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Character Amenability of Lipschitz Algebras

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Abstract. Let $\mathcal{X}$ be a locally compact metric space and let $\mathcal{A}$ be any of the Lipschitz algebras $\mathrm{Lip}_\alpha \mathcal{X}$, $\mathrm{lip}_\alpha \mathcal{X}$, or $\mathrm{lip}^0_\alpha \mathcal{X}$. In this paper, we show, as a consequence of rather more general results on Banach algebras, that $\mathcal{A}$ is C-character amenable if and only if $\mathcal{X}$ is uniformly discrete.

1 Introduction

Johnson [6] introduced the important concept of amenability for Banach algebras in 1972. In fact, he defined the amenability of a Banach algebra $\mathcal{A}$ through vanishing of the first cohomology group of $\mathcal{A}$ with coefficients in a dual Banach $\mathcal{A}$-bimodule. Many papers have considered the implications of amenability and some other related concepts for various Banach algebras such as group algebras and Lipschitz algebras.

Ülger [13] showed that amenability of $\mathcal{A}$ implies that $\Delta(\mathcal{A})$, the spectrum of $\mathcal{A}$, is discrete with respect to the weak topology induced by $\mathcal{A}^{**}$. He also observed that when $\mathcal{A}$ is commutative and an ideal in $\mathcal{A}^{**}$, the weak and weak* topologies agree on $\Delta(\mathcal{A})$. In particular, if $\mathcal{A}$ is commutative and amenable, and an ideal in $\mathcal{A}^{**}$, then $\Delta(\mathcal{A})$ is necessarily discrete with respect to the weak* topology.

On the other hand, for $\phi \in \Delta(\mathcal{A})$, Kaniuth, Lau, and Pym [7, 8] introduced and studied the concept of ϕ-amenability for Banach algebras. In fact, $\mathcal{A}$ is called *ϕ-amenable* if there exists a bounded linear functional m on $\mathcal{A}^*$ satisfying

$$m(\phi) = 1 \quad \text{and} \quad m(f \cdot a) = m(f)\phi(a)$$

for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, where $f \cdot a \in \mathcal{A}^*$ is defined by $(f \cdot a)(b) = f(ab)$ for all $b \in \mathcal{A}$. Any such m is called a *ϕ-mean*. Moreover, for some $C > 0$, $\mathcal{A}$ is called *C-ϕ-amenable* if there exists a ϕ-mean bounded by C; see Hu, Monfared, and Traynor [5]. The notion of *(right) character amenability* was introduced and studied by Monfared [9]. Character amenability of $\mathcal{A}$ is equivalent to $\mathcal{A}$ being ϕ-amenable for all $\phi \in \Delta(\mathcal{A})$ and $\mathcal{A}$ having a bounded right approximate identity. The concept of *C-character amenability* is defined similarly; see [5] for details.

Our purpose here is to consider when the Lipschitz algebras, $\mathrm{Lip}_\alpha\mathcal{X}$, $\mathrm{lip}_\alpha\mathcal{X}$, and $\mathrm{lip}^0_\alpha\mathcal{X}$ on a locally compact metric space $\mathcal{X}$, where $0 < \alpha$, are C-character amenable. These interesting Banach algebras were first considered by Schebert [12]; see also Bishop [2]. Gourdeau [3] discussed amenability of Lipschitz algebras by showing that if a Banach algebra $\mathcal{A}$ is amenable, then $\Delta(\mathcal{A})$ is uniformly discrete with respect

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to norm topology induced by $\mathcal{A}^*$; see also Bade, Curtis, and Dales [1], Gourdeau [4], and Zhang [14].

For this purpose, we discuss the relation between C-ϕ-amenability of $\mathcal{A}$ and its spectrum. We show that C-ϕ-amenability of $\mathcal{A}$ for all $\phi \in \Delta(\mathcal{A})$ implies that $\Delta(\mathcal{A})$ is uniformly discrete. We also show that $\Delta(\mathcal{A})$ is discrete with respect to the weak* topology for a certain Banach algebra $\mathcal{A}$. Finally, we show that C-character amenability of $\mathrm{Lip}_\alpha \mathcal{X}$, $\mathrm{lip}_\alpha \mathcal{X}$, and $\mathrm{lip}_\alpha^0 \mathcal{X}$ is equivalent to uniform discreteness of the underlying locally compact metric space $\mathcal{X}$.

2 The Spectrum of ϕ-amenable Banach Algebras

Let $\mathcal{A}$ be a Banach algebra. Kaniuth, Lau, and Pym [8, Remark 5.1] brought a necessary condition for discreteness of $\Delta(\mathcal{A})$ with respect to the weak topology. Indeed, they showed that $\Delta(\mathcal{A})$ is discrete with respect to the weak topology induced by $\mathcal{A}^{**}$ if $\mathcal{A}$ is ϕ-amenable for all $\phi \in \Delta(\mathcal{A})$. In this section, we present necessary conditions for uniform discreteness and discreteness of $\Delta(\mathcal{A})$ with respect to the weak* topology.

Let us commence with the following result, which we need in the sequel and follows from an observation in [8, Remark 5.1].

