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Spaces of Lorentz Multipliers

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Abstract. We study when the spaces of Lorentz multipliers from $L^{p,t} \to L^{p,s}$ are distinct. Our main interest is the case when $s < t$, the Lorentz-improving multipliers. We prove, for example, that the space of multipliers which map $L^{p,t} \to L^{p,s}$ is different from those mapping $L^{r,v} \to L^{r,u}$ if either $r = p$ or p' and $1/s - 1/t \neq 1/u - 1/v$, or $r \neq p$ or p'. These results are obtained by making careful estimates of the Lorentz multiplier norms of certain linear combinations of Fejer or Dirichlet kernels. For the case when the first indices are different the linear combination we analyze is in the spirit of a Rudin-Shapiro polynomial.

1 Introduction

The Lorentz spaces, $L^{p,q}(G)$, for G an infinite, compact, abelian group, are Banach spaces which generalize the classical spaces $L^p(G)$, and are intermediate to them in the sense that whenever $1 \leq q \leq p \leq r \leq \infty$ then

$$\bigcup_{t>p} L^t(G) \subseteq L^{p,q}(G) \subseteq L^p(G) = L^{p,p}(G) \subseteq L^{p,r}(G) \subseteq \bigcap_{s<p} L^s(G).$$

By a *Lorentz multiplier*, or convolution operator, we mean a bounded linear map from $L^{p,q}(G)$ to $L^{r,s}(G)$, for some p, q, r, s, which commutes with translation. The action of convolution by a measure is an example of a multiplier from $L^{p,x}(G)$ to $L^{p,x}(G)$, while convolution operators of strong (weak) type (p, p) are Lorentz multipliers from $L^{p,p}$ to $L^{p,p}$ (or $L^{p,\infty}$, respectively). We will denote by $M(p, q; r, s)$ the space of multipliers from $L^{p,q}(G)$ to $L^{r,s}(G)$ (or simply $M(p; r)$ if $p = q$ and $r = s$).

Many authors have considered the problem of which Lorentz multiplier spaces are included, or not included, within others. For example, in [5] Gaudry, improving upon work of Price [12], showed that $M(p; q) \subsetneq M(r; s)$ if $1 < p < r < 2$ and $1/p - 1/q = 1/r - 1/s$, while in [15] Zafran proved that for the circle group, (and certain other locally compact abelian groups) and each $1 < p < 2$, there existed a multiplier of weak type (p, p) which was not of strong type (p, p). This result was improved by Cowling and Fournier in [4] who showed more generally that if $p \neq 1, 2, \infty$ then $M(p, 1; p, \infty) \supsetneq M(p, q; p, r)$ if $r < \infty$ or $q > 1$. In their extensive investigation they also proved strict inclusions such as

$$M(p, q; p, r) \subsetneq M(p, q; p, t)$$

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when $p \neq 1, 2, \infty$, $q < 2$ and $1 \leq r < t \leq q'$ (here q' is the conjugate index to q).

Cowling and Fournier mainly derived their results by estimating the norms of certain operators in $M(p, q; p, r)$ where $r \geq q$. In contrast, our primary focus is on the spaces $M(p, q; p, r)$ with $r < q$. The operators in these spaces are also known as Lorentz-improving multipliers, a generalization of the notion of L^p-improving multipliers. One reason for the current interest in Lorentz multipliers is that they arise in the study of the smoothing property of measures on flat curves (*cf.* [1]).

One of our main results is that if $1 < p < \infty$ and $0 < 1/s - 1/t \neq 1/r - 1/q$ then $M(p, t; p, s) \neq M(p_1, q; p_1, r)$ if p_1 is either p or p'. In particular, $M(p, q; p, r) \subsetneq M(p, q; p, t)$ whenever $1 \leq r < \min(t, q)$. We also prove that if $1 < p, r < \infty$ and $r \neq p, p'$, then $M(p, t; p, s) \neq M(r, v; r, u)$ when $s \leq t$, $u \leq v$.

Our method is constructive: for instance, for any given $\tau > 0$ we produce examples of integrable functions which belong to $M(p, t; p, s)$ for all $1 < p < \infty$ and $1/s - 1/t = \tau$, but do not belong to any space $M(p, q; p, r)$ for $1/r - 1/q > \tau$. These examples are formed from linear combinations of Fejer or Dirichlet kernels. We also produce examples to prove the non-equality of multiplier spaces when the first index is different. These latter examples are analogues of Rudin-Shapiro polynomials or measures, a key technique used in both [5] and [4]. Norm estimates are found first for our operators on the circle group in Section 3, and then for groups whose duals have infinitely many elements of finite order in Section 4. The results for arbitrary infinite, compact, abelian groups are obtained in section 5 by essentially reducing the problem to one of the first two cases.

We begin in Section 2 by reviewing facts about Lorentz spaces and Lorentz-improving multipliers.

2 Lorentz Spaces and Lorentz-Improving Multipliers

2.1 Properties of Lorentz Spaces

Throughout this paper m will denote normalized Haar measure on G and c will denote constants which may vary from one line to another.

We will briefly review the basic properties of Lorentz spaces. Most of these definitions and facts can be found in either [10] or [8].

Given a measurable function f on G, the distribution function of f is defined by

$$m_f(y) \equiv m\{x \in G : |f(x)| > y\} \quad \text{for } y \geq 0$$

and the non-increasing rearrangement of f is the function f^* defined by

$$f^*(t) \equiv \inf\{y > 0 : m_f(y) \leq t\} \quad \text{for } t \geq 0.$$

The Lorentz space $L^{p,q}(G)$ is the space of functions f for which $\|f\|^*_{p,q} < \infty$ where

$$\|f\|^*_{p,q} \equiv \begin{cases} \left(\frac{q}{p} \int_0^1 \left(x^{1/p} f^*(x) \right)^q \frac{dx}{x} \right)^{1/q} & \text{if } 1 \leq p, q < \infty \\ \sup_x x^{1/p} f^*(x) & \text{if } 1 \leq p \leq \infty, q = \infty. \end{cases}$$

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As f^* and f have the same distribution function, it follows that $\|f\|^*_{p,p} = \|f\|_p$, so the Lorentz space $L^{p,p}(G) = L^p(G)$.

Because the function $\|\cdot\|^*_{p,q}$ is not a norm it is useful to define the function $f^{**}(x) = \frac{1}{x}\int_0^x f^*(s)\,ds$ and the norm

$$\|f\|_{(p,q)} \equiv \begin{cases} \left(\int_0^1 \left(x^{1/p} f^{**}(x)\right)^q \frac{dx}{x}\right)^{1/q} & \text{if } 1 \le p, q < \infty \\ \sup_x x^{1/p} f^{**}(x) & \text{if } 1 \le p \le \infty, q = \infty. \end{cases}$$

If $p = q = 1$, $p = q = \infty$ or $1 < p < \infty$ and $1 \le q \le \infty$, then $L^{p,q}(G)$ is a Banach space with dual space $L^{p',q'}(G)$ (provided $p, q \ne \infty$) where, as usual, $1/p + 1/p' = 1$. The space $M(p,q;p,r)$ is a Banach space when given the operator norm

$$\|T\|_{M(p,q;p,r)} \equiv \sup\{\|T(f)\|_{(p,r)} : \|f\|_{(p,q)} \le 1\}.$$

By duality, $M(p,q;p,r) = M(p',r';p',q')$ [7].

The norm and quasi-norm are comparable. Indeed,

$$\left(\frac{p}{q}\right)^{1/q} \|f\|^*_{p,q} \le \|f\|_{(p,q)} \le p' \left(\frac{p}{q}\right)^{1/q} \|f\|^*_{p,q}$$

(where $(p/q)^{1/q} = 1$ if $q = \infty$). If $r > q$ then

$$\|f\|^*_{p,r} \le \|f\|^*_{p,q} \quad \text{and} \quad \|f\|_{(p,r)} \le \left(\frac{q}{p}\right)^{1/q - 1/r} \|f\|_{(p,q)},$$

while if $1 < p_1 < p_2 < \infty$ and $1 \le q < \infty$ then

$$\|f\|^*_{p_1,q} \le \left(\frac{p_2}{p_2 - p_1}\right)^{1/q} \|f\|^*_{p_2,\infty}.$$

From these inequalities it is clear that if we define a total ordering on $(1,\infty) \times [1,\infty]$ by $(p,t) > (r,s)$ if $p > r$ or $p = r$ and $t < s$ then

$$L^{p,t}(G) \subseteq L^{r,s}(G) \quad \text{if } (p,t) > (r,s).$$

Moreover, this inclusion is strict. It follows trivially that we have the inclusions

$$M(p,t;q,s) \subseteq M(r,u;w,v) \quad \text{if } (p,t) \le (r,u) \quad \text{and} \quad (q,s) \ge (w,v).$$

Furthermore, the point mass measure at the identity is not in $M(p,t;q,s)$ if $(p,t) < (q,s)$ and this implies, in particular, that $M(p,t;p,s) \ne M(p;p)$ if $t > s$.

There is one other inequality relating Lorentz norms with different first indices which we will need.

