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THE MEASURE SPECTRUM OF A UNIFORM ALGEBRA AND SUBHARMONICITY

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Introduction. Let A be a uniform algebra on a compact Hausdorff space X. The spectrum, or the maximal ideal space, M_A, of A is given by

$$M_A = \{\phi \in A^* | \ \phi(1) = 1; \phi(fg) = \phi(f)\,\phi(g); f, g \in A\}.$$

We define the measure spectrum, S_A, of A by

$$S_A = \{\mu \in C(X)^* | \mu \geqq 0; \int fg d\mu = \int f d\mu \int g d\mu; f, g \in A\}.$$

S_A is the set of all representing measures on X for all $\phi \in M_A$. (A representing measure for $\phi \in M_A$ is a probability measure μ on X satisfying $\phi(f) = \int_X f d\mu$ for each $f \in A$.)

The concept of representing measure continues to be an effective tool in the study of uniform algebras. See for example [**12**, Chapters 2 and 3], [**5**, pp. 15–22] and [**3**]. Most of the known results on the subject of representing measures, however, concern measures associated with a single homomorphism. In this paper we treat the structure and behavior of the space consisting of all representing measures associated with a uniform algebra, i.e., the measure spectrum.

The subharmonicity of a certain class of functions arising from a uniform algebra has been useful in exhibiting analytic structure in the maximal ideal space. See for example, [**14**, p. 139], [**2**, pp. 99–106]. These functions have the following form: Let $g \in A$. (By this we mean the Gelfand transform of g.) For $\lambda \in \mathbf{C}$, let

$$g^{-1}(\lambda) = \{\phi \in M_A | \ g(\phi) = \lambda\}.$$

If u is a set function mapping compact subsets of $\mathbf{C}$ into $\mathbf{R} \cup \{-\infty\}$ and if $f \in A$ then we can associate a function U on a component of $\mathbf{C}\backslash g(X)$ by

$$(1) \quad U : \lambda \rightarrow u\{f[g^{-}(\lambda)]\}.$$

Various techniques for proving subharmonicity of functions of this type have been developed recently [**7**], [**10**], [**11**]. In Section 1 of this paper we define functions of type (1) arising from $f \in C_R(X)$ and $g \in A$. Using the properties of measure spectrum we study the subharmonicity

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of these functions. One example of U given by Wermer in [**15**] is

$$Z(\lambda) = \max_{g^{-1}(\lambda)} |f|.$$

We define, for $f \in C_R(X)$ and $g \in A$,

$$(2) \quad H_f(\lambda) = \max_{\phi \in g^{-1}(\lambda)} \hat{f}(\phi)$$

where $\hat{f}$ denotes the extension of f into M_A via

$$\hat{f}(\phi) = \min_{\mu \in M\phi} \int_X f d\mu.$$

Using Wermer's result, we show in Theorem 1.5 that the upper regularization of H_f is subharmonic. We include an interesting example where H_f (without the upper regularization) is subharmonic.

In Section 2 we construct the measure spectrum for two concrete examples: the annulus algebra and the generalized annulus algebra with two holes. (The results obtained here can readily be generalized to the case of n holes.) These algebras are prototypes of hypo-Dirichlet algebras whose measure spectrum can be regarded as a finite codimensional extension of the maximal ideal space. Hence they provide a natural setting in which to begin our investigation of S_A. Our computations for these examples are based on the classical results on the inner normal derivative of the Green's function [**8**] and Royden's work on annihilating measures [**9**].

In Section 3 we examine subharmonicity properties of the S_A's constructed in Section 2. Each $\mu \in S_A$ is assigned a unique set of coordinate(s) and for each $\phi \in M_A$ a notion of minimum measure is defined by the minimulty of the coordinate(s). In Theorem 3.2 we show that the coordinates of the minimum measures vary subharmonically in the interior of M_A. Using the above results we prove in Theorem 3.3 that the extension, $\hat{f}$, of $f \in C_R(X)$ to M_A by

$$\hat{f}(z) = \min_{\mu \in m_z} \int f d\mu$$

is subharmonic in the interior of M_A.

