Here the first term $H^{1/2}K^{1/2}$ is the heuristically expected (and dominant) main term in the ranges of H, K of greatest interest, crucial for the application. If however K is a multiple of q then $q^{-1/2}HK$ gives the true order of magnitude for every $H \geqq 1$. If on the other hand $H = 1$ and $1 \leqq K \leqq \frac{1}{2}\sqrt{q/a}$ then K gives the correct order. Similarly the remaining terms $H^{3/4}$ and $q^{-1/2}K^2$ account for other more complex cases.

In fact although C** leads to the stronger result, we expect that it may be possible using the methods of [**3**] to make a non-trivial step in this direction (which would however be too weak to yield an unconditional improvement of Heilbronn's result). It seems more difficult to make such a step toward C*.

The main idea in the proof of the above results is the estimation of sums

$$\sum_m e(\theta m^2)$$

using the approximation of θ by rationals p/q in conjunction with the explicit evaluation of the Gauss sum

$$\sum_m e\left(\frac{p}{q}m^2\right).$$

This idea is also applied in a somewhat different setting by Bombieri and Iwaniec [**2**] in the development of a new method for the estimation of exponential sums which leads, in particular, to improved estimates for the Riemann zeta function.

2. Gauss sums. By definition

$$G(a, b; q) = \sum_{d(\mathrm{mod}\, q)} e\left(\frac{ad^2 + bd}{q}\right).$$

In the sequel we appeal to the following elementary properties.

PROPOSITIONS. *We have* $G(a, b; q) = 0$ *unless* $(a, q)\,|b$ *and in this case*

$$(1)\quad G(a, b; q) = (a, q)G\left(\frac{a}{(a, q)}, \frac{b}{(a, q)}; \frac{q}{(a, q)}\right).$$

If $(a, q) = 1$ *and* q *is odd we have*

$$(2)\quad G(a, b; q) = e\left(-\frac{\overline{4a}b^2}{q}\right)G(a, 0; q),$$

$$(3)\quad G(a, 0; q) = \left(\frac{a}{q}\right)G(1, 0; q),$$

and

$$(4)\quad |G(1, 0; q)| = q^{1/2}.$$

LEMMA 1. *Let* q *be odd,* $(p, q) = 1$ *and* $H \geqq 1$. *We have*

$$(5)\quad \sum_{\substack{1 \leqq h \leqq H \\ (h,q)=1}} G(hp, k; q) \ll (H^{1/2}q^{1/2} + H)(k^2, q)^{1/2}q^{\epsilon}$$

the constant implied in $\ll$ *depending on* ϵ *at most.*

Proof. This follows from Propositions 2 and 3 and Conjecture (C*).

LEMMA 2. *The estimate* (5) *of Lemma* 1 *still holds if the condition* $(h, q) = 1$ *is removed.*

Proof. The sum in question is by Proposition 1

$$\sum_{\substack{1 \leqq m \leqq H \\ m|k, m|q}} m \sum_{\substack{1 \leqq h_1 \leqq H/m \\ (h_1, q/m)=1}} G\left(h_1 p, \frac{k}{m}; \frac{q}{m}\right) \ll (H^{1/2}q^{1/2} + H)(k^2, q)^{1/2}q^{\epsilon}.$$

Define for $0 < \Delta < \frac{1}{2}$ and $h \neq 0$

$$c(h) = \Delta\left(\frac{\sin \pi\Delta h}{\pi\Delta h}\right)^2.$$

LEMMA 3. *Let q be odd,* $(p, q) = 1$ *and* $0 < \Delta < \frac{1}{2}$. *We have*

$$\sum_{h=1}^{\infty} c(h)G(hp; k; q) \ll [(\Delta q)^{-1} + (\Delta q)^{1/2}](k^2, q)^{1/2}q^{\epsilon}.$$

Proof. We apply Lemma 2 and partial summation getting

$$\sum_{1 \leqq h \leqq q^2} = \int_0^{q^2} c(H)d\left(\sum_{1 \leqq h \leqq H} G(hp, k; q)\right)$$
$$\ll q^{\epsilon}(k^2, q)^{1/2} \int_0^{q^2} |c'(H)|(H^{1/2}q^{1/2} + H)dH + (\Delta q)^{-1}.$$

Computing,

$$|c'(H)| \ll H^{-2}|\sin(\pi\Delta H)|$$

this yields what was claimed. For the remaining range $h > q^2$ we apply the trivial estimates

$$c(h) \ll \Delta^{-1}h^{-2} \quad \text{and} \quad G(hp, k; q) \ll q$$

getting an even better bound.

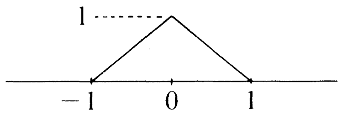
3. Proof of theorem. Main estimates. Let q be odd, $(p, q) = 1$, $0 < \Delta < \frac{1}{2}$. Consider

$$S_{p/q}(\Delta, N) = \#\left\{n \leqq N; \left\|\frac{p}{q}n^2\right\| < \Delta\right\}.$$

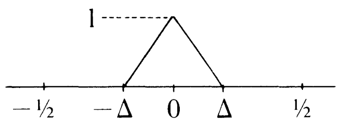
LEMMA 4. *We have*

$$S_{p/q}(\Delta, N) \geqq \frac{1}{2}\Delta N + O([(\Delta q)^{-1} + (\Delta q)^{1/2} + q^{-1/2}N]q^{\epsilon}).$$

Proof. Let $f(x)$ be the function whose graph is



and $g(x)$ the function of period one whose graph on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ is