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Proposition 2.1 *Suppose* $1 < p_1 < p_2 < \infty$ *and* $1 \le q_1 < q_2 < \infty$. *Then*

$$\|f\|^*_{p_1,q_1} \le \left(\frac{q_1}{p_1}\right)^{1/q_1} \left(\frac{p_2}{q_2}\right)^{1/q_2} \left(\frac{p_1p_2(q_2-q_1)}{q_1q_2(p_2-p_1)}\right)^{1/q_1-1/q_2} \|f\|^*_{p_2,q_2}.$$

Proof As $q_2/q_1 > 1$, Hölder's inequality implies

$$\begin{aligned}\|f\|^*_{p_1,q_1} &= \left(\frac{q_1}{p_1}\int_0^1 \left(x^{1/p_1} f^*(x)\right)^{q_1}\frac{dx}{x}\right)^{1/q_1}\\ &\le \left(\frac{q_1}{p_1}\right)^{1/q_1}\left(\int_0^1 \left(x^{1/p_2} f^*(x)\right)^{q_2}\frac{dx}{x}\right)^{1/q_2}\left(\int_0^1 x^{\frac{q_1q_2(p_2-p_1)}{p_1p_2(q_2-q_1)}}\frac{dx}{x}\right)^{1/q_1-1/q_2}.\end{aligned}$$

Because $q_1q_2(p_2-p_1)/p_1p_2(q_2-q_1) > 0$,

$$\int_0^1 x^{\frac{q_1q_2(p_2-p_1)}{p_1p_2(q_2-q_1)}}\frac{dx}{x} = \frac{p_1p_2(q_2-q_1)}{q_1q_2(p_2-p_1)}.$$

Using the definition of $\|f\|^*_{p_2,q_2}$ completes the proof. ■

As with the classical L^p spaces, the trigonometric polynomials are dense in $L^{p,q}(G)$ whenever $1 < p < \infty$ and $q < \infty$. Also, as the spaces $L^{p,q}(G)$ for $1 < p < \infty, q < \infty$ are homogeneous Banach spaces, any bounded approximate identity for L^1, say $\{k_\alpha\}$, satisfies

$$\|k_\alpha * f\|_{(p,q)} \to \|f\|_{(p,q)}.$$

Two very important theorems for the study of the spaces $M(p,q;r,s)$ are the weak and strong interpolation theorems which are similar to the Riesz-Thorin interpolation theorem for the L^p spaces. We will state them here for the convenience of the reader. Proofs may be found in [10].

Notation Given p_j, q_j, r_j, s_j and $0 \le \theta \le 1$ define $p_\theta, q_\theta, r_\theta, s_\theta$ by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$
$$\frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} \qquad \frac{1}{s_\theta} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}.$$

Weak Interpolation Theorem Suppose T is quasi-linear and for $p_0 < p_1$, $r_0 \ne r_1$ we have

$$\|Tf\|^*_{r_i,s_i} \le B_i\|f\|^*_{p_i,q_i} \quad \text{for all } f \in L^{p_i,q_i}, \quad i = 0, 1.$$

If $q \le s$ then

$$\|Tf\|^*_{r_\theta,s} \le B_\theta\|f\|^*_{p_\theta,q}.$$

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Strong Interpolation Theorem Suppose T is a sublinear operator and

$$\|Tf\|^*_{r_i,s_i} \le B_i \|f\|^*_{p_i,q_i} \quad \text{for all } f \in L^{p_i,q_i}, \quad i = 0, 1.$$

Then

$$\|Tf\|^*_{r_\theta,s_\theta} \le BB_0^{1-\theta} B_1^\theta \|f\|^*_{p_\theta,q_\theta}.$$

An immediate corollary of the weak interpolation theorem is that convolution by a measure on G is an operator in $M(p, x; p, x)$ for all $1 < p < \infty$.

2.2 Lorentz-Improving Multipliers

Multipliers in $M(p, t; q, s)$ with $(p, t) < (q, s)$ are interesting since not all measures, or even all L^1 functions, belong to these spaces (see [6] and [8]).

When there exists some $p > 2$ such that the multiplier T maps $L^2(G)$ to $L^p(G)$ then T is said to be L^p*-improving*. When there exists some $1 < p < \infty$ and $s < t$ such that T maps $L^{p,t}(G)$ to $L^{p,s}(G)$ then T is called *Lorentz-improving*. L^p-improving multipliers have been studied by many authors. Examples include all L^q functions for $q > 1$, Riesz products and some singular measures on curves (*cf.* [13], [3], [9] and [2], as well as the references cited therein).

Lorentz-improving multipliers are a refinement of the notion of L^p-improving multipliers; indeed, all L^p-improving multipliers are Lorentz-improving but not the converse. In fact, there exist integrable functions which are not L^p-improving, but belong to $M(p, \infty; p, 1)$ for all $1 < p < \infty$ [8].

The inclusions of the Lorentz spaces imply that $M(r, v; w, u) \subseteq M(r + \varepsilon; w - \varepsilon)$ for all $\varepsilon > 0$, hence if $r < w$ then the operators in $M(r, v; w, u)$ are L^p-improving multipliers. Together with the previous remark this implies that $M(r, v; w, u) \ne M(p, t; p, s)$ for any $s < t$. Hence our interest is in determining when the spaces $M(r, v; r, u)$ and $M(p, t; p, s)$ are distinct.

3 Circle Group

In this section we will prove non-equalities of classes of Lorentz space multipliers for the case when G is the circle group T. But first we obtain a result which is valid for all compact, abelian groups.

Proposition 3.1 *Let G be any compact, abelian group and let P be a trigonometric polynomial. Suppose that for all $1 < p < \infty$ there are constants $A(p)$ and $B(p)$ such that*

$$\|P\|^*_{p',1} \le A(p) |\operatorname{supp} \widehat{P}|^{1/p}$$

and $P\colon L^p \to L^p$ has multiplier norm at most $B(p)$. Then for any $1 \le s \le t \le \infty$ there is a constant $c = c(p)$ such that

$$\|P\|_{M(p,t;p,s)} \le c(\log |\operatorname{supp} \widehat{P}|)^{1/s-1/t}.$$

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Remark 3.1 Of course, the interest here is not that the polynomials of this proposition are $L^{p,t} \to L^{p,s}$ multipliers, as all polynomials are, but rather the estimates on the magnitude of their multiplier norms in terms of the size of the support of the Fourier transform.

Proof First assume $t < \infty$ and let $f \in L^{p,t}$. We want to use Proposition 2.1 with $p_1 = p$, $q_1 = s$, $p_2 > p$ defined by the equation

$$\frac{1}{p} - \frac{1}{p_2} = \frac{1}{\log|\operatorname{supp}\widehat{P}|},$$

and $q_2 = tp_2/p$. Notice the proposition does indeed apply since $1 < p < p_2 < \infty$ and $1 \le s = q_1 \le t < q_2 < \infty$, and hence yields the bound

(3.1)

$$\begin{aligned}\|P * f\|^*_{p,s} &\le \left(\frac{s}{p}\right)^{1/s}\left(\frac{p_2}{q_2}\right)^{1/q_2}\left(\frac{pp_2(q_2-s)}{sq_2(p_2-p)}\right)^{1/s-1/q_2}\|P * f\|^*_{p_2,q_2}\\ &\le \left(\frac{s}{p}\right)^{1/s}\left(\frac{p_2}{q_2}\right)^{1/q_2}\left(\frac{1}{s}-\frac{1}{q_2}\right)^{1/s-1/q_2}\left(\frac{1}{\frac{1}{p}-\frac{1}{p_2}}\right)^{1/s-1/q_2}\|P * f\|^*_{p_2,q_2}.\end{aligned}$$

Now,

$$\frac{1}{s} - \frac{1}{q_2} = \frac{1}{s} - \frac{1}{t} + \frac{p}{t\log|\operatorname{supp}\widehat{P}|}$$

hence

$$\left(\frac{1}{\frac{1}{p}-\frac{1}{p_2}}\right)^{1/s-1/q_2} \le (\log|\operatorname{supp}\widehat{P}|)^{1/s-1/t}e^{p/t}.$$

Since $(\frac{1}{s} - \frac{1}{q_2}) \le 1$, we may simplify (3.1) to

$$\|P * f\|^*_{p,s} \le \left(\frac{s}{p}\right)^{1/s}\left(\frac{p_2}{q_2}\right)^{1/q_2}(\log|\operatorname{supp}\widehat{P}|)^{1/s-1/t}e^{p/t}\|P * f\|^*_{p_2,q_2}.$$

Take $\alpha = p/p_2$. Since $\big((P * f)^*\big)^{\alpha} = \big((P * f)^{\alpha}\big)^*$, $\alpha p_2 = p$ and $\alpha q_2 = t$, it is a routine calculation to show

$$\|P * f\|^*_{p_2,q_2} \le (\sup|P * f|)^{1-\alpha}(\|P * f\|^*_{p,t})^{\alpha}.$$

It follows from [10] and the hypotheses of the proposition that

$$\sup|P * f| \le p'\|P\|^*_{p',1}\,\|f\|^*_{p,\infty} \le p'A(p)|\operatorname{supp}\widehat{P}|^{1/p}\,\|f\|^*_{p,t}.$$