1. We begin our exposition by describing some basic properties of S_A. The measure spectrum, S_A, of a uniform algebra A is a compact Hausdorff space when it is endowed with the topology it inherits as a subspace of the dual space, $C(X)^*$, (given the weak-*topology). Let π be a restriction map of $C(X)^*$ onto A^*. Then $\pi(S_A) = M_A$. For each $\phi \in A^*$, $\pi^{-1}(\phi) = \{$the set of all representing measures for $\phi\}$. X can be regarded as a subspace of S_A by identifying each $x \in X$ with δ_x, the point mass measure concentrated at x. Each $f \in C_R(X)$ can be extended to $C_R(S_A)$ by the Hahn–Banach theorem.

Suppose A is a Dirichlet algebra, i.e., that ReA is dense in $C_R(X)$. Then every element of M_A has a unique representing measure. In this case $\pi : C(X)^* \to A^*$ induces a homeomorphism of S_A onto M_A, and S_A "inherits" the structures that M_A has. For example, consider the Gleason parts for A. Every Gleason part for a Dirichlet algebra is either trivial or an analytic disc. Suppose P is a nontrivial Gleason part of M_A. Then there is a continuous one to one function Φ from $\{|z| < 1\}$ onto P such that for all $f \in A$, $f \circ \Phi$ is holomorphic [**4**]. An analogous situation exists for S_A: Suppose $u \in C_R(X)$. Extend u to $\tilde{u}$ on S_A by defining $\tilde{u}(m) = \int_X u dm$. Then it is not hard to see that $\tilde{u} \circ \pi^{-1} \circ \Phi$ is harmonic. If P is an analytic disc $\pi^{-1}P$ is a "harmonic disc".

The above example suggests the possibility of extending the properties of M_A such as the Gleason part, the Shilov boundary, etc. to S_A when A is an arbitrary uniform algebra. In particular, the concept of S_A may prove to be useful in the problems of analytic structure in the maximal ideal space.

Using the method of Oka, J. Wermer proved the following theorem [**15**], which has become a key tool in the study of the analytic structure.

THEOREM 1.1. *Let A be a uniform algebra on X. Suppose $g \in A$ and let Ω be an open subset of $\mathbf{C}\backslash g(X)$. Choose $f \in A$. Define on Ω,*

$$Z_f(\zeta) = \sup\nolimits_{g^{-1}(\zeta)} |f|.$$

Then Z_f is logarithmically subharmonic.

We define a function H_f modeled after Z_f for $f \in C_R(X)$ and $g \in A$. In Theorem 1, which is our main theorem, we prove using the properties of S_A that the upper regularization of H_f is subharmonic. We shall need the following lemmas.

LEMMA 1.2. *Let Ω be an open subset of $\mathbf{C}\backslash g(X)$. Then for each $h \in$ ReA, $\zeta \mapsto \sup_{\varphi \in g^{-1}(\zeta)} h(\varphi)$ is subharmonic on Ω.*

Proof. $e^{h+ik} \in A$ for some $k \in C_R(X)$. By Theorem 1.1,

$$\sup\nolimits_{\varphi \in g^{-1}(\zeta)} \log|e^{h+ik}| = \sup\nolimits_{\varphi \in g^{-1}(\zeta)} h(\varphi)$$

is subharmonic.

LEMMA 1.3. *Using notation as above,*

$$\sup_{\varphi \in g^{-1}(\zeta)} \min_{\mu \in m_\varphi} \int_X f d\mu = \sup_{\substack{h \in \mathrm{Re}A \\ h \leqq f}} \left\{ \sup_{\varphi \in g^{-1}(\zeta)} h(\varphi) \right\}.$$

Proof. First, note that

$$\sup_{\varphi \in g^{-1}(\zeta)} \left\{ \sup_{\substack{h \in \mathrm{Re}A \\ h \leqq f}} \right\} = \sup_{\substack{h \in \mathrm{Re}A \\ h \leqq f}} \left\{ \sup_{\varphi \in g^{-1}(\zeta)} h(\varphi) \right\}.$$

It is well known [**3**, p. 32] that

$$\min_{\mu \in m_\varphi} \int_X f d\mu = \sup_{\substack{h \in \mathrm{Re}\, A \\ h \leqq f}} h(\varphi).$$

Hence Lemma 1.3 follows.

