NEW RESULTS ON THE COMMENSURABILITY
CASES OF THE PROBLEM SUN-JUPITER-ASTEROID

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ABSTRACT

The short-period terms are removed by averaging from special equations of motion for commensurability cases of the three-dimensional, elliptic, restricted three-body problem. Some examples of retrograde motion corresponding to the -1/1 commensurability, and an application to Hilda-type motion demonstrate the possibilities given by the method.

1. LONG-PERIOD EFFECTS STUDIED BY AVERAGING OF THE EQUATIONS OF MOTION

In 1963 the late celestial mechanician Imre G. Izsak invited me to work at the Smithsonian Astrophysical Observatory on long-period effects in commensurability cases according to the ideas of Poincaré (1902). I did so on the basis of the circular, restricted three-body problem given by the sun, Jupiter, and a small body. However, I replaced Poincaré's way of removing the short-period terms from the basic equations of the problem, by an averaging procedure (Schubart, 1964, 1966). The work done by Message (1966) is closer to the way proposed by Poincaré, but I had the advantage of including the treatment of very eccentric orbits in my work.

Later on D. Brouwer asked me to work on the 3/2 commensurability case that is represented by an asteroid of Hilda-type and Jupiter. For this I generalized my method of averaging to the elliptic, restricted three-body problem, but to the planar case only. The inclination of real orbits had to be neglected (Schubart, 1968). The inclination of (153) Hilda, for instance, is not very large, but when Giffen (1973) applied the same method of averaging to the 2/1 case, he had to neglect a much larger inclination to obtain a model for the motion of (1362) Griqua. As a consequence, the model gave only a rough approximation to real motion. Recently, I have dropped the restriction to the planar case, and I present some first results obtained in this

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way in the present paper.

In the mean time, Scholl and Froeschlé (1974, 1975) used the method of averaging for the planar case in a treatment of the 3/1, 5/2, 7/3, and 2/1 resonances with respect to Jupiter. They tested a collision hypothesis for the formation of the Kirkwood gaps in the asteroid belt at these resonances. As an application of their results, they will present a paper "The Kirkwood Gaps as an Asteroidal Source for Meteorites" at IAU Colloquium No. 39 at Lyon. In a recent paper, Froeschlé and Scholl (1976) confirmed former results obtained by Scholl and Giffen (1974) with respect to a conjecture by Giffen (1973), which is evidently not true.

R. Bien, working on a dissertation at the Rechen-Institut, is treating the 1/1 commensurability of the planar, elliptic restricted problem by the method of averaging. He made a search in a wide range of phase space for orbits with interesting long-period effects in the orbital elements, especially in e and $\tilde{\omega}$. Orbits of Trojan-type appear in his material, but also other orbits that represent a kind of very remote, retrograde satellites of Jupiter.

2. THE METHOD OF AVERAGING FOR THE THREE-DIMENSIONAL, ELLIPTIC, RESTRICTED THREE-BODY PROBLEM

In generalizing the computer program for the planar, elliptic problem (Schubart, 1968) to the three-dimensional case, I retained the basic definitions, units and constants. The reader is referred to the former paper (Schubart, 1968) for details, and especially, for the definition of the averaging process applied to the differential equations of the problem, and for the way of numerical integration. An IBM/360-44 computer was available for the recent computations.

As before, a commensurability case is described by the approximate ratio of the mean motions of an asteroid and Jupiter, given by (p+q)/p, where p and q are relative prime integers, and p > 0. The six variables to be determined from averaged differential equations by integration, are now :

$$G = a^{1/2}\ (1-e^2)^{1/2}$$

$$\mu = 1-l_J\ (p+q)/p$$

$$\psi_1 = e\ \cos\tilde{\omega}$$

$$\psi_2 = e\ \sin\tilde{\omega}$$

$$\psi_3 = \mathrm{tg}\ (i/2)\ \cos\ \Omega$$

$$\psi_4 = \mathrm{tg}\ (i/2)\ \sin\ \Omega$$

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Here, l and l_J are the mean longitudes of the asteroid and Jupiter. a, e, $\tilde{\omega} = \omega + \Omega$, i, and Ω are the usual designations of the osculating elements of the asteroid, but the orbital plane of Jupiter is the plane of reference, and the longitude of the node, Ω, is counted in this plane from the fixed direction of the perihelion of Jupiter. This direction is optional, if the eccentricity of Jupiter, e_J, is neglected. e_J, $a_J = 1$, and $\tilde{\omega}_J = 0^o$ describe the orbit of Jupiter.