The assumption that $\|P\|_{M(r;r)} \le B(r)$ for all r, together with the weak interpolation theorem, ensures that $\|P * f\|^*_{p,t} \le B_1(p)\|f\|^*_{p,t}$ for all p and t. Thus

$$\|P * f\|^*_{p_2,q_2} \le c(p)|\operatorname{supp}\widehat{P}|^{(1-\alpha)/p}\,\|f\|^*_{p,t}$$

for some constant $c(p)$. But $(1-\alpha)/p = (\log|\operatorname{supp}\widehat{P}|)^{-1}$, so $|\operatorname{supp}\widehat{P}|^{(1-\alpha)/p}$ is bounded. This means

$$\|P * f\|^*_{p_2,q_2} \le c(p)\|f\|^*_{p,t}.$$

Returning to (3.1) we see this gives

$$\begin{aligned}\|P * f\|^*_{p,s} &\le c(p)\left(\frac{p_2}{q_2}\right)^{1/q_2}(s)^{1/s}(\log|\operatorname{supp}\widehat{P}|)^{1/s-1/t}\|f\|^*_{p,t}\\ &\le c(p)(\log|\operatorname{supp}\widehat{P}|)^{1/s-1/t}\|f\|^*_{p,t}\end{aligned}$$

with the finally inequality resulting from the observation that $s^{1/s} \le 1$ and $p_2/q_2 \le p$. This proves that if $1 < p < \infty$ and $s \le t < \infty$ then

$$\|P\|_{M(p,t;p,s)} \le c(p)(\log|\operatorname{supp}\widehat{P}|)^{1/s-1/t}.$$

In particular, if $s \ne 1$ a duality argument proves

$$\|P\|_{M(p,\infty;p,s)} = \|P\|_{M(p',s';p',1)} \le c(p)(\log|\operatorname{supp}\widehat{P}|)^{1-1/s'} = c(p)(\log|\operatorname{supp}\widehat{P}|)^{1/s}.$$

It only remains to consider the case $t = \infty$ and $s = 1$. For this, note that it is shown in [8] that $\|f\|_{(p,1)} = \lim_{s_n\to 1}\|f\|_{(p,s_n)}$ and therefore

$$\begin{aligned}\|P * f\|_{(p,1)} &\le \limsup_{s_n\to 1}\|P\|_{M(p,\infty;p,s_n)}\|f\|_{(p,\infty)}\\ &\le c(p)(\log|\operatorname{supp}\widehat{P}|)\|f\|_{(p,\infty)}\end{aligned}$$

which completes the proof. ∎

Interesting classes of examples in the setting of the circle include the Dirichlet, Fejer and de la Vallée Poussin kernels.

Proposition 3.2 *Let G be the circle group. Suppose λ is an integer, $1 < p < \infty$ and $1 \le s \le t \le \infty$. Let P denote either the Dirichlet kernel of degree λ^N, the Fejer of degree λ^{8N} or the de la Vallée Poussin kernel of degree $2\lambda^{8N}+1$. There is a constant $c = c(\lambda, p)$ (but independent of N) such that*

$$\|P\|_{M(p,t;p,s)} \le cN^{1/s-1/t}.$$

Proof It is well known that for any of these kernels the $L^p \to L^p$ multiplier norms are bounded independently of N, thus we only need check the Lorentz norms, $L^{p,1}$. We will do this for the Fejer kernel, K_N, of degree λ^{8N}. The other two kernels are similar.

Observe that

$$\begin{aligned}\frac{1}{n+1}\left(\frac{\sin(n+1)x/2}{\sin x/2}\right)^2\chi_{[0,\pi]} &\le \left(\frac{\pi}{2}\right)^2(n+1)\chi_{[0,\pi/n]} + \frac{1}{n+1}\left(\frac{\pi}{x}\right)^2\chi_{[\pi/n,\pi]}\\ &\equiv f_1 + f_2.\end{aligned}$$

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Since the function f_1 is non-increasing we can easily determine its Lorentz norms:

$$\begin{aligned}\|f_1\|_{p,1}^* &\leq c\int_0^{\pi/n} t^{1/p}(n+1)\frac{dt}{t}\\ &\leq c(n+1)n^{-1/p} = cn^{1/p'}.\end{aligned}$$

The function f_2 has the same distribution as the decreasing function

$$\frac{1}{n+1}\left(\frac{\pi}{x+\pi/n}\right)^2 \chi_{[0,\pi-\pi/n]}.$$

Consequently

$$\begin{aligned}\|f_2\|_{p,1}^* &\leq \frac{c}{n+1}\int_0^{\pi-\pi/n} t^{1/p}\left(\frac{\pi}{t+\pi/n}\right)^2\frac{dt}{t}\\ &\leq \frac{c}{n+1}\sum_{k=0}^{n-2}\int_{\pi k/n}^{\pi(k+1)/n} t^{1/p}\left(\frac{\pi}{t+\pi/n}\right)^2\frac{dt}{t}\\ &\leq \frac{c}{n+1}\sum_{k=0}^{n-2}\left(\frac{\pi}{\pi(k+1)/n}\right)^2\left(\frac{k+1}{n}\right)^{1/p}\\ &\leq cn^{1/p'}\sum_{k=1}^{n-1}k^{-2+1/p} \leq cn^{1/p'}.\end{aligned}$$

Combining these results and taking $n = \lambda^{8N}$ shows that $\|K_N\|_{p,1}^* \leq c\lambda^{8N/p'} = c|\operatorname{supp}\widehat{K_N}|^{1/p'}$ as required. ∎

Our next goal is to prove that the upper bound given in this proposition is the order of magnitude of the multiplier K_N, for sufficiently large λ. This will be accomplished by finding a suitable test function. First we will describe the test function and establish some basic properties of it.

Test Function Let λ be a large integer ($\lambda = 1000$ will suffice) and for convenience set $M_N = 2\lambda^N + 1$. Let D_N be the Dirichlet kernel of degree λ^N. Set $x_j = 2(j-1)/\sqrt{M_N}$ for $j = 1, \ldots, 2^N$ and set $z_k = 3^N k/\sqrt{M_N}$ for $k = 1, \ldots, N$. Define $D_{j,k}(x) = D_N\big(x - (x_j + z_k)\big)$ and

$$\widetilde{D_{j,k}(x)} = \begin{cases} D_{j,k}(x) & \text{if } x \in [\frac{-2}{M_N}, \frac{2}{M_N}] + x_j + z_k\\ 0 & \text{else.}\end{cases}$$

Notice that if N is sufficiently large than the functions $\widetilde{D_{j,k}(x)}$ are disjointly supported. The test function will be

$$F_N(x) = \frac{1}{M_N}\sum_{k=1}^{N} 2^{-k/p}\sum_{j=1}^{2^k}\widetilde{D_{j,k}(x)}. \tag{3.2}$$

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Proposition 3.3 *Let $1 < p < \infty$ and $1 \le t \le \infty$. There is a constant $c = c(p)$ such that*

$$\|F_N\|^*_{p,t} \le cN^{1/t} M_N^{-1/p}.$$

Proof Without loss of generality we may assume N is sufficiently large that the functions $\widetilde{D_{j,k}(x)}$ are disjointly supported. As $|\widetilde{D_{j,k}(x)}| \le M_N$ this means that if $g_k \equiv \sum_{j=1}^{2^k} \widetilde{D_{j,k}(x)}$ then $|g_k| \le M_N$. Also, if n is (temporarily) fixed and $x \in \operatorname{supp} g_k$ for some $k > n$, then

$$|F_N(x)| \le 2^{-k/p} \le 2^{-(n+1)/p}.$$

Therefore

$$m\{x : |F_N(x)| \ge 2^{-n/p}\} \le m\Bigl(\bigcup_{k=1}^{n} \operatorname{supp} g_k\Bigr) \le \sum_{k=1}^{n} 2^k \frac{4}{M_N} \le \frac{2^{n+3}}{M_N}.$$

Furthermore, $|F_N(x)| \le 2^{-1/p}$ for all x and

$$m\{x : |F_N(x)| \ne 0\} \le \sum_{k=1}^{N} 2^k \frac{4}{M_N} \le \frac{2^{N+3}}{M_N}.$$

These calculations show

(1) $F_N^*(0) \le 2^{-1/p}$
(2) $F_N^*(2^{n+3}/M_N) \le 2^{-n/p}$ for $n = 1, \dots, N$
(3) $F_N^*(y) = 0$ for $y > 2^{N+3}/M_N$.

As F_N^* is a non-increasing function these three properties imply that if $t < \infty$ then

$$\begin{aligned}(\|F_N\|^*_{p,t})^t &\le c\Bigl(\int_0^{16/M_N} \bigl(x^{1/p} F_N^*(x)\bigr)^t \frac{dx}{x} + \sum_{n=1}^{N-1} \int_{2^{n+3}/M_N}^{2^{n+4}/M_N} \bigl(x^{1/p} F_N^*(x)\bigr)^t \frac{dx}{x}\Bigr) \\ &\le \frac{c}{M_N^{t/p}} \Bigl(2^{-t/p} + \sum_{n=1}^{N-1} 2^{-nt/p} (2^{(n+4)t/p} - 2^{(n+3)t/p})\Bigr) \le \frac{c2^{4t/p} N}{M_N^{t/p}}.\end{aligned}$$

Therefore $\|F_N\|^*_{p,t} \le cN^{1/t} M_N^{-1/p}$.

Similarly,

$$\|F_N\|^*_{p,\infty} \le \sup_n 2^{-n/p} \left(\frac{2^{n+4}}{M_N}\right)^{1/p} = \frac{c}{M_N^{1/p}}.$$ ■

In order to obtain lower bounds on the Lorentz norm of $K_N * F_N$ we will need to make delicate estimates which depend upon some elementary facts about the Dirichlet and Fejer kernels. These are recorded here for the convenience of the reader.

Lemma 3.4 *Let d_n be the Dirichlet kernel of degree n and k_n the Fejer kernel of degree n^8. Let $\widetilde{d_n}$ denote d_n restricted to the interval $[\frac{-2}{2n+1}, \frac{2}{2n+1}]$. For n sufficiently large the following are true:*

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(1) *If* $|t| \le 1/n^3$ *then* $|d_n(x+t) - d_n(x)| \le 1$.
(2) *If* $|t| \ge 1/n^3$ *then* $|k_n(t)| \le \pi^2/n^2$.
(3) *Given* $0 < \delta < 1$ *there exists* $\alpha > 0$ *such that whenever* $|t| \le \alpha/(2n+1)$ *then* $|d_n(t)| \ge \delta(2n+1)$.
(4) *For* $|t| \le 1/(2n+1)$, $|\widetilde{d_n} * k_n(t) - \widetilde{d_n}(t)| \le 2$.
(5) *Given* $1 < p < \infty$, *there exists* $\alpha > 0$ *such that whenever* $|t| \le \alpha/(2n+1)$ *then*

$$|\widetilde{d_n} * k_n(t)| \ge 2^{-1/2p}(2n+1).$$

Proof The first three items are entirely routine exercises involving trigonometric inequalities.