Let Ω be a domain in $\mathbf{C}$. Suppose u maps Ω into $[-\infty, \infty]$. The *upper regularization* $u^\uparrow$ of u is a function on Ω defined by

$$u^\uparrow(z_0) = \lim_{\delta \to 0} \sup_{\substack{|z - z_0| < \delta \\ z \in \Omega}} u(z).$$

Clearly u is upper semicontinous.

LEMMA 1.4. *For an arbitrary index set, I, let $\{u_\alpha\}_{\alpha \in I}$ be a bounded right directed set of subharmonic functions on Ω. Let $u = \sup_\alpha u_\alpha$. Then the upper regularization, $u^\uparrow$, of u is subharmonic on Ω.*

Proof. See [**6**, p. 68].

We are ready to prove our main theorem.

THEOREM 1.5. *Let A, X, and M_A be as in Theorem 1.1. For $f \in C_R(X)$ let $\hat{f}$ be the extension of f to M_A given by*

$$\hat{f}(\phi) = \min_{\mu \in M_\phi} \int_X f d\mu.$$

Fix $g \in A$ and let Ω be an open subset of $\mathbf{C} \backslash g(X)$. Define:

$$H_f(\zeta) = \max_{\phi \in g^{-1}(\zeta)} \hat{f}(\phi),\ \zeta \in \Omega.$$

The upper regularization of H_f is subharmonic.

Proof. Let $\{u_\alpha\}_{\alpha \in I}$ consist of functions in $\mathrm{Re}\, A$ with $u_\alpha \leqq f$ for all $\alpha \in I$. Set

$$U_\alpha(\zeta) = \sup_{\phi \in g^{-1}(\zeta)} u_\alpha(\phi).$$

By Corollary 1.2, U_α is subharmonic for each α. Let

$$\mathscr{F} = \{\max\{u_\alpha\}_{\alpha \in F} | F\text{: a finite subset of } I\}.$$

It is easy to see that $\mathscr{F}$ is right directed. Each element of $\mathscr{F}$, being the maximum of a finite number of subharmonic functions, is subharmonic. Note that $H_f = \sup_{F \subseteq I} \mathscr{F}$. By Lemma 1.4 the upper regularization, $H_f^\uparrow$, of H_f is subharmonic.

Although $H_f^\uparrow$ is subharmonic for arbitrary uniform algebras, H_f fails to be subharmonic in general. We exhibit an interesting example of a uniform algebra which gives rise to subharmonic H_f (without the upper regularization).

Example.

$$Y = \{z \in \mathbf{C} | \tfrac{1}{2} \leqq |z| \leqq 1\}$$

$$X = \partial Y \times [0, 1].$$

$A = \{g \in C(X) |$ for each t, $g(z, t)$ is a uniform limit of polynomials in z and $z^{-1}\}$.

Note $M_A = Y \times [0, 1]$. Choose $f \in C_R(X)$ and $f \geqq 0$. Let $g(z, t) = z$. Then

$$H_f(\zeta) = \max_{(z,t) \in g^{-1}(\zeta)} \min_{\mu \in m_{(z,t)}} \int_X f d\mu$$

is subharmonic on Ω, an open subset of $\mathbf{C} \backslash g(X)$.

Let $(z_0, t_0) \in M_A$. We shall show that every representing measure for (z_0, t_0) is supported on $\partial Y \times t_0$. Set $w(z, t) = 1 - |t - t_0|$. Then, $w \in A$ and $w(z, t_0) = 1$.

$$1 = w^n(z, t_0) = \int_{\partial Y \times I} |w|^n d\mu$$

$$= \int_{\partial Y \times t_0} |1 - |t - t_0||^n d\mu + \int_{\partial Y \times I \backslash \{t_0\}} |1 - |t - t_0||^n d\mu.$$

We have

$$\lim_{n \to \infty} \int_{\partial Y \times t_0} |1 - |t - t_0||^n d\mu = \mu(\partial Y \times t_0)$$

while

$$\lim_{n \to \infty} \int_{\partial Y \times I \backslash \{t_0\}} |1 - |t - t_0||^n d\mu = 0.$$

