Canad. Math. Bull. Vol. **25** (1), 1982

FUNCTIONS IN ALL H^p SPACES, $p<\infty$

BY
DOUGLAS M. CAMPBELL

ABSTRACT. Let $\hat{H}$ denote the class of functions analytic in $|z|<1$ which are in every H^p class, $0<p<\infty$. The class $\hat{H}$ strictly contains H^∞ and consists of those functions that are 'almost in H^∞' in the sense of integration. L. Hansen and W. Hayman have given simple geometric conditions for a function to belong to $\hat{H}$. The purpose of this note is to show that Hansen and Hayman's conditions are far from necessary. Using techniques from normal functions, gap series, characterizations of BMOA, subordination, Bloch functions, and VMOA, six completely different examples of functions in $\hat{H}$ are given which 'fill the plane'.

A function $f(z)$ analytic in $D=\{|z|<1\}$ is said to be in $H^p, 0<p<\infty$, if $\sup_{r<1}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta<\infty$ and in H^∞, if $\sup_{|z|<1}|f(z)|<\infty$. Let $\hat{H}$ denote the class of functions analytic in D which are in every H^p class, $0<p<\infty$. The class $\hat{H}$ consists of those functions that are 'almost in H^∞'; the class $\hat{H}$ strictly contains H^∞.

There is a simple geometric condition that guarantees an analytic function belongs to $\hat{H}$. Let

$$A(R)=\text{Area of }\{|w|\le R\cap f(D)\}.$$

Hansen proved in 1974 [3] that $\lim_{R\to\infty}A(R)R^{-2}\log R=0$ implies f belongs to $\hat{H}$. Later Hansen and Hayman [4] removed the $\log R$ factor and proved $A(R)R^{-2}\to 0$ implies f belongs to $\hat{H}$. Informally, f is in $\hat{H}$ if f misses a substantial part of the plane. In fact when asked to exhibit functions in $\hat{H}-H^\infty$ most people cite the canonical examples of an unbounded function of finite area or an unbounded function of infinite area such as $\log(1-z)$; both of these examples fill an infinitesimal part of the plane.

Hansen observed in [3] that his condition 'is almost best possible since the inequality $A(R)\le\pi R^2$ holds for any complex valued function f'. The purpose of this note is to show that Hansen and Hayman's condition is far from being necessary. Using techniques from normal functions, gap series, characterizations of BMOA, subordination, Bloch functions, and VMOA, we will produce six completely different examples of functions in $\hat{H}$ which satisfy the extremal

AMS (1980) Subject Classification 30D55, 30D45, 30C45, 30B10

Received by the editors July 4, 1979 and in revised form, September 26, 1980 and November 5, 1980

condition $A(R)\equiv\pi R^2$, $0\le R<\infty$. Informally, we produce functions in $\hat{H}$ that fill the plane. Since the material comes from so many different areas of function theory, accessible rather than original sources have been cited.

EXAMPLE 1. *A non-normal function in $\hat{H}$ which takes on every value except* 0 *infinitely often.* Let $f(z)=z^{-1}\log(1-z)\cdot\exp[(z+1)/(z-1)]$. The function $\log(1-z)$ is contained in a strip of width π. Thus $A(R)\le 2\pi R$ and Hayman and Hansen's condition implies $\log(1-z)$ is in $\hat{H}$. The vanishing of $\log(1-z)$ at $z=0$ guarantees that $z^{-1}\log(1-z)$ is also in $\hat{H}$. Since $\hat{H}$ is an algebra and $\exp[(z+1)/(z-1)]$ is bounded we conclude that $f(z)=z^{-1}\log(1-z)\exp[(z+1)/(z-1)]$ is in $\hat{H}$. Rather than try to determine the range of f directly, we observe that $f(z)\to 0$ as $z\to 1$ radially, while $f(z)\to\infty$ as $z\to 1$ on the semi-circle $r=\cos\theta$, $0<r<1$. The point $z=1$ has two different asymptotic values which implies that $f(z)$ is not a normal function [8, p. 268]. A non-normal function takes on all but at most two values of the Riemann sphere infinitely often. Since f omits 0 and ∞ we see that f is a non-normal function in $\hat{H}$ which takes on every complex number except 0 infinitely often.