I omit a listing of the six differential equations that follow from the corresponding equations of the orbital elements, but I want to remark, that comparatively simple equations result for the derivatives of ψ_3 and ψ_4 with respect to t, compare my former treatment of Hill's Problem (Schubart, 1963).

The set of the six variables is not suitable for cases of retrograde motion, if such cases are described by $i > 90^o$, and if i is close to 180^o. However, if a decreasing mean longitude, or a negative mean motion, $n < 0$, is introduced, such a case can be described by $i < 90^o$, and $i = 0^o$ is not an exceptional case then. If the sign of $(p+q)$ is changed together with the sign of n, μ will vary slowly as before. The new computer program can integrate many retrograde cases in two ways, either by $i > 90^o$, or by a negative mean motion. In the latter case, a negative starting value of G causes $a^{1/2} < 0$ and thus $n = a^{-3/2} < 0$.

Following Poincaré (1902), I had used quantities σ and τ in my studies of the circular problem (Schubart, 1964). They are given by :

$$\sigma = l - \tilde{\omega} - (l-l_J)\cdot(p+q)/q = -\tilde{\omega} - \mu \cdot p/q$$

$$\tau = l - \Omega - (l-l_J)\cdot(p+q)/q = -\Omega - \mu \cdot p/q$$

The new program can list both $q\sigma$ and $q\tau$, as well as $\omega = \tilde{\omega} - \Omega = \tau - \sigma$ and other quantities, as functions of t. Libration of σ appears in many cases of commensurability, and these librations are an important way for many asteroids and some other objects to avoid close approaches to a disturbing body (see, for instance, Schubart, 1968).

3. SOME SPECIAL CASES OF THE CIRCULAR RESTRICTED PROBLEM

It was one of my first questions to the new program to find out, whether τ can be equally important for an object to avoid close approaches to Jupiter. It is sufficient to consider the circular restricted problem for a first answer to this question. I knew from former studies of nonplanar motion corresponding to the 3/1 case, that libration of τ is possible. If q is even, as in this case, $\psi_1 = \psi_2 = 0$ is a particular solution of the differential equations. This allows a comparatively simple study of the long-range effects in quantities corresponding to i and τ (Schubart, 1964). However, the vanishing eccentricity is sufficient to avoid close approaches, if a is small enough.

TABLE I

Starting values of eight orbits

- The sign of n equals the sign of (p+q)/p -

No.	(p+q)/p	e_J	a	e	$\tilde{\omega}$	μ	i	Ω
1	-1/1	0.0	1.0	0.0	0.0°	0.0°	30.0°	90.0°
2	-1/1	0.0	1.00091140	0.0	0.0	0.0	30.0	90.0
3	-1/1	0.0	1.0	0.4	0.0	0.0	0.0	0.0
4	-1/1	0.0	1.00340782	0.4	0.0	0.0	0.0	0.0
5	3/2	0.048	0.763806	0.14881	261.94	65.75	8.72	221.54
6	3/2	0.0	0.762	0.15	0.0	0.0	30.0	270.0
7	3/2	0.0	0.76266322	0.15	0.0	0.0	30.0	270.0
8	3/2	0.0	0.7625	0.44	0.0	0.0	30.0	270.0

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Therefore, I increased a to 1, the value corresponding to Jupiter. Libration of μ is important for direct motion in the range of the 1/1 commensurability, but I changed to retrograde motion in using p = 1, q = - 2, and a negative mean motion. The first orbit in Table I has a starting value $\tau = -90^{\circ}$, which will give the asteroid a distance of 90° from both nodes at a moment of conjunction with Jupiter, $l = l_J$. The integration of the orbit shows, that τ librates around the starting value with an amplitude of about 5°, and that the asteroid will not come closer to Jupiter than 2.69 a.u., on the basis of the averaged circular problem. In this case the libration of τ prevents a close approach. I did not study the effects caused by a variation of the starting value of e in this case, but I varied the one of a. In this way I found solution No. 2 (see Table I) with values of a, e = 0, i, and $\tau = -90^{\circ}$, which are constant as functions of t.