Hence, $\mu(\partial Y \times t_0) = 1 = \mu(\partial Y \times I) = \mu(X)$.

$$\hat{f}(z, t) = \min_{\mu \in m_{(z,t)}} \int_X f d\mu = \min_{\mu \in m_{(z,t)}} \int_{\partial Y \times t} f d\mu.$$

Denote the uniform algebra, $R(\partial Y)$, by B. Any continuous real valued function f on X can be expressed as

$$f(z, t) = \varphi(t) \cdot \{h(z) + \beta \log |z|\}$$

for some $h \in \overline{\operatorname{Re} B}$ and for some $\beta \in \mathbf{R}$. $\mu \in S_A$ representing (z, t) is also an element of S_B representing z. For simplicity of notation denote

$$h(z) + \beta \log |z| = v(z).$$

Then,

$$\hat{f}(z, t) = \varphi(t) \min_{\mu \in m_z} \int_{\partial Y} v(z) d\mu = \varphi(t) \hat{v}(z).$$

We shall show in Theorem 3.3 that $\hat{v}(z)$ is subharmonic. Note that

$$H_f(\zeta) = \begin{cases} [\max \varphi(t)] \cdot \hat{v}(z) & \text{if } \hat{v}(z) \geqq 0 \\ [\min \varphi(t)] \cdot \hat{v}(z) & \text{if } \hat{v}(z) < 0;\ (z, t) \in g^{-1}(\zeta). \end{cases}$$

Therefore,

$$H_f(\zeta) = \max \{\max \varphi(t) \cdot \hat{v}(z), \min \varphi(t)\hat{v}(z)\};\ (z, t) \in g^{-1}(\zeta)$$

and $H_f(\zeta)$ is subharmonic in Ω.

2. In this section we construct measure spectra for two specific algebras; the annulus algebra and the generalized annulus algebra with two holes. While the measure spectrum of a Dirichlet algebra is homeomorphic to the maximal ideal space the measure spectrum of a hypo-Dirichlet algebra may be regarded as a finite dimensional extension of the maximal ideal space. Our computation will be based on the following observations. Consider a domain $K \subseteq \mathbf{C}$, bounded by finitely many mutually disjoint smooth simple closed curves and let $\bar{K}$ be its closure. Denote by $A(\bar{K})$ the algebra of all functions continuous on $\bar{K}$ and analytic on K. Algebras of this type are prototypes of hypo-Dirichlet algebras. For such algebras it is well known [**13**], that if z_0 is an interior point of K, then, for all $f \in A(\bar{K})$

$$f(z_0) = \frac{1}{2\pi} \int_{\partial R} f \frac{\partial}{\partial n} G(z_0, \zeta) ds,$$

where: G is the Green's function for K singular at z_0; $\partial G/\partial n$ is its inner normal derivative on the boundary, and ds denotes the arc length. Since $\partial G/\partial n$ is always positive, $1/2\pi\ [\partial G(z_0, \zeta)/\partial n]ds$ is a representing measure for z_0. Any two representing measures for z_0 differ by an annihilating measure for $A(\bar{K})$. The annihilating measures for $A(\bar{K})$ are the holomorphic differentials on K which are real on the boundary of K [**1**]. The dimension of the space of annihilating measures is $N - 1$ if N is the number of components in the complements of K.

Example 1. Let X_1 be two circles, $\{|z| = 1\}$ and $\{|z| = \rho\}$, $\rho < 1$, in $\mathbf{C}$. Let A_1 be the subalgebra of $C(X_1)$ consisting of all functions which are uniform limits on X_1 of rational functions in z with poles off X_1. Then, the maximal ideal space, M_1 of A_1 is: $M_1 = \{\rho \leqq |z| \leqq 1\}$. Each $f \in A_1$ has a unique extension which is analytic in the interior of M_1. It is well known [**13**, p. 59] that any representing measure for z in the interior of M_1 is given by

$$\frac{1}{2\pi}\left\{\frac{\partial G}{\partial n}(z, \zeta) + aT(\zeta)\right\}ds$$