To prove (4) we use the fact if $|t| \le 1/(2n+1)$ and $|x| \le 1/n^3$ then $|t+x| \le 2/(2n+1)$, so that $\widetilde{d_n}(t+x) = d_n(t+x)$. Since $|d_n(x+t) - d_n(t)| \le 1$ when $|x| \le 1/n^3$ it follows that

$$\left| \int_{|x| \le 1/n^3} \big(\widetilde{d_n}(t+x) - \widetilde{d_n}(t)\big)\, k_n(x)\, dm(x) \right| \le \int k_n = 1.$$

Also,

$$\begin{aligned} \left| \int_{|x| > 1/n^3} \big(\widetilde{d_n}(t+x) - \widetilde{d_n}(t)\big)\, k_n(x)\, dm(x) \right| &\le 2(2n+1) \int_{|x|>1/n^3} k_n(x)\, dm(x) \\ &\le 2(2n+1)\pi^2/n^2 \le 1. \end{aligned}$$

Combining these estimates gives

$$|\widetilde{d_n} * k_n(t) - \widetilde{d_n}(t)| \le \left| \int \big(\widetilde{d_n}(t+x) - \widetilde{d_n}(t)\big)\, k_n(x)\, dm(x) \right| \le 2.$$

Item (5) is a corollary of (3) and (4). ■

We continue to use the notation D_N for the Dirichlet kernel of degree λ^N, K_N for the Fejer kernel of degree λ^{8N} and $M_N = 2\lambda^N + 1$.

Lemma 3.5 *Let* $1 < p < \infty$. *There is a constant* $\alpha > 0$ *such that for any* $n = 2, \dots, N$

$$m\{x : |F_N * K_N(x)| \ge 2^{-n/p}\} \ge \alpha 2^n / M_N.$$

Proof Choose $\alpha > 0$ as in item (5) such that whenever $|t| \le \alpha/M_N$ then $|\widetilde{D_N} * K_N(t)| \ge 2^{-1/2p} M_N$. Without loss of generality we may assume $\alpha < 1$. Temporarily fix j, k and consider $\widetilde{D_{j,k}(x)} * K_N$. If $|x - (x_j + z_k)| \le \alpha/M_N$ then by (5) we must have

$$|\widetilde{D}_{j,k} * K_N(x)| \ge 2^{-1/2p} M_N,$$

and hence

$$|F_N * K_N(x)| \ge \frac{2^{-k/p}}{M_N} |\widetilde{D}_{j,k} * K_N| \ge 2^{-(k+1)/p}.$$

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Thus

$$\{x : |F_N * K_N(x)| \geq 2^{-n/p}\} \supseteq \bigcup_{k=1}^{n-1} \bigcup_{j=1}^{2^k} \{x : |x - (x_j + z_k)| \leq \alpha/M_N\}$$

and adding the measure of these disjoint intervals completes the proof of the lemma. ■

Proposition 3.6 *Let $1 < p < \infty$ and $1 \leq s < t \leq \infty$. There is a constant $c = c(p)$ such that*

$$\|K_N\|_{M(p,t;p,s)} \geq cN^{1/s-1/t}.$$

Proof From the lemma we have that $(F_N * K_N)^*(\alpha 2^n/M_N) \geq 2^{-n/p}$ for $n = 2, \dots, N$. Hence for $s \neq \infty$

$$\begin{aligned}(\|F_N * K_N\|_{p,s}^*)^s &\geq \frac{s}{p} \sum_{n=2}^{N-1} \int_{\alpha 2^n/M_N}^{\alpha 2^{n+1}/M_N} \left(x^{1/p}(F_N * K_N)^*(x)\right)^s \frac{dx}{x} \\ &\geq \frac{c\alpha^{s/p} N}{M_N^{s/p}}.\end{aligned}$$

Combining this with the upper bound on the norm of F_N we obtain

$$\|K_N\|_{M(p,t;p,s)} \geq \frac{\|F_N * K_N\|_{(p,s)}}{\|F_N\|_{(p,t)}} \geq c\frac{N^{1/s} M_N^{-1/p}}{N^{1/t} M_N^{-1/p}} = cN^{1/s-1/t}$$ ■

Together, Propositions 3.2 and 3.6 establish the order of magnitude of the multipliers K_N and show that the $M(p,t;p,s)$ and $M(p,q;p,r)$ norms are not comparable if $1/s - 1/t \neq 1/r - 1/q$. This certainly ensures that the spaces are different; indeed, we now use these kernels to produce examples of multipliers which belong to certain Lorentz spaces, but not to others.

Theorem 3.7 *Let $\delta > \varepsilon > 0$. There is an integrable function F, on the circle, such that whenever $1 < p < \infty$, $0 \leq 1/s - 1/t \leq \varepsilon$ and $1/r - 1/q \geq \delta$ then $F \in M(p,t;p,s)$, but $F \notin M(p,q;p,r)$.*

Proof Set $2\tau = \delta + \varepsilon$. Let K_{2^N} and d_N denote the Fejer and Dirichlet kernels of degree $\lambda^{2^N 8}$ respectively, and let $K'_{2^N}(x) = K_{2^N}(x)e^{iL_N x}$ for $N = 1, 2, \dots$ where the integers L_N are chosen inductively to ensure that the Fourier transforms $\widehat{K'_{2^N}}$ have disjoint support. We take

$$F = \sum_N 2^{-\tau N} K'_{2^N}.$$

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The function F belongs to $L^1(T)$ since $\sum_N 2^{-\tau N} < \infty$. If $1/s - 1/t \leq \varepsilon$ then F belongs to $M(p,t;p,s)$ since

$$\|F\|_{M(p,t;p,s)} \leq \sum_N 2^{-\tau N} \|K_{2^N}\|_{M(p,t;p,s)} \leq c \sum_N 2^{-\tau N} 2^{N(1/s-1/t)} < \infty$$

as $\tau > 1/s - 1/t$.

On the other hand, in the proof of the previous proposition we saw that there were functions F_{2^N} satisfying $\|K_{2^N} * F_{2^N}\|_{(p,r)} \geq c2^{N(1/r-1/q)}\|F_{2^N}\|_{(p,q)}$. Let $f_{2^N} = F_{2^N} * d_N$. Because $\operatorname{supp} \widehat{f_{2^N}} \subseteq \operatorname{supp} \widehat{K_{2^N}}$ and the functions $\widehat{K'_{2^N}}$ have disjoint support, it follows that $|F * f_{2^N}(x)e^{iL_N x}| = |K_{2^N} * f_{2^N}|2^{-\tau N}$. Moreover, as $\|d_N\|_{M(p,q;p,q)} \leq c$ for some constant $c = c(p)$ which is independent of N, it follows that $\|f_{2^N}\|_{(p,q)} \leq c\|F_{2^N}\|_{(p,q)}$. Thus if $F \in M(p,q;p,r)$ then we would have

$$\infty > \|F\|_{M(p,q;p,r)} \geq \sup_N \frac{2^{-\tau N}\|K_{2^N} * f_{2^N}\|_{(p,r)}}{\|f_{2^N}\|_{(p,q)}} \geq c \sup_N 2^{-\tau N} 2^{N(1/r-1/q)},$$

and this is a contradiction since $\tau < 1/r - 1/q$. ■

Remark 3.2 It would be interesting to know if this could be done by a shorter method, perhaps using Young-type convolution inequalities (see [10] or [11]) and the fact that an even function $\sum a_n \cos nx$ with $a_n \downarrow 0$ belongs to $L(p,q)$ for $1 < p < \infty$ if and only if $\sum a_n^q n^{q/p'-1} < \infty$ [14]. For our Lorentz-improving multiplier problem one would seem to need to study the limiting behaviour as $p \to 1$, in the latter theorem.

Remark 3.3 In Section 5 we will use this result to help prove the stronger Theorem 5.4 which is valid for all infinite, compact, abelian groups.

Corollary 3.8 *$M(p,t;p,s) \cap L^1(T) \neq M(p,q;p,r)$ if $0 < 1/s - 1/t \neq 1/r - 1/q$.*

Proof If $r < q$ then this is obvious from the theorem. If $r \geq q$ it follows trivially from the fact that not all integrable functions are Lorentz-improving. (See Section 2.) ■

Remark 3.4 We do not know if $M(p,t;p,s) = M(p,q;p,r)$ when $1/s - 1/t = 1/r - 1/q$ (other than for the trivial case $M(2,t;2,s) = M(2,s';2,t')$).

Spaces of Lorentz-improving multipliers can also be shown to be distinct when the first indices are different. For this we construct Rudin-Shapiro type polynomials built from Fejer kernels and determine their multiplier norms.

Rudin-Shapiro Type Polynomials Choose $y_1, \ldots, y_N$ such that the intervals

$$\sum_{j=1}^N \varepsilon_j y_j + \left[\frac{-4}{\lambda^{N/3}}, \frac{4}{\lambda^{N/3}}\right]$$

are disjoint for $\varepsilon_j = 0, 1$. Let $L(y_j)$ denote translation by y_j. Set $\rho_0 = \sigma_0 = K_N$ (the λ^{8N}-Fejer kernel, as before) and inductively define the Rudin-Shapiro polynomials ρ_{n+1} and σ_{n+1} by

$$\rho_{n+1} = \rho_n - L(y_{n+1})\sigma_n$$
$$\sigma_{n+1} = \rho_n + L(y_{n+1})\sigma_n.$$

Then ρ_n is a linear combination of translated Fejer kernels, with coefficients equal to ± 1.