Proposition 2.1 *Let $\mathcal{A}$ be a Banach algebra and let $\phi \in \Delta(\mathcal{A})$. If there exists $C > 0$ such that $\mathcal{A}$ is C-ϕ-amenable, then*

$$\|\phi - \psi\|_{\sup} \geq C^{-1} \quad \text{for all} \quad \psi \in \Delta(\mathcal{A}) \setminus \{\phi\}.$$

Proof Suppose that $\mathcal{A}$ is C-ϕ-amenable. Then there exists an element $m \in \mathcal{A}^{**}$ with $\|m\| \leq C$ such that

$$m(\phi) = 1 \quad \text{and} \quad m(f \cdot a) = m(f)\phi(a)$$

for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. Let $\psi \in \Delta(\mathcal{A}) \setminus \{\phi\}$. Then, by [8, Remark 5.1], we have $m(\psi) = 0$ for all $\psi \in \Delta(\mathcal{A}) \setminus \{\phi\}$. So,

$$1 = |m(\phi - \psi)| \leq \|m\| \, \|\phi - \psi\| \leq C \, \|\phi - \psi\|,$$

and, consequently, $\|\phi - \psi\| \geq C^{-1}$ for all $\psi \in \Delta(\mathcal{A}) \setminus \{\phi\}$. ■

The following result is an immediate consequence of Proposition 2.1. First, let us recall that, for a metric space $\mathcal{X}$ with a metric d, a subset $\mathcal{Y}$ of $\mathcal{X}$ is called *uniformly discrete* if there exists $\epsilon > 0$ such that $d(x, y) > \epsilon$ for all distinct elements $x, y \in \mathcal{Y}$.

Corollary 2.2 *Let $\mathcal{A}$ be a Banach algebra. If there exists $C > 0$ such that $\mathcal{A}$ is C-ϕ-amenable for all $\phi \in \Delta(\mathcal{A})$, then $\Delta(\mathcal{A})$ is a uniformly discrete subset of $\mathcal{A}^*$.*

Recall that a Banach algebra $\mathcal{A}$ is *ϕ-contractible* if for any Banach $\mathcal{A}$-bimodule $\mathcal{X}$ with right module action of $\mathcal{A}$ on $\mathcal{X}$ defined by

$$x \cdot a = \phi(a)x \; (a \in \mathcal{A}, \; x \in \mathcal{X}),$$

every continuous derivation $D\colon \mathcal{A} \to \mathfrak{X}$ is inner. This notion was recently introduced and studied by Hu, Monfared, and Traynor [5] as right ϕ-contractibility. Later on, the second and third authors [10] showed, as a consequence of rather more general results, that ϕ-contractibility of $\mathcal{A}$ is equivalent to existence of an element $m \in \mathcal{A}$ such that $\phi(m) = 1$ and $am = \phi(a)m$ for all $a \in \mathcal{A}$.

Proposition 2.3 *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}$ is ϕ-contractible for all $\phi \in \Delta(\mathcal{A})$, then $\Delta(\mathcal{A})$ is discrete with respect to the weak* topology induced by $\mathcal{A}$.*

Proof Let $\phi \in \Delta(\mathcal{A})$. Since $\mathcal{A}$ is ϕ-contractible, there exists an element $m \in \mathcal{A}$ such that

$$\phi(m) = 1 \quad \text{and} \quad am = \phi(a)m$$

for all $a \in \mathcal{A}$. By [8, Remark 5.1] again, we have $\psi(m) = 0$ for all $\psi \in \Delta(\mathcal{A}) \setminus \{\phi\}$. Therefore, $\Delta(\mathcal{A})$ is $\sigma(\mathcal{A}^*, \mathcal{A})$-discrete. ■

In [13, Corollary 3.2], Ülger proved that if $\mathcal{A}$ is a commutative amenable Banach algebra that is an ideal in its second dual, then $\Delta(\mathcal{A})$ is discrete with respect to the weak* topology induced by $\mathcal{A}$. Related to this result, we have the following consequence of Proposition 2.3.

Corollary 2.4 *Let $\mathcal{A}$ be a Banach algebra that is an ideal in its second dual. If $\mathcal{A}$ is ϕ-amenable for all $\phi \in \Delta(\mathcal{A})$, then $\Delta(\mathcal{A})$ is discrete with respect to the weak* topology induced by $\mathcal{A}$.*

Proof Fix $\phi \in \Delta(\mathcal{A})$. By assumption, $\mathcal{A}$ is ϕ-amenable and is an ideal in $\mathcal{A}^{**}$. Then $\mathcal{A}$ is ϕ-contractible by [10, Corollary 3.6]. Hence, by the preceding proposition, $\Delta(\mathcal{A})$ is discrete with respect to the weak* topology. ■

Next we present some interesting examples to which our preceding results apply.

Example 2.5 (i) Let G be a locally compact amenable group. Then the group algebra $L^1(G)$ and the generalized Fourier algebra $A_p(G)$, $1 < p < \infty$, are 1-character amenable; see [5]. So, their spectra are discrete with respect to the weak topology and are uniformly discrete.