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where $T(\zeta)$ is an annihilating measure and a is a real number for which

$$\frac{\partial G}{\partial n}(z, \zeta) + aT(\zeta) \geqq 0$$

for all $\zeta \in X_1$. The set of such a's forms an interval. Denote this interval by $[a_m(z), a_M(z)]$. Using Cauchy's theorem T can be computed to be

$$T = \begin{cases} 1 & \text{on } |\zeta| = 1 \\ -\dfrac{1}{\rho} & \text{on } |\zeta| = \rho. \end{cases}$$

If $z \in X_1$ then the point mass measure, δ_z, is the only measure representing z. Thus, we have

$$S_A = \bigcup_{z \in M_1{}^0} \left\{ \frac{1}{2\pi} \left[\frac{\partial G}{\partial n}(z, \zeta) + aT(\zeta) \right] dS | a_m(z) \leqq a \leqq a_M(z) \right\} \cup \bigcup_{z \in X_1} S_z.$$

S_{A_1} is, therefore, a solid torus.

Example 2. Consider three circles in $\mathbf{C}$: $\Gamma_0 = \{|z| = 1\}$, $\Gamma_1 = \{|z - \frac{1}{2}| = \rho\}$; $\Gamma_2 = \{|z + \frac{1}{2}| = \rho\}$; $0 < \rho < \frac{1}{4}$. Let $X_2 = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ and let $A_2 = R(X_2)$. I.e., A_2 consists of all functions which are uniform limits on X_2 of rational functions in z with poles off X_2. Then, the maximal ideal space, M_2, of A_2 is

$$M_2 = \{z \in \mathbf{C} | \, |z| \leqq 1\} \backslash [\{|z + \tfrac{1}{2}| < \rho\} \cup \{|z - \tfrac{1}{2}| < \rho\}].$$

Let z be an interior point of M_2. By the preceding remarks any representing measure for z is expressed as

$$(4) \quad \frac{1}{2\pi} \left[\frac{\partial G}{\partial n}(z, \zeta) ds + aT_u + bT_v \right] ds$$

where T_u and T_v are linearly independent annihilating measures for A_2 and a, b are real numbers such that (4) is nonnegative. The space of annihilating measures for A_2 can be obtained as follows. Set

$$T_u = -{}^*du + idu \text{ and } T_v = -{}^*dv + idv$$

where u and v are the harmonic functions on M_2 with the boundary values:

$$u \equiv \begin{cases} 0 \text{ on } \Gamma_1 \\ 1 \text{ on } \Gamma_0 \cup \Gamma_2 \end{cases} \quad \text{and} \quad v \equiv \begin{cases} 0 \text{ on } \Gamma_2 \\ 1 \text{ on } \Gamma_0 \cup \Gamma_1. \end{cases}$$

Clearly, T_u and T_v are holomorphic differentials which are real on X_2. It is not hard to see that T_u and T_v are linearly independent. By the

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preceding remark T_u and T_v generate the space of annihilating measures for A_2. Denote

$$K_z = \left\{ (a, b) \in \mathbf{R}^2 \middle| \frac{\partial G}{\partial n}(z, \zeta)ds + aT_u + bT_v \geqq 0 \right\}.$$

S_{A_2} is, then, given by:

$$S_{A_2} = \bigcup_{z \in MA_2^0} \left\{ \frac{1}{2\pi} \left[\frac{\partial G}{\partial n}(z, \zeta)ds + aT_u + bT_v \middle| (a, b) \in K_z \right] \right\} \cup \bigcup_{z \in X_2} \delta_z.$$

Thus, S_{A_2} can be identified with a subset of $\mathbf{C} \times \mathbf{R}^2$.

More generally, if A is a generalized annulus algebra with n holes, then S_A can be realized as a subset of $\mathbf{C} \times \mathbf{R}^n$.