Thus

$$1 + 2S_{p/q}(\Delta, N) \geqq \sum_{n=-\infty}^{\infty} f\left(\frac{n}{N}\right)g\left(\frac{p}{q}n^2\right).$$

The function g has Fourier expansion

$$g(x) = \sum_h c(h)e(hx)$$

where $c(h)$ is the function occurring in Lemma 3. The constant term $h = 0$ contributes

$$c(0)\sum_n f\left(\frac{n}{N}\right) = \Delta N$$

for N an integer. The non-zero terms contribute

$$R = \sum_{h \neq 0} c(h) \sum_n e\left(\frac{hpn^2}{q}\right)f\left(\frac{n}{N}\right).$$

Applying Poisson summation the inner sum becomes

$$\sum_n = \sum_{d(\bmod q)} e\left(\frac{hpd^2}{q}\right) \sum_{n \equiv d(\bmod q)} f\left(\frac{n}{N}\right)$$

$$= \frac{N}{q}\sum_k G(hp, k; q)\hat{f}\left(\frac{kN}{q}\right)$$

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where

$$\hat{f}(y) = (\sin(\pi y)/\pi y)^2$$

is the Fourier transform. Then

$$R = \frac{N}{q} \sum_k \left(\frac{\sin(\pi kN/q)}{\pi kN/q}\right)^2 \sum_{h \neq 0} c(h) G(hp, k; q).$$

For $k = 0$ we use the trivial result

$$|G(hp, 0; q)| = (h, q)^{1/2} q^{1/2},$$

which follows from (1), (3) and (4) and also the estimate

$$\sum_{h \neq 0} c(h)(h, q)^{1/2} \ll q^{\epsilon}.$$

For $k \neq 0$ we use Lemma 3 together with the estimate

$$\sum_{k \neq 0} \left(\frac{\sin \pi kN/q}{\pi kN/q}\right)^2 (k^2, q)^{1/2} \ll \frac{q^{1+\epsilon}}{N}.$$

This completes the proof of Lemma 4.

4. Proof of theorem. Conclusion. In view of the last result we wish to approximate θ by rational numbers p/q where $(p, q) = 1$ and q is odd. To this end we prove

LEMMA 5. *Given a positive integer s and an irrational α there are infinitely many p/q with $(q, ps) = 1$ such that*

$$\left|\alpha - \frac{p}{q}\right| < s^2 q^{-2}.$$

Proof. For $s = 1$ the result is well known. Consider a sequence $q_1 \to \infty$ such that

$$\left|\alpha - \frac{p_1}{q_1}\right| < q_1^{-2};$$

we may assume $(q_1, s) > 1$ for all q_1 in the sequence. For given q_1 take q such that $0 \leqq q < (s - 1)q_1$,

$$qp_1 \equiv 1 \pmod{q_1} \quad \text{and} \quad (q, s) = 1.$$

Then $p = (qp_1 - 1)/q_1$ is an integer prime to q. We find that as $q_1 \to \infty$ we get infinitely many distinct values of q for each of which

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q_1^2} + \frac{1}{qq_1} \leqq \frac{s^2}{q^2}.$$

Remark. s^2 can be replaced by $v(v + 1)$ where

$$v = s\tau(s)/\phi(s)$$

using the sieve of Eratosthenes.

Now we consider

$$S_\theta(\Delta, N) = \#\{n \leqq N; \|\theta n^2\| < \Delta\}.$$

In view of Lemma 5

$$\|\theta n^2\| \leqq \left\|\frac{p}{q} n^2\right\| + \frac{4N^2}{q^2},$$

so

$$S_\theta(\Delta, N) \geqq S_{p/q}\left(\frac{\Delta}{2}, N\right)$$

provided that $8N^2 \leqq \Delta q^2$. Hence by Lemma 4

$$S_\theta(\Delta, N) \geqq \frac{1}{4}\Delta N + O([(\Delta q)^{-1} + (\Delta q)^{1/2} + q^{-1/2}N]q^\epsilon)$$

$$\geqq \frac{1}{8}\Delta N$$

provided that $\Delta N^2 > \delta(\epsilon)q^{1+2\epsilon}$, and $\Delta > q^{-1/2+\epsilon}$. Choose

$$N = [q^{3/4}] \quad \text{and} \quad \Delta = N^{-2/3+4\epsilon}.$$

The conditions being satisfied we get the theorem.

REFERENCES

1. R. C. Baker, *Recent results on fractional parts of polynomials*, Number Theory, Carbondale (1979), 10-18, Lecture Notes in Math. *751* (Springer, Berlin, 1979).
2. E. Bombieri and H. Iwaniec, *On the order of* $\zeta(\frac{1}{2} + it)$, Ann. Sc. Abrm. Sup. Pisa (to appear).
3. J. B. Friedlander and H. Iwaniec, *Incomplete Kloosterman sums and a divisor problem*, Annals of Math. *121* (1985), 319-350.
4. H. Heilbronn, *On the distribution of the sequence* $n^2\theta$ (mod 1), Quart. J. Math. (Oxford) *19* (1948), 249-256.
5. C. Hooley, *On the greatest prime factor of a cubic polynomial*, J. Reine Angew. Math. *303/304* (1978), 21-50.
6. ——— *On a new technique and its applications to the theory of numbers*, Proc. London Math. Soc. (3) *38* (1979), 115-151.
7. W. M. Schmidt, *Small fractional parts of polynomials*, Regional Conference Series in Math. *32* (A.M.S., Providence, 1977).
8. G. Tenenbaum, *Sur la concentration moyenne des diviseurs*, Comm. Math. Helv. *60* (1985), 411-428.

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