EXAMPLE 2. *A normal function in $\hat{H}$ which takes on every value infinitely often.* Let $f(z)=\sum_{k=1}^{\infty}a_kz^{n_k}$, where $n_{k+1}/n_k\ge q$ $(q\ge 100)$, $\sum_{k=1}^{\infty}|a_k|=\infty$, $\sum_{k=1}^{\infty}|a_k|^2<\infty$ (for example $a_k=1/k$, $n_k=1000^k$). Since f is a gap series with $\sum|a_k|^2<\infty$, f belongs to $\hat{H}$ (the proof of [11, p. 215] is clearer than the more recent edition [12, p. 213]). Since the coefficients go to zero and f is a gap series, f is in B_0, hence is Bloch, hence is a normal function. Finally, by G. and M. Weiss' theorem for gap series with $\sum|a_k|=\infty$, q sufficiently large, we know that f takes on every value infinitely often [10].

Perhaps the reader has been lulled into thinking that the key ingredient in the above examples has been the infinite valence of f. We therefore turn to the other extreme and produce functions in $\hat{H}$ which are univalent but satisfy $A(R)\equiv\pi R^2$, $0\le R<\infty$. Since f is univalent it will be forced to omit a continuum of values, but these will be made to have zero area.

EXAMPLE 3. *A univalent close-to-convex function in $\hat{H}$ with $A(R)\equiv\pi R^2$.* Let Ω be the set of complex numbers not equal to $n+iy$, $|y|\ge 1$, $n=0,\pm1,\pm2,\ldots$ Let f be the analytic univalent function mapping $|z|<1$ onto Ω, $f(0)=0$, $f'(0)>0$. The function f is close-to-convex [8, p. 52]. Clearly $A(R)\equiv\pi R^2$, $0\le R<\infty$. The Hayman–Pommerenke–Stegenga criteria for BMOA functions [5] states: A domain $G\subset\mathbb{C}$ has the property that every function $f(z)$ analytic in D with values in G belongs to BMOA if and only if there exist constants R and $\delta>0$ such that cap $((\mathbb{C}-G)\cap\{|w-w_0|\le R\}>\delta$ for every w_0 in G, where cap denotes the logarithmic capacity. For every point w_0 in Ω the disc $|w-w_0|\le 20$ contains a line segment of length at least 1 in the complement of Ω. Therefore the capacity of $(\mathbb{C}-\Omega)\cap\{|w-w_0|\le 20\}$ is bounded below by $\frac{1}{4}$ [8,

p. 335]. By the Hayman–Pommerenke–Stegenga criterion for BMO functions f is in BMOA. But every BMOA function is in $\hat{H}$ and we are done.

EXAMPLE 4. *A starlike, non-BMOA, non-Bloch, function in* $\hat{H}$ with $A(R) \equiv \pi R^2$. Let Ω be the region formed from $\mathbb{C}$ by performing a countable number of operations, the nth of which is the removal of 2^{n+2} infinite radial slits whose end points have modulus 4^n and whose arguments are $\pi k \cdot 2^{-n-1}$, $k = 1, 2, 3, \ldots, 2^{n+2}$. Clearly Ω is a starlike domain containing the origin. Let f map $|z|<1$ into Ω, $f(0)=0$, $f'(0)>0$. It is well known [7] that a function analytic in D is Bloch if and only if $\sup\{d_f(z): z \in D\} < \infty$, where $d_f(z)$ is the radius of the largest schlicht disc around $f(z)$ on the Riemann image surface of $f(D)$. Since Ω contains a disc of radius approximately 2^{n-1} at the point $(4^n - 2^n)\exp(\pi i 2^{-n-1})$, the function f is not Bloch. All BMOA functions are Bloch [9, p. 593]; hence f is not BMOA. The function

$$g_n(z) = cz \Big/ \prod_{v=1}^{2^{n+2}} (1 - e^{\pi i v/2^{n+1}} z)^{1/2^{n+1}}$$

maps $|z|<1$ onto the plane with 2^{n+2} infinite symmetrically placed slits with end points of equal modulus. Furthermore

$$\int_0^{2\pi} |g_n(re^{i\theta})|^{2^n} \, d\theta \le |c|^{2^n} \int_0^{2\pi} \prod_{v=1}^{2^{n+2}} |1 - e^{\pi i v/2^{n+1}} e^{i\theta} r|^{-1/2} \, d\theta$$

obviously remains finite as $r \to 1$. Any function subordinate to an H^q function is also in H^q [1, p. 10]. To conclude that f is in $\hat{H}$ it suffices to note that for every n the function f is subordinate to the H^{2^n} function g_n. The function f is normal since it is univalent [8, p. 262].