3. In this section we shall examine various subharmonicity properties associated with S_{A_1} and S_{A_2}.

Suppose μ is a representing measure for z in the interior of M_1 (recall that M_1 is the maximal ideal space of the annulus algebra, A_1.) Then, there is a real number a such that

$$\mu = \frac{1}{2\pi} \left[\frac{\partial G}{\partial n}(z, \zeta) + aT(\zeta) \right] ds.$$

We call a the *coordinate* of μ (with respect to T). The set of all coordinates of the representing measures for z is the interval $[a_m(z), a_M(z)]$. The representing measure for z with the coordinate $a_m(z)$ is called the *minimum measure* for z and the representing measure with $a_M(z)$ is called the *maximum measure* for z. We shall show that $a_m(z)$ and $a_M(z)$ are subharmonic and superharmonic functions of z respectively. First we need the following characterizations of $a_m(z)$ and $a_M(z)$.

LEMMA 3.1. *We use notation as above. For* $z \in M^0$,

$$a_m(z) = -\min_{|\zeta|=1} \frac{\partial G}{\partial n}(z, \zeta),$$

$$a_M(z) = \rho \min_{|\zeta|=\rho} \frac{\partial G}{\partial n}(z, \zeta).$$

Proof. For simplicity of notation we set

$$\alpha_z = \min_{|\zeta|=1} \frac{\partial G}{\partial n}(z, \zeta) \quad \text{and}$$

$$\beta_z = \min_{|\zeta|=\rho} \frac{\partial G}{\partial n}(z, \zeta).$$

Since the inner normal derivative of the Green's function is always nonnegative we have $\alpha_z \geqq 0$ and $\beta_z \geqq 0$. Note that

$$aT(\zeta) = \begin{cases} -a/\rho & \text{on } |\zeta| = \rho \\ a & \text{on } |\zeta| = 1. \end{cases}$$

(i) Consider the case when $a \geqq 0$; then $aT(\zeta) \leqq 0$ on $\{|\zeta| = \rho\}$, and we need $a/\rho < \beta_z$ in order to satisfy the nonnegativity in (3). Consequently, $a \leqq \rho\beta_z$.

(ii) Assume $a < 0$. Then $aT(\zeta) \leqq 0$ on $\{|\zeta| = 1\}$. By (3) we have $-\alpha_z < a \leqq \rho\beta_z$. So, we conclude that

$$a_M(z) = \rho\beta_z = \rho \cdot \left\{\min_{|\xi|=\rho} \frac{\partial G}{\partial n}(z, \zeta)\right\}$$

$$a_m(z) = -\alpha_z = -\min_{|\zeta|=1} \frac{\partial G}{\partial n}(z, \zeta).$$

THEOREM 3.2. *We use notation as above. The function, $z \to a_m(z)$ is subharmonic in M_1^0.*

Proof. In view of Lemma 3.1 it suffices to show that $\partial G(z, \zeta)/\partial n$ is a harmonic function of z for all $\zeta \in X_1$.

Recall that the Green's function $G(z, \zeta)$ is harmonic in each variable on $M_1^0 \times M_1^0\backslash$diagonal. Fix $z \in M_1^0$. Define h_z on M_1^0 by

$$h_z(\zeta) = G(z, \zeta).$$

h_z is harmonic on $M_1^0\backslash\{z\}$ and $h_z \equiv 0$ on X_1 by the definition of G. So, h_z extends, by the reflection principle, to a harmonic function, H_z, defined on a region $B \supset M_1$. Let Δ be a disc contained in M_1. Define on $\Delta \times B\backslash$diagonal,

$$\tilde{G}(z, \zeta) = H_z(\zeta).$$

Note that $\tilde{G}$ is harmonic in each variable. We shall prove that $\partial\tilde{G}(z, \zeta)/\partial n$ is a harmonic function of z for any $\zeta \in X_1$ by showing

$$\Delta_z\left\{\frac{\partial \tilde{G}}{\partial n}(z, \zeta)\right\} = 0.$$

Let $\{\eta_n\}$ be a sequence of points in M_1^0 converging to ζ along the normal.

$$\Delta_z\left\{\frac{\partial \tilde{G}}{\partial n}(z, \zeta)\right\} = \lim_{\eta_n \to \xi} \Delta_z\left\{\frac{\tilde{G}(z, \eta_n) - \tilde{G}(z, \zeta)}{|\xi - \eta_n|}\right\}$$

$$= \lim_{\eta_n \to \zeta} \Delta_z\left\{\frac{G(z, \eta_n) - 0}{|\xi - \eta_n|}\right\} = 0,$$

since $\tilde{G}(z, \eta_n) = G(z, \eta_n)$ and G is harmonic in z. Hence both α_z and β_z

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are subharmonic. This concludes the proof of Theorem 3.2. We have also proved:

COROLLARY 3.3. *$a_M(z)$ is superharmonic in M_1^0.*

Theorem 3.4 is an application of Theorem 3.2 and Corollary 3.3.