Also, let $\widetilde{K_N}$ be K_N restricted to $[-\lambda^{-3N}, \lambda^{-3N}]$, set $\widetilde{\rho_0} = \widetilde{\sigma_0} = \widetilde{K_N}$ and define $\widetilde{\rho_n}$ and $\widetilde{\sigma_n}$ correspondingly.

Lemma 3.9 *For ρ_n, defined as above, the $L^2 \to L^2$ multiplier norm is at most $c2^{n/2}$. For $1 < p < 2$ and $x \in [1, \infty]$ the $L^{p,x} \to L^{p,x}$ multiplier norm is at most $c2^{n/p}$.*

Proof Since ρ_n is a sum of 2^n trigonometric polynomials of L^1 norm one, we clearly have $\|\rho_n\|_1 \le 2^n$ and hence the $L^1 \to L^1$ multiplier norm is at most 2^n. Also, the parallelogram law gives that

$$\|\rho_n * f\|_2^2 + \|\sigma_n * f\|_2^2 \le 2^{n+1}\|\rho_0\|_1^2 \, \|f\|_2^2,$$

thus the $L^2 \to L^2$ multiplier norm is at most $c2^{n/2}$. An application of the weak interpolation theorem yields the desired result when $1 < p < 2$. ■

Upper bounds on the multiplier norms of ρ_N are now easy to obtain.

Proposition 3.10 *For $1 < p < 2$ and $1 \le s \le t \le \infty$*

$$\|\rho_N\|_{M(p,t;p,s)} \le c2^{N/p} N^{1/s-1/t}.$$

For $p = 2$ and $1 \le s \le t \le \infty$

$$\|\rho_N\|_{M(2,t;2,s)} \le \begin{cases} c2^{N/2}N^{1/s-1/t} & \text{if } s \le 2 \le t \\ c2^{N/2}N^{1/2-1/t} & \text{if } 2 \le s \le t \\ c2^{N/2}N^{1/s-1/2} & \text{if } s \le t \le 2. \end{cases}$$

Proof Let V_N denote the de la Vallée Poussin kernel of degree λ^{8N}. For any $f \in L^{p,t}$ Proposition 3.2 implies

$$\|V_N * f\|_{p,s}^* \le cN^{1/s-1/t}\|f\|_{p,t}^*.$$

If $1 < p < 2$ the lemma above yields

$$\|\rho_N * V_N * f\|_{p,s}^* \le c2^{N/p}\|V_N * f\|_{p,s}^*.$$

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As ρ_N is a trigonometric polynomial with Fourier transform supported on $\{-\lambda^{8N}, \dots, \lambda^{8N}\}$, and $\widehat{V_N} = 1$ on this set, we have $\rho_N * V_N = \rho_N$. Hence

$$\|\rho_N * f\|^*_{p,s} = \|\rho_N * V_N * f\|^*_{p,s}.$$

The proof is now obvious if $p \neq 2$.

The argument is slightly different if $p = 2$. Here we observe that $\|\rho_N\|_{M(2;2)} \leq c2^{N/2}$ means that $|\widehat{\rho_N}(k)| \leq c2^{N/2}$ for all k. Moreover, $\widehat{\rho_N}(k) = 0$ if $|k| > \lambda^{8N}$. Thus if $f \in L^{2,q}$ for $q \geq 2$ then

$$\begin{aligned}\|\rho_N * f\|^*_{2,2} &= \Big(\sum_k |\widehat{\rho_N}(k)\widehat{f}(k)|^2\Big)^{1/2} \leq c2^{N/2}\|V_N * f\|^*_{2,2} \\ &\leq c2^{N/2}N^{1/2-1/q}\|f\|^*_{2,q}.\end{aligned}$$

With this estimate the remainder of the argument for the case $s \leq 2 \leq t$ is similar to the one above:

$$\begin{aligned}\|\rho_N * f\|^*_{2,s} &= \|\rho_N * V_N * f\|^*_{2,s} \\ &\leq c\|\rho_N\|_{M(2,2;2,s)}\,\|V_N\|_{M(2,t;2,2)}\,\|f\|^*_{2,t} \\ &\leq c2^{N/2}N^{1/s-1/2}N^{1/2-1/t}\|f\|^*_{2,t}.\end{aligned}$$

When $2 \leq s \leq t$ note that

$$\|\rho_N\|_{M(2,t;2,s)} \leq \|\rho_N\|_{M(2,t;2,2)} \leq c2^{N/2}N^{1/2-1/t}.$$

The case $s \leq t \leq 2$ is similar. ■

Next, we find lower bounds on the multiplier norms. Notice that these are valid for all s, t.

Proposition 3.11 *For* $1 < p \leq 2$ *and* $1 \leq s, t \leq \infty$

$$\|\rho_N\|_{M(p,t;p,s)} \geq c2^{N/p}N^{1/s-1/t}.$$

Proof We again use the test function F_N of Proposition 3.3. Notice that $|K_N - \widetilde{K_N}| \leq \pi^2\lambda^{-2N}$ (item (2) of Lemma 3.4), thus if N is sufficiently large and if $|K_N * F_N| \geq 2^{-n/p}$ then $|\widetilde{K_N} * F_N| \geq 2^{-(n+1)/p}$. By Lemma 3.5

$$m\{x : |\widetilde{K_N} * F_N| \geq 2^{-n/p}\} \geq \alpha 2^{n-1}/M_N.$$

Note that for N sufficiently large, $\operatorname{supp} F_N \subseteq [-\lambda^{-N/3}, \lambda^{-N/3}]$. Thus

$$\operatorname{supp} \widetilde{K_N} * F_N \subseteq [-2\lambda^{-N/3}, 2\lambda^{-N/3}]$$

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and hence the choice of y_j ensures that $\widetilde{\rho_N} * F_N$ is a sum of 2^N disjointly supported translates of $\widetilde{K_N} * F_N$. This means

$$m\{x : |\widetilde{\rho_N} * F_N| \geq 2^{-n/p}\} \geq 2^N m\{x : |\widetilde{K_N} * F_N| \geq 2^{-n/p}\} \geq 2^N \alpha 2^{n-1}/M_N.$$

Thus $(\widetilde{\rho_N} * F_N)^*(2^N \alpha 2^{n-1}/M_N) \geq 2^{-n/p}$ and the usual integration estimates give

$$\|\widetilde{\rho_N} * F_N\|^*_{p,s} \geq c2^{N/p} N^{1/s} M_N^{-1/p}.$$

Since we already know $\|F_N\|^*_{p,t} \leq cN^{1/t} M_N^{-1/p}$ it follows that

$$\|\widetilde{\rho_N}\|_{M(p,t;p,s)} \geq c2^{N/p} N^{1/s-1/t}.$$

As

$$\|\rho_N - \widetilde{\rho_N}\|_\infty \leq 2^N \|K_N - \widetilde{K_N}\|_\infty \leq c2^N \lambda^{-2N}$$

it is routine to check that we also have

$$\|\rho_N\|_{M(p,t;p,s)} \geq c2^{N/p} N^{1/s-1/t}$$

(for a different constant c). ■

The previous two propositions show that Lorentz-improving multiplier spaces with first indices $r < p \leq 2$ are distinct. Indeed, we can now obtain the following results.

Theorem 3.12 *Let G be the circle group, $1 < p \leq 2$ and $1 \leq s \leq t \leq \infty$. There exists a multiplier $F \in M(p,t;p,s)$ which does not belong to $M(r,v;r,u)$ for any $1 < r < p$ and $1 \leq u, v \leq \infty$.*

Corollary 3.13 *Let G be the circle group. Suppose $1 < r, p < \infty$ and $1 \leq s, t, u, v \leq \infty$, with $s \leq t$ and $u \leq v$. If $r \neq p, p'$ then*

$$M(p,t;p,s) \neq M(r,v;r,u).$$

Proof This is immediate from the theorem since $M(p,t;p,s) = M(p',s';p',t')$ and $M(r,v;r,u) = M(r',u';r',v')$. ■

Proof of Theorem Define

$$F = \sum_N \rho'_N N^{-3} 2^{-N/p}$$

where the functions ρ'_N are suitable translates of ρ_N whose Fourier transforms have disjoint support. By Proposition 3.10

$$\|F\|_{M(p,t;p,s)} \leq \sum cN^\alpha N^{-3}$$

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for

$$\alpha = \max\{1/s - 1/t, 1/2 - 1/t, 1/s - 1/2\}.$$

As $\alpha \le 1$, $\|F\|_{M(p,t;p,s)} < \infty$ and thus $F \in M(p,t;p,s)$.

Similar arguments to those used in the proof of Theorem 3.7 show that if $r < p$ then

$$\|F\|_{M(r,v;r,u)} \ge \sup_N \|\rho_N\|_{M(r,v;r,u)} N^{-3} 2^{-N/p}$$
$$\ge \sup_N cN^{-4} 2^{-N/p} 2^{N/r}.$$

But this supremum is infinite since $1/r - 1/p > 0$, and therefore $F \notin M(r,v;r,u)$ for any $r < p$. ■

This theorem is also strengthened in Section 5.