The next two orbits, No. 3 and 4, belong to the retrograde 1/1 case as well, but the eccentricity is different from zero instead of the inclination. Libration of σ around 0° causes the small body to be close to perihelion or aphelion at a conjunction with Jupiter, which prevents close approaches again. σ librates with an amplitude of $8\overset{\circ}{.}4$ in case of No. 3, while it stays at 0° in the next case. There are probably no real objects on such retrograde orbits, but the orbits demonstrate the possibilities given by the program.

4. APPLICATION TO HILDA-TYPE MOTION

The remaining orbits in Table I belong to the 3/2 commensurability, especially No. 5 represents a model for the asteroid (153) Hilda. This model is an extension of my former model for Hilda (Schubart, 1968). The model is based on numerical results by Akiyama (1962). The angular elements were transformed to the orbit of Jupiter, which is the plane of reference. Orbit No. 5 was integrated forward and backward, so that a total period of about 12 000 yr is covered. This corresponds to more than 180° of retrograde motion of ☊ , as it is demonstrated at the bottom of Fig. 1. The moment t = 0 corresponds to the year 1963. Fig. 1 demonstrates the resulting variations of σ with increasing time in analogy to the corresponding figure for the former model. Since the interval in t is much larger now, I did not draw a curve, but I plotted the successive maxima and minima, caused by the period of libration. The crosses correspond to the maxima. The period of libration equals 275 yr now. The period of perihelion that equals 2650 yr, causes strong variations in the subsequent maxima, or minima, as it is known from my former model. The backward revolution of the node which follows a period of about 22 300 yr, causes only small effects in the variations of σ, as it appears from Fig. 1. The range of these variations is only slightly larger than in case of the former model.

The period of revolution of ω, on the average, equals about 3000 yr. This and other periods cause variations in i and in the speed of motion of ☊ . However, i remains close to $9\overset{\circ}{.}0$. The deviations are less than $0\overset{\circ}{.}4$.