THEOREM 3.4. *Suppose A_1 is the annulus algebra defined on X_1 as before. For each $f \in C_R(X_1)$, the extension, $\hat{f}$, of f to M_1 defined by*

$$\hat{f}(z) = \min_{\mu \in m_z} \int_{X_1} f d\mu$$

is subharmonic in the interior of M_1.

Proof. Note that f may be expressed as

$$f(\zeta) = g(\zeta) + \beta \log |\zeta|$$

for some $g \in \overline{\operatorname{Re} A_1}$ and for some $\beta \in \mathbf{R}$. Choose $z \in M_1$. If μ is a representing measure for z, then there is a real number, a, such that

$$\mu = \frac{1}{2\pi}\left[\frac{\partial G}{\partial n}(z, \zeta) + aT(\zeta)\right] ds.$$

Each $f \in C_R(X_1)$ can be extended to $C_R(S_{A_1})$ by $\tilde{f}(\mu) = \int_{X_1} f d\mu$. We claim

$$\tilde{f}(\mu) = f(z) - \beta a \log \zeta.$$

Proof of the claim. Note that

$$\tilde{f}(\mu) = \int_{X_1} g(\zeta) d\mu + \int_{X_1} \beta \log |\zeta| d\mu.$$

From $g \in \overline{\operatorname{Re} A_1}$, we get $\int_{X_1} g d\mu = g(z)$. Since $1/2\pi \ \partial G(z, \zeta)/\partial n \, ds$ is a logmodular measure we have

$$\frac{1}{2\pi}\int_{X_1} \beta \log |\zeta| \frac{\partial G}{\partial n}(z, \zeta) ds = \beta \log |z|.$$

On the other hand,

$$\frac{1}{2\pi}\int_{X_1} \beta \log |\zeta| a T(\zeta) ds = \frac{a}{2\pi}\int_{|\zeta|=1} \beta \log |\zeta| ds$$
$$+ \frac{a}{2\pi}\int_{|\zeta|=\rho} \beta \log |\zeta| \left(-\frac{1}{\rho}\right) \frac{\rho d\zeta}{\zeta} = 0 - \beta \cdot a \log \rho.$$

We conclude that

$$\tilde{f}(\mu) = f(z) - \beta \cdot a \log \rho.$$

So,

$$\hat{f}(z) = \min_{\mu \in m_z} \int f d\mu = \min_a \{ f(z) - \beta a \log \rho \}.$$

Case i. $\beta \geqq 0$. In this case $\min_{\mu \in m_z} \int f d\mu$ is attained when μ is the minimum measure for z; i.e.,

$$\min_{\mu \in m_z} \int f d\mu = f(z) - \beta a_m(z) \log \rho.$$

By Theorem 3.2, $z \to a_m(z)$ is subharmonic. $f(z)$ is harmonic and $\log \rho < 0$. So, $z \to \min_{\mu \in m_z} \int f d\mu$ is subharmonic.

Case ii. $\beta < 0$. In this case $\min_{\mu \in m_z} \int f d\mu$ is attained when μ is the maximum measure. By an argument similar to the proof of Case i and Corollary 3.3, we have the desired subharmonicity. This concludes the proof of Theorem 3.4.

Now let A_2 be the generalized annulus algebra of Example 2 in Section 2. Suppose z is an interior point of M_2 and $\mu \in S_{A_2}$ represents z. Then, there exists $(a, b) \in K_z$ satisfying

$$\mu = \frac{1}{2\pi} \frac{\partial G}{\partial n} (z, \zeta) ds + aT_u + bT_v.$$

Define $a_m(z)$ to be the minimum value of a such that there exists b with $(a, b) \in K_z$. The value of b such that $(a_m(z), b) \in K_z$ is uniquely determined for each z. We denote this unique value by $b_m(z)$.