4 Groups Whose Duals Have Elements of Finite Order

Throughout this section we will assume G is an infinite, compact, abelian group and X is a finite subgroup of $\widehat{G}$. We denote by H the annihilator of X. By replacing the Dirichlet and Fejer kernels on the circle with the Dirichlet kernel $D_H = \chi_H/m(H)$ we will be able to prove the same non-equalities of Lorentz multiplier spaces.

First we find bounds on the Lorentz multiplier norms of D_H which depend on the size of the finite group X. For this we need to calculate Lorentz norms of an appropriate test function.

Test Function Assume $|X| \ge 100^N$. We claim first that there exist

$$\{x_1, \dots, x_N, z_1, \dots, z_N\} \subseteq G$$

such that:

(i) $\{\sum_1^N \varepsilon_j x_j + H : \varepsilon_j = 0, 1\}$ are pairwise disjoint, and
(ii) $\{\sum_1^n \varepsilon_j x_j + z_n + H : \varepsilon_j = 0, 1; 1 \le n \le N\}$ are pairwise disjoint.

This claim can be proved by induction: To begin, choose any $x_1 \notin H$. Assume $\{x_1, \dots, x_{n-1}\} \subseteq G$, for $n \le N$, are chosen such that the sets $\{\sum_1^{n-1} \varepsilon_j x_j + H\}_{\varepsilon_j = 0,1}$ are pairwise disjoint. As $|G/H| = |X| > 3^N$, there exists $x_n \in G$ with $x_n \notin \{\sum_1^{n-1} \beta_j x_j + H : \beta_j = 0, 1, -1\}$ (here $\beta_j = 0, 1$ if the order of $x_j = 2$). This produces $\{x_1, \dots, x_N\}$ satisfying (i). Now further assume that $\{z_1, \dots, z_{n-1}\} \subseteq G$, for $n \le N$, have been chosen such that $\{\sum_1^k \varepsilon_j x_j + z_k + H : \varepsilon_j = 0, 1; 1 \le k \le n-1\}$ are pairwise disjoint. Since

$$\Big| \Big\{ \sum_1^N \beta_j x_j + z_k : \beta_j = 0, 1, -1; 1 \le k \le n-1 \Big\} \Big| \le N 3^N$$

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and $|G/H| > 10^N$, there exists $z_n \in G$ such that

$$z_n \notin \bigcup_{k=1}^{n-1} \Big\{ \sum_1^N \beta_j x_j + z_k + H : \beta_j = 0, 1, -1 \Big\}$$

(where $\beta_j = 0, 1$ if the order of $x_j = 2$). In this way we obtain $\{z_1, \dots, z_N\}$ satisfying (ii).

For $1 \le n \le N$ define a function $J_n(x)$ on G by

$$J_n(x) = \sum_{k=1}^{2^n} \chi_{v_k + H} \quad \text{where } \{v_1, \dots, v_{2^n}\} = \{\sum_{j=1}^{n} \varepsilon_j x_j : \varepsilon_j = 0, 1\}.$$

By (i), J_n is a sum of 2^n disjoint translates of χ_H. As

$$\widehat{\chi_{v_k + H}}(\gamma) = m(H)\overline{\gamma}(v_k)\chi_X,$$

$J_n(x)$ is a trigonometric polynomial on G. We also define

$$J'_n(x) = J_n(x - z_n) \quad \text{for } 1 \le n \le N$$

and finally, our test function,

$$F_N(x) = \sum_{n=1}^{N} 2^{-n/p} J'_n(x). \tag{4.1}$$

It is useful to note that the sets $\{\operatorname{supp} J'_n\}_{1 \le n \le N}$ are pairwise disjoint by (ii), and $\operatorname{supp} \widehat{F_N} \subseteq X$.

Lemma 4.1 *For the function F_N defined above, there are constants $A(p)$ and $B(p)$ such that for $1 < p < \infty$ and $1 \le q \le \infty$ we have*

$$A(p) m(H)^{1/p} N^{1/q} \le \|F_N\|_{(p,q)} \le B(p) m(H)^{1/p} N^{1/q}.$$

Proof This is a straightforward calculation: Since

$$m\{|F_N| > t\} = \begin{cases} \sum_{k=1}^{n} 2^k m(H) & \text{if } 2^{-(n+1)/p} \le t < 2^{-n/p} \\ 0 & \text{if } t \ge 2^{-1/p}, \end{cases}$$

we obtain

$$F_N^*(u) = \begin{cases} 2^{-1/p} & \text{if } u < 2m(H) \\ 2^{-n/p} & \text{if } \sum_{k=1}^{n-1} 2^k m(H) \le u < \sum_{k=1}^{n} 2^k m(H) \\ 0 & \text{if } u \ge \sum_{k=1}^{N} 2^k m(H). \end{cases}$$

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Thus if $q < \infty$ then $\|F_N\|^*_{p,q}$ equals

$$\left(\frac{q}{p}\left(\int_0^{2m(H)}(t^{1/p}2^{-1/p})^q\frac{dt}{t}+\sum_{n=2}^{N}\int_{\sum_{k=1}^{n-1}2^k m(H)}^{\sum_{k=1}^{n}2^k m(H)}(t^{1/p}2^{-n/p})^q\frac{dt}{t}\right)\right)^{1/q}$$

while

$$\|F_N\|^*_{p,\infty} = \sup_{1\le n\le N} 2^{-n/p}(2^{n+1}-2)^{1/p}m(H)^{1/p}.$$

Simplifying gives the stated result. ∎

The Lorentz multiplier norms of the Dirichlet kernels $D_H = \chi_H/m(H)$ can now be easily calculated.

Proposition 4.2 *Let $1 < p < \infty$ and $1 \le s < t \le \infty$. There are constants c_1 and c_2 which depend on p (but not s, t or X) such that*

$$c_1(\log|X|)^{1/s-1/t} \le \|D_H\|_{M(p,t;p,s)} \le c_2(\log|X|)^{1/s-1/t}.$$

Proof We will show first that the right hand inequality follows from Proposition 3.1. As $\|D_H\|_1 = 1$, the L^p multiplier norms are certainly bounded independently of H.

Because

$$m\{x : D_H(x) > t\} = \begin{cases} m(H) & \text{if } 0 \le t < m(H)^{-1} \\ 0 & \text{if } t \ge m(H)^{-1} \end{cases}$$

we have $D_H^*(y) = m(H)^{-1}$ if $0 \le y < m(H)$ and $D_H^*(y) = 0$ if $y \ge m(H)$. Thus

$$\|D_H\|^*_{p,1} = m(H)^{-1/p'} = |\operatorname{supp}\widehat{D_H}|^{1/p'} = |X|^{1/p'}.$$

By Proposition 3.1 we obtain

$$\|D_H\|_{M(p,t;p,s)} = \|D_H\|_{M(p',s';p',t')} \le c_2(\log|X|)^{1/s-1/t}.$$

To prove the left hand inequality we set $N = [\frac{1}{7}\log|X|]$ so that $|X| \ge 100^N$ (without loss of generality N is very large) and use the test function F_N defined above. Recall that $\operatorname{supp}\widehat{F_N} \subseteq X$ and $\widehat{D_H} = \chi_X$, thus $D_H * F_N = F_N$. Hence by the previous lemma

$$\|D_H\|_{M(p,t;p,s)} \ge \frac{\|D_H * F_N\|_{(p,s)}}{\|F_N\|_{(p,t)}} = \frac{\|F_N\|_{(p,s)}}{\|F_N\|_{(p,t)}} \ge \frac{A(p)}{B(p)}N^{1/s-1/t}.$$

As $N \ge c\log|X|$ this completes the proof. ∎

We are now ready to prove the non-equality of certain Lorentz spaces when $\widehat{G}$ contains infinitely many elements of finite order.

Theorem 4.3 *Let G be an infinite, compact, abelian group and suppose that $\widehat{G}$ contains infinitely many elements of finite order. Let $\delta > \varepsilon > 0$. There is an $f \in L^1(G)$ such that whenever $1 < p < \infty$, $1/s - 1/t \le \varepsilon$ and $1/r - 1/q \ge \delta$, then $f \in M(p,t;p,s)$ but $f \notin M(p,q;p,r)$.*

Remark 4.1 This theorem, which parallels Theorem 3.7, is also used to prove the stronger Theorem 5.4.

Proof Since $\widehat{G}$ contains infinitely many elements of finite order there exist finite subgroups X_n of $\widehat{G}$ such that $|X_n| \uparrow \infty$ as $n \to \infty$. Without loss of generality we may assume that $\{k_n\}$ is a strictly increasing sequence of positive integers with $k_n > n$ and $2^{2^{k_n}} \le |X_n| < 2^{2^{k_n+1}}$. Let H_n be the annihilator of X_n and choose $\{\gamma_n\} \subseteq \widehat{G}$ such that $\{X_n + \gamma_n\}$ are disjoint. We define D'_{H_n} by

$$\widehat{D'_{H_n}}(\gamma) = \widehat{D_{H_n}}(\gamma - \gamma_n)$$

and set $2\tau = \delta + \varepsilon$. The desired function is

$$f(x) = \sum_{n=1}^{\infty} 2^{-\tau k_n} D'_{H_n}.$$

In fact, since $\|D'_{H_n}\|_1 = \|D_{H_n}\|_1 = 1$, it is easy to see that $f \in L^1(G)$, and as

$$\|D'_{H_n}\|_{M(p,t;p,s)} = \|D_{H_n}\|_{M(p,t;p,s)},$$

Proposition 4.2 implies that for any $1 \le s < t \le \infty$

$$\begin{aligned}\|f\|_{M(p,t;p,s)} &\le \sum_{n=1}^{\infty} 2^{-\tau k_n} \|D'_{H_n}\|_{M(p,t;p,s)} \le \sum_{n=1}^{\infty} 2^{-\tau k_n} B(p) (\log |X_n|)^{1/s-1/t} \\ &\le c \sum_{n=1}^{\infty} 2^{(1/s-1/t-\tau)k_n}.\end{aligned}$$

The final sum is finite since $k_n > n$ and $\tau > 1/s - 1/t$, and hence $f \in M(p,t;p,s)$.