(ii) The Fourier–Stieltjes algebra $B(G)$ of a compact group G, is 1-character amenable; see [5]. So, $\Delta(B(G))$ is discrete with respect to the weak topology and is also uniformly discrete.

We end this section with the following counter example that shows that the C-ϕ-amenability (ϕ-contractibility) for all $\phi \in \Delta(\mathcal{A})$, although sufficient, is not necessary for the space $\Delta(\mathcal{A})$ to be uniformly discrete (discrete with respect to the weak* topology). In fact, it shows that the converse of Corollary 2.2, Proposition 2.3, and Corollary 2.4 are not valid.

Example 2.6 Let $\mathcal{A}$ be the Banach algebra of all upper-triangular 3×3 matrices over $\mathbb{C}$. Then $\Delta(\mathcal{A}) = \{\phi_1, \phi_2, \phi_3\}$, where for $k = 1, 2, 3$, ϕ_k is defined by

$$\phi_k\big([a_{ij}]\big) = a_{kk};$$

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see [5, Example 6.5]. It is clear that $\Delta(\mathcal{A})$ is discrete with respect to the weak* topology induced by $\mathcal{A}$; moreover, $\Delta(\mathcal{A})$ is uniformly discrete. Whereas, as proved in [5], $\mathcal{A}$ is not ϕ_2-amenable, it is therefore not ϕ_2-contractible.

3 An Application to Lipschitz Algebras

Let $\mathcal{X}$ be a metric space with metric d, and take α with $\alpha > 0$. Recall that $\mathrm{Lip}_\alpha\mathcal{X}$ is the space of bounded complex-valued functions f on $\mathcal{X}$ such that

$$p_\alpha(f) = \sup\Big\{ \frac{|f(x) - f(y)|}{d(x,y)^\alpha} : \quad x, y \in \mathcal{X}, \quad x \neq y \Big\} < \infty.$$

It is known that $\mathrm{Lip}_\alpha\mathcal{X}$ endowed with the norm $\|\cdot\|_\alpha$ given by

$$\|f\|_\alpha = p_\alpha(f) + \|f\|_{\sup},$$

and pointwise product is a Banach algebra called a Lipschitz algebra. Moreover, $\mathrm{lip}_\alpha\mathcal{X}$ is the subalgebra of functions $f \in \mathrm{Lip}_\alpha\mathcal{X}$ such that

$$\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \to 0 \quad \text{as} \quad d(x,y) \to 0.$$

If $\mathcal{X}$ is a locally compact metric space, then $\mathrm{lip}^0_\alpha\mathcal{X}$ is the subalgebra of $\mathrm{lip}_\alpha\mathcal{X}$ consisting of those functions tending to zero at infinity.

Recently, character amenability of Lipschitz algebras have been investigated by Hu, Monfared, and Traynor [5]. They showed, among other things, that when $\mathcal{X}$ is an infinite compact metric space and $0 < \alpha < 1$, $\mathrm{Lip}_\alpha\mathcal{X}$ is not character amenable.

In our last result, we characterize C-character amenability of Lipschitz algebras.

Theorem 3.1 *Let $\mathcal{X}$ be a locally compact metric space and let $\mathcal{A}$ be any of the Lipschitz algebras $\mathrm{Lip}_\alpha\mathcal{X}$, $\mathrm{lip}_\alpha\mathcal{X}$ or $\mathrm{lip}^0_\alpha\mathcal{X}$. Then the following statements are equivalent.*

(i) *$\mathcal{A}$ is C-character amenable, for some $C > 0$.*
(ii) *$\mathcal{A}$ is amenable.*
(iii) *$\mathcal{X}$ is uniformly discrete.*

Proof (i)⇒(iii). Since $\mathcal{A}$ is C-character amenable, for some $C > 0$, it follows from Corollary 2.2 that $\Delta(\mathcal{A})$ is uniformly discrete; that is, there is $\epsilon > 0$ such that

$$\|\phi - \psi\| > \epsilon$$

for all distinct elements $\phi, \psi \in \Delta(\mathcal{A})$. In particular, $\|\phi_x - \phi_y\| > \epsilon$ for all distinct elements $x, y \in \mathcal{X}$, where ϕ_x denotes the character on $\mathcal{A}$ defined by $\phi_x(f) = f(x)$ for all $f \in \mathcal{A}$. But

$$\|\phi_x - \phi_y\| = \sup_{\|f\|_\alpha \le 1} |\phi_x(f) - \phi_y(f)| = \sup_{\|f\|_\alpha \le 1} |f(x) - f(y)| \le d(x,y)^\alpha$$

for all $x, y \in \mathcal{X}$, where d is the metric of $\mathcal{X}$. This yields that $d(x,y)^\alpha > \epsilon$ for all distinct elements $x, y \in \mathcal{X}$ whence $\mathcal{X}$ is uniformly discrete.

(iii)⇒(ii). This follows from [3, Theorem 3].

(ii)⇒(i). Since $\mathcal{A}$ is amenable, it has an approximate diagonal bounded by some $C > 0$; see [11]. So, $\mathcal{A}$ is C-character amenable by [5, Theorem 2.9]. ■

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