THEOREM 3.5. *We use notation as above. $a_m(z)$ is subharmonic in the interior of M_2.*

Proof. First we show that $a_m(z)$ has the submean value property. Fix $\zeta \in X_2$. Let $z_0 \in M_2{}^0$ and let $\bar{\Delta}(z_0, \delta)$ be the closed disc centered at z_0 with radius δ. Suppose $\bar{\Delta}(z_0, \delta) \subseteq M_A{}^0$. For all z with $|z - z_0| = \delta$, we have:

$$\frac{\partial G}{\partial n} (z_0 + \delta e^{i\theta}, \zeta) ds + a_m(z) T_u + b_m(z) T_v \geqq 0.$$

Hence,

$$(5) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial G}{\partial n} (z_0 + \delta e^{i\theta}, \zeta) d\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} a_m(z_0 + \delta e^{i\theta}) T_u d\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} b_m(z_0 + \delta e^{i\theta}) T_v d\theta \geqq 0.$$

Let

$$\overline{a_m(z)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} a_m(z_0 + \delta e^{i\theta}) T_u d\theta;$$

the average of $a_m(z)$. Then (5) becomes

$$\frac{\partial G}{\partial n} (z_0, \zeta) + \overline{a_m(z)} T_u + \overline{b_m(z)} T_u \geqq 0$$

which shows that $\overline{(a_m(z)}, \overline{b_m(z))} \in K_{z_0}$. Hence,

$$a_m(z_0) \leqq \overline{a_m(z)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} a_m(z_0 + \delta e^{i\theta})d\theta.$$

It remains to be shown that $a_m(z)$ is upper semicontinuous. Note that $a_m(z) < 0$ for all z. Let z_0 be an interior point of M_2. Choose a disc, $\overline{\Delta(z_0, \delta)} \subset M_2{}^0$. Denote

$$c = \max_{\bar{\Delta}\times X_2} \frac{\partial G}{\partial n} > 0.$$

Given any $\epsilon > 0$, set

$$\epsilon_1 = -c \cdot \epsilon/(a_m(z_0)) > 0.$$

Without loss of generality, assume $\epsilon_1/c < 1$. By the continuity of $\partial G(z, \zeta)/\partial n$ on $\bar{\Delta} \times X_2$ there exists $\delta' < \delta$ such that for all z with $|z - z_0| < \delta'$ and for all $\zeta \in X_2$,

$$\frac{\partial G}{\partial n}(z_0, \zeta) \leqq \frac{\partial G}{\partial n}(z, \zeta) + \epsilon_1$$

from which we get,

$$\text{(6)} \quad \frac{\partial G}{\partial n}(z, \zeta) \geqq \frac{\partial G}{\partial n}(z_0, \zeta)\left(1 - \frac{\epsilon_1}{c}\right).$$

Choose b such that

$$\frac{\partial G}{\partial n}(z_0, \zeta) + a_m(z_0)T_u(\zeta) + bT_v(\zeta) \geqq 0 \quad \text{for all } \zeta \in X.$$

Then,

$$\text{(7)} \quad \left(1 - \frac{\epsilon_1}{c}\right)\frac{\partial G}{\partial n}(z_0, \zeta)ds + \left(1 - \frac{\epsilon_1}{c}\right)a_m(z_0)T_u + \left(1 - \frac{\epsilon_1}{c}\right)bT_v \geqq 0.$$

From (6) and (7)

$$\frac{\partial G}{\partial n}(z, \zeta) + \left(1 - \frac{\epsilon_1}{c}\right)a_m(z_0)T_u + \left(1 - \frac{\epsilon_1}{c}\right)bT_v \geqq 0.$$

Hence,

$$\left(1 - \frac{\epsilon_1}{c}\right)a_m(z_0) \geqq a_m(z)$$

by the definition of $a_m(z)$. Consequently,

$$a_m(z_0) \geqq a_m(z) - \epsilon.$$

Theorem 3.5 is proved.

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