On the other hand, for n suitably large and $N(n) = [\log |X_n|/7]$ we consider the test functions $F_{N(n)}$ defined in (4.1) (with $X = X_n$) and define $F'_{N(n)}$ by

$$\widehat{F'_{N(n)}}(\gamma) = \widehat{F_{N(n)}}(\gamma - \gamma_n).$$

Since $\operatorname{supp} \widehat{F_{N(n)}} \subseteq X_n$ it follows from the choice of $\{\gamma_n\}$ that

$$|f * F'_{N(n)}| = 2^{-\tau k_n} |D'_{H_n} * F'_{N(n)}| = 2^{-\tau k_n} |D_{H_n} * F_{N(n)}|.$$

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Thus if $f \in M(p,q;p,r)$ then

$$\infty > \|f\|_{M(p,q;p,r)} \geq \sup_n \frac{2^{-\tau k_n}\|D_{H_n} * F_{N(n)}\|_{(p,r)}}{\|F_{N(n)}\|_{(p,q)}}$$
$$\geq \sup_n c2^{-\tau k_n} N(n)^{(1/r-1/q)}$$

from Lemma 4.1. But this is a contradiction because $N(n) \geq c2^{k_n}$, $k_n \to \infty$, and $\tau < 1/r - 1/q$. ∎

Corollary 4.4 *If G is as in the theorem and $0 < 1/s - 1/t \neq 1/r - 1/q$ then $M(p,t;p,s) \cap L^1(G) \neq M(p,q;p,r)$.*

Next we consider the spaces $M(p_j,t_j;p_j,s_j)$ for $j = 1,2$ and $p_1 \neq p_2, p_2'$.

Proposition 4.5 *Let G be an infinite, compact, abelian group, $1 < p < 2$ and let X_N be a finite subgroup with $|X_N| \geq 100^N$. There exist constants c_1 and c_2 depending on p, s, t and a trigonometric polynomial ρ_N such that*

$$c_1 2^{N/p} N^{1/s-1/t} \leq \|\rho_N\|_{M(p,t;p,s)} \leq c_2 2^{N/p} N^{1/s-1/t}.$$

Proof As usual let H_N denote the annihilator of X_N. The same type of arguments as used in the construction of the test function allow one to show there exist

$$\{x_1,\dots,x_N,y_1,\dots,y_N,z_1,\dots,z_N\} \subseteq G$$

such that

$$\Big\{\sum_1^n \varepsilon_j x_j + z_n + \sum_1^n \varepsilon_j' y_j + H_N : \varepsilon_j, \varepsilon_j' = 0,1; 1 \leq n \leq N\Big\}$$

are pairwise disjoint. Construct Rudin-Shapiro type polynomials as in Section 3 with this choice for $\{y_1,\dots,y_N\}$ and taking $\rho_0 = \sigma_0 = D_{H_N}$. Then for

$$\{u_1,\dots,u_{2^N}\} = \Big\{\sum_1^N \varepsilon_j y_j : \varepsilon_j = 0,1\Big\}$$

and a suitable choice of signs, $r_k = \pm 1$, we have

$$\rho_N(x) = \frac{1}{m(H_N)} \sum_{k=1}^{2^N} r_k \chi_{u_k + H_N}(x).$$

Notice ρ_N is a trigonometric polynomial and $\operatorname{supp} \widehat{\rho_N} \subseteq X_N$. As in Proposition 3.10,

$$\|\rho_N\|_{M(p,t;p,s)} \leq c_2 2^{N/p} N^{1/s-1/t}.$$

For the left hand inequality we again make use of our test function F_N of (4.1). Since $\chi_{H_N} * \chi_{H_N} = m(H_N)\chi_{H_N}$, it follows that

$$\rho_N * F_N = \sum_{n=1}^{N} 2^{-n/p} \sum_{k=1}^{2^n} \sum_{l=1}^{2^N} r_k \chi_{z_n+v_k+u_l+H_N}(x)$$

where

$$\{v_1, \dots, v_{2^N}\} = \Big\{ \sum_{1}^{n} \varepsilon_j x_j : \varepsilon_j = 0, 1 \Big\}.$$

The choice of z_i, x_j and y_k ensures that the sets

$$\{z_n + v_k + u_l + H_N : 1 \le k, l \le 2^N, 1 \le n \le N\}$$

are pairwise disjoint. Thus

$$m\{|\rho_N * F_N| > t\} = \begin{cases} \sum_{k=1}^{n} 2^{k+N} m(H_N) & \text{if } 2^{-(n+1)/p} \le t < 2^{-n/p} \\ 0 & \text{if } t \ge 2^{-1/p}, \end{cases}$$

and so we obtain

$$(\rho_N * F_N)^*(u) = \begin{cases} 2^{-1/p} & \text{if } u < 2^{1+N} m(H_N) \\ 2^{-n/p} & \text{if } \sum_{k=1}^{n-1} 2^{k+N} m(H_N) \le u < \sum_{k=1}^{n} 2^{k+N} m(H_N) \\ 0 & \text{if } u \ge \sum_{k=1}^{N} 2^{k+N} m(H_N). \end{cases}$$

Similar calculations to those in the proof of Lemma 4.1 show that

$$\|\rho_N * F_N\|_{(p,s)} \ge c m(H_N)^{1/p} 2^{N/p} N^{1/s}.$$

But we previously saw that

$$\|F_N\|_{(p,t)} \le c m(H_N)^{1/p} N^{1/t},$$

thus

$$\|\rho_N\|_{M(p,t;p,s)} \ge \frac{\|\rho_N * F_N\|_{(p,s)}}{\|F_N\|_{(p,t)}} \ge c 2^{N/p} N^{1/s-1/t}.$$

Therefore ρ_N is the desired function. ∎

Remark 4.2 Note that the lower bound,

$$\|\rho_N\|_{M(p,t;p,s)} \ge c 2^{N/p} N^{1/s-1/t}$$

remains true when $p = 2$. For the upper bound one can show, as was done in the circle group case (Proposition 3.10), that

$$\|\rho_N\|_{M(2,t;2,s)} \le \begin{cases} c 2^{N/2} N^{1/s-1/t} & \text{if } s \le 2 \le t \\ c 2^{N/2} N^{1/2-1/t} & \text{if } 2 \le s < t \\ c 2^{N/2} N^{1/s-1/2} & \text{if } s < t \le 2. \end{cases}$$

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Armed with these results we can now prove the following corollary in essentially the same manner as the proof of Theorem 3.12.

Corollary 4.6 *Let G be an infinite, compact, abelian group and suppose that $\widehat{G}$ contains infinitely many elements of finite order. Suppose $1 < r$, $p < \infty$ and $1 \leq s, t, u, v \leq \infty$ with $s \leq t$ and $u \leq v$. If $r \neq p, p'$ then*

$$M(p, t; p, s) \neq M(r, v; r, u).$$

5 Non-Inclusions for Arbitrary Compact Abelian Groups

The key step needed to obtain similar results for general compact abelian groups is the following proposition about homomorphic images.

Proposition 5.1 *Let G and H be compact abelian groups and $\pi\colon G \to H$ a continuous, onto homomorphism. Let $F \in L^1(H)$ and define a function $\widetilde{F}$ on G by $\widetilde{F} = F \circ \pi$. Then $\widetilde{F} \in L^1(G)$ and for any p, q, r, s,*

$$\|\widetilde{F}\|_{L^{p,q}(G)} = \|F\|_{L^{p,q}(H)} \quad \textit{and} \quad \|F\|_{M(p,q;r,s)} \leq \|\widetilde{F}\|_{M(p,q;r,s)}.$$

Proof In [7] it is shown that for any measurable function f on H we have $\|f\|_{L^{p,q}(H)} = \|f \circ \pi\|_{L^{p,q}(G)}$. Thus $\widetilde{F} \in L^1(G)$ and $\|\widetilde{F}\|_{L^{p,q}(G)} = \|F\|_{L^{p,q}(H)}$.

Suppose $g \in L^{p,q}(H)$. Set $f = g \circ \pi$. A change of variables argument proves that $\widetilde{F} * f = (F * g) \circ \pi$ and hence

$$\|F * g\|_{L^{r,s}(H)} = \|\widetilde{F} * f\|_{L^{r,s}(G)} \leq \|\widetilde{F}\|_{M(p,q;r,s)} \|f\|_{L^{p,q}(G)}.$$

But $\|f\|_{L^{p,q}(G)} = \|g\|_{L^{p,q}(H)}$ and this certainly suffices to prove $\|F\|_{M(p,q;r,s)} \leq \|\widetilde{F}\|_{M(p,q;r,s)}$. ■

This proposition enables us to transfer results from the circle group to certain other compact, abelian groups.