EXAMPLE 5. *A starlike univalent BMOA function in* $\hat{H}$, *with* $A(R) \equiv \pi R^2$. Let Ω be the starlike region formed by removing from $\mathbb{C}$ all infinite radial rays which begin at an integral lattice point of $\mathbb{C}$ (except $(0,0)$). Let f map $|z|<1$ univalently onto Ω, $f(0)=0$, $f'(0)>0$. Clearly $A(R) \equiv \pi R^2$. Since f contains no disc of radius bigger than 1, it is a Bloch function [7]. All Bloch univalent functions are BMOA [9, p. 592] hence in $\hat{H}$.

EXAMPLE 6. *A locally univalent, normal function in* $\hat{H}$ *which takes on every value except* 0 *infinitely often.* We first construct a simply connected region Ω with two special properties. First, if w_n is in Ω and $w_n \to \partial\Omega$, then the radius of the largest schlicht disc contained in Ω and centered at w_n will be required to go to zero as $n \to \infty$. Second, for every real number x there must be an infinite *ray*, whose real part is x, contained in Ω. We proceed as follows:

In the half-strip $\{x+iy: 0<x<1,\ 0<y<\infty\}$ we perform a countable number of operations, the nth of which is the removal from $\{x+iy: 0<x<1,\ 0<y<\infty\}$ of 2^{n-1} infinite slits parallel to the imaginary axis whose initial points are $2\pi i n + k2^{-n}$, $k = 1, 3, 5, \ldots, 2^n - 1$.

In the half-strip $\{x+iy: m<x<m+1,\ 0<y<\infty\}$, $m=\pm1, \pm2, \ldots$ we perform a countable number of operations, the nth of which is the removal of 2^{n-1} infinite slits parallel to the imaginary axis whose initial points are $2\pi in/|m| + k2^{-n}+m$, $k=1,3,5,\ldots,2^n-1$.

In the half-strip $\{x+iy: \sqrt{2}<x<1+\sqrt{2},\ -\infty<y<0\}$ we perform a countable number of operations, the nth of which is the removal of 2^{n-1} infinite slits parallel to the imaginary axis whose initial points are $2\pi in+k2^{-n}+\sqrt{2}$, $k=1,3,\ldots,2^n-1$.

In the half-strip $\{x+iy: \sqrt{2}+m<x<\sqrt{2}+m+1,\ -\infty<y<0\}$, $m=\pm1,\pm2,\ldots$ we perform a countable number of operations, the nth of which is the removal of 2^{n-1} infinite slits parallel to the imaginary axis whose initial points are $2\pi in+k2^{-n}+\sqrt{2}$, $k=1,3,\ldots,2^n-1$.

Let g map $|z|<1$ univalently into Ω, $g(0)=0$, $g'(0)>0$. Since the radius of any schlicht disc which approaches $\partial\Omega$ goes to zero, we conclude that g is in B_0. Let $f(z)=\exp(g(z))$. Univalent B_0 functions are in VMO [9, p. 593] and the exponential of any VMO function is in $\hat{H}$ [9, p. 596]. Since g is univalent, f is locally univalent. Since g is Bloch, f is normal. We constructed Ω so that for any real number x there is an infinite ray whose real part is x which is contained in Ω. Thus f takes on every value expect 0 infinitely often.

References

1. P. L. Duren, *Theory of H^p spaces*, Academic Press, 1970.
2. L. J. Hansen, *Hardy classes and ranges of functions*, Mich. Math. J. **17** (1970), 235–248.
3. L. J. Hansen, *The Hardy class of a function with slowly growing area*, Proc. Amer. Math. Soc. **45** (1974), 409–410.
4. L. J. Hansen and W. K. Hayman, *On the growth of functions omitting large sets*, Journal d'Analyse Math. **30** (1976), 208–214.
5. W. K. Hayman and Ch. Pommerenke, *On analytic functions of bounded mean oscillation*, Bull. London Math. Soc. **10** (1978), 219–224.
6. O. Lehto and K. I. Virtanen, *Boundary behavior and normal meromorphic functions*, Acta Math. **97** (1957), 47–65.
7. Ch. Pommerenke, *On Bloch Functions*, J. London Math. Soc. (2) **2** (1970), 689–695.
8. Ch. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.
9. Ch. Pommerenke, *Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation*, Comm. Math. Helvetici **52** (1977), 591–602.
10. M. Weiss and G. Weiss, *On the Picard property of lacunary power series*, Studia Math. **22** (1963), 221–245.
11. A. Zygmund, *Trigonometrical Series*, Warsaw, 1935.
12. A. Zygmund, *Trigonometric Series*, Vol. 1, Cambridge, 1958.

BRIGHAM YOUNG UNIVERSITY
PROVO, UTAH, 84602.