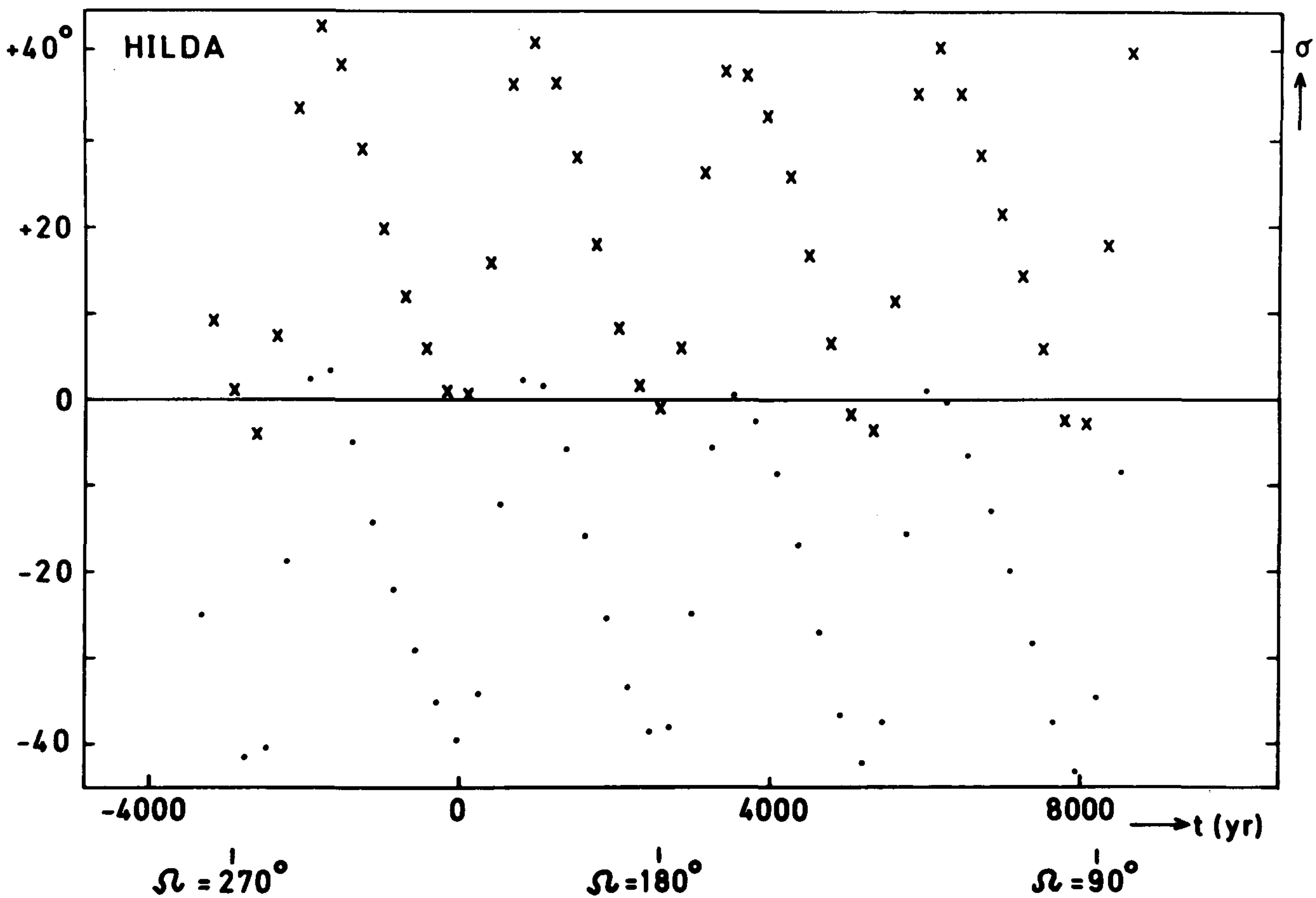

Fig. 1. Variations of σ during 12 000 yr for asteroid (153) Hilda. Instead of a curve, the figure shows the succession of the maxima (crosses) and minima (dots) of σ (t), which are caused by the period of libration. Special marks indicate the motion of Ω .

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The node deviates from linear motion by not more than $2\overset{\circ}{.}5$ during the interval considered. As a whole, all the results obtained for this orbit show that a planar model is meaningful in a case like Hilda, but that additional effects are revealed by the three-dimensional model.

I have not yet considered real asteroids with a larger inclination, but I studied some theoretical examples with $i = 30^\circ$ that belong to the 3/2 case of the circular restricted problem. Orbit No. 6, started at $\omega = 90^\circ$, shows a libration of σ around 0°. The period of libration gives an effect in σ, but the period of the retrograde revolution of ω causes a much stronger effect. In case of orbit No. 7 there is almost no influence of the period of libration, but the revolution of ω causes an amplitude of 15° in σ. The variations of e and ω can be demonstrated in rectangular coordinates $\xi = e \cos 2\omega$ and $\eta = e \sin 2\omega$ in this case : The point ξ, η moves with a nearly constant velocity along a nearly circular curve, that has its center on the positive ξ-axis.

Finally, I selected orbit No. 8, because I suspected a libration of ω in this case, according to the information given in a paper by Jefferys and Standish (1972). A limited integration indicates indeed, that both σ and ω librate around 0° and 90°, respectively, with different main periods. The amplitudes are small in both cases. Since $\tau = \sigma + \omega$, τ is in libration as well. According to this twofold libration, an asteroid on this orbit will not come much closer to Jupiter than 3.4 a.u..

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