Corollary 5.2 *Suppose G is a compact abelian group and $\pi\colon G \to T$ is a continuous, onto homomorphism. If $F \in L^1(T)$ and $F \notin M(p, q; p, r)$ then $\widetilde{F} \equiv F \circ \pi \in L^1(G)$ and $\widetilde{F} \notin M(p, q; p, r)$.*

Corollary 5.3 *Suppose G is a compact abelian group whose dual contains an element of infinite order. Let $\delta > \varepsilon > 0$. There is a function $\widetilde{F} \in L^1(G)$ such that if $1 < p < \infty$ and $1/s - 1/t \leq \varepsilon$ then $\widetilde{F} \in M(p, t; p, s)$, while if $1/r - 1/q \geq \delta$ then $\widetilde{F} \notin M(p, q; p, r)$.*

Proof Since the dual of G contains an element of infinite order the circle group T is a homomorphic image of G. Let π be the canonical map from G to T. Let F be the integrable function on T constructed in the proof of Theorem 3.7 and set $\widetilde{F} = F \circ \pi$. Clearly $\widetilde{F} \in L^1(G)$, and if $1 < p < \infty$ then $\widetilde{F} \notin M(p, q; p, r)$ since $F \notin M(p, q; p, r)$ for any p.

Now $\widetilde{F} = \sum_N 2^{-\tau N} K'_{2^N} \circ \pi$ for $2\tau = \delta + \varepsilon$. Because $K'_{2^N} \circ \pi$ has the same Lorentz norms as K_{2^N}, Proposition 3.1 implies that $\|K'_{2^N} \circ \pi\|_{M(p,t;p,s)} \le c2^{N(1/s-1/t)}$. As $\tau > 1/s - 1/t$,

$$\|\widetilde{F}\|_{M(p,t;p,s)} \le \sum_N 2^{-\tau N} \|K'_{2^N} \circ \pi\|_{M(p,t;p,s)} \le \sum_N 2^{-\tau N} 2^{N(1/s-1/t)} < \infty$$

and hence $\widetilde{F} \in M(p,t;p,s)$ for every $1 < p < \infty$. ■

We can now improve Theorems 3.7 and 4.3.

Theorem 5.4 *Let G be an infinite, compact, abelian group and let* $0 < \tau < 1$. *There is a function* $f \in L^1(G)$ *such that*

$$f \in \bigcap_{\substack{1<p<\infty,\\ 1/s-1/t=\tau}} M(p,t;p,s) \quad \textit{but} \quad f \notin \bigcup_{\substack{1<p<\infty,\\ 1/r-1/q>\tau}} M(p,q;p,r).$$

Proof Choose a sequence $\{p_k\}$ dense in $(1,\infty)$. Let $\{(q_n,r_n)\}$ be a listing of the rational pairs satisfying $1/r_n - 1/q_n > \tau$. Form a sequence $\{(Q_n,R_n)\}$ such that each (Q_n,R_n) equals some (q_m,r_m), and each pair (q_m,r_m) occurs infinitely often in $\{(Q_n,R_n)\}$. Take u such that $1 - 1/u = \tau$.

If the dual of G has an element of infinite order then Corollary 5.3 implies that for every n there is a function $f_n \in L^1(G)$ such that $f_n \in M(p,u;p,1)$ for all $1 < p < \infty$, but $f_n \notin M(p,Q_n;p,R_n)$ for any p. Otherwise, all elements of $\widehat{G}$ are of finite order and Theorem 4.3 implies such a sequence of functions exist.

Let $c_1(n) = \|f_n\|_1$ and let

$$c_2(n) = \max_{1\le k\le n} \|f_n\|_{M(p_k,u;p_k,1)}.$$

Without loss of generality we may assume $c_1(n)$ and $c_2(n) \ge 1$.

Since $f_n \notin M(p_k,Q_n;p_k,R_n)$, for each n,k we can choose trigonometric polynomials $P_{n,k}$ of L^{p_k,Q_n} norm one such that

$$\|f_n * P_{n,k}\|_{(p_k,R_n)} > n2^n c_1(n)c_2(n).$$

Let $\{J_n\}$ be a bounded approximate identity with finite support of $\widehat{J_n}$, $\|J_n\|_1 \le C$ and

$$\|f_n * J_n * P_{n,k}\|_{(p_k,R_n)} > n2^n c_1(n)c_2(n) \quad \text{for all } k \le n.$$

Multiply by appropriate characters γ_n so that the functions $F_n = (f_n * J_n)\gamma_n$ have disjoint Fourier transforms and let

$$f = \sum_n \frac{F_n}{2^n c_1(n)c_2(n)}.$$

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Since $\|f_n * J_n\|_1 \leq Cc_1(n)$ it is clear that $f \in L^1(G)$. Temporarily fix k. If $n \geq k$ then

$$\|f_n * J_n\|_{M(p_k,u;p_k,1)} \leq C\|f_n\|_{M(p_k,u;p_k,1)} \leq Cc_2(n),$$

and thus

$$\Big\|\sum_{n=k}^{\infty} \frac{F_n}{2^n c_1(n)c_2(n)}\Big\|_{M(p_k,u;p_k,1)} \leq C.$$

As $\sum_{n=1}^{k-1} F_n/\big(2^n c_1(n)c_2(n)\big)$ is a trigonometric polynomial it follows that $f \in M(p_k, u; p_k, 1)$ for all k. Since the sequence $\{p_k\}$ is dense, interpolating gives that $f \in M(p, u; p, 1)$ for all $1 < p < \infty$. By duality $f \in M(p, \infty; p, u')$ for all p, and a further interpolation argument yields that $f \in M(p, t; p, s)$ whenever $1/s - 1/t = \tau$.

Because the Fourier transforms of the functions F_n are disjointly supported it follows that if k is fixed and $n \geq k$,

$$\|f\|_{M(p_k,Q_n;p_k,R_n)} \geq \frac{\|f_n * J_n * P_{n,k}\|_{(p_k,R_n)}}{2^n c_1(n)c_2(n)} > n.$$

Since $(q_m, r_m) = (Q_n, R_n)$ for infinitely many n's, $f \notin M(p_k, q_m; p_k, r_m)$ for any k, m.

Now suppose (q, r) is any pair with $1/r - 1/q > \tau$. Choose (q_m, r_m) such that $r \leq r_m$, $q \geq q_m$, but still $1/r_m - 1/q_m > \tau$. Since $f \notin M(p_k, q_m; p_k, r_m)$ and $M(p_k, q_m; p_k, r_m) \supseteq M(p_k, q; p_k, r)$ it follows that $f \notin M(p_k, q; p_k, r)$ for any k.

Finally, assume $f \in M(p, q; p, r)$ for some p and pair q, r satisfying $1/r - 1/q > \tau$. Pick $p_k > p$ such that if $p/p_k = \beta$ then $\beta(1/r - 1/q) > \tau$. Since $f \in L^1$ we have $f \in M(\infty, \infty; \infty, \infty)$, and interpolating gives $f \in M(p_k, a_k; p_k, b_k)$ for $r/b_k = q/a_k = \beta$. But as $1/b_k - 1/a_k > \tau$ this contradicts the previous paragraph. Hence f does not belong to $M(p, q; p, r)$ for any $1 < p < \infty$ and (r, q) such that $1/r - 1/q > \tau$. ∎

Corollary 5.5 *Suppose G is any infinite, compact, abelian group, $1 < p < \infty$ and $0 < 1/s - 1/t \neq 1/r - 1/q$. Then $M(p, t; p, s) \neq M(p, q; p, r)$ and $M(p, t; p, s) \neq M(p', q; p', r)$.*

Proof We only need observe that $M(p', q; p', r) = M(p, r'; p, q')$, $1/r - 1/q = 1/q' - 1/r'$, and then argue as in Corollary 3.8. ∎

Corollary 5.6 *Suppose G is any infinite, compact, abelian group and $1 < p < \infty$.*

(1) If $1 \leq r < \min(t, q) \leq \infty$ then $M(p, q; p, r) \subsetneq M(p, q; p, t)$.
(2) If $1 \leq \max(r, q) < v \leq \infty$ then $M(p, v; p, r) \subsetneq M(p, q; p, r)$.

Proof For (1), suppose r_1 is chosen such that $r < r_1 < \min(t, q)$. We clearly have the inclusions

$$M(p, q; p, r) \subseteq M(p, q; p, r_1) \subseteq M(p, q; p, t),$$

and the theorem proves $M(p, q; p, r) \neq M(p, q; p, r_1)$.

The proof for (2) is similar. ∎

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Remark 5.1 The reader should see [4, Theorems 24–26] for related results.

Corollary 5.7 *Let G be an infinite, compact, abelian group and suppose* $1 < p < \infty$. *If* $1 < q, r, u, v < \infty$, $1/u - 1/v > 1/r - 1/q \geq 0$, $v/u > q/r$ *and* $u'/v' > r'/q'$, *then*

$$M(p, v; p, u) \cap L^1(G) \subsetneq M(p, q; p, r).$$

Proof The theorem clearly shows that $M(p, v; p, u) \cap L^1(G) \neq M(p, q; p, r)$ thus we need only prove the inclusion. Observe that this is obvious from the inclusions of the Lorentz spaces if both $q \leq v$ and $r \geq u$, so we assume otherwise.

Let $F \in M(p, v; p, u) \cap L^1(G)$ and choose $\alpha \in (0, 1]$ such that

$$1/r - 1/q = (1 - \alpha)(1/u - 1/v).$$

First we claim that $1/q > (1 - \alpha)/v$. To see this observe that

$$\begin{aligned} \frac{1}{q} - \frac{1-\alpha}{v} &= \frac{1}{q} - \left(\frac{1}{r} - \frac{1}{q}\right) / \left(\frac{v}{u} - 1\right) \\ &= \frac{1}{q}\left(\frac{v}{u} - \frac{q}{r}\right) / \left(\frac{v}{u} - 1\right) > 0 \end{aligned}$$

by the assumptions.

If $q \geq v$ then choose $x > 0$ satisfying $1/q = \alpha/x + (1 - \alpha)/v$. The choice of α ensures that $1/r = \alpha/x + (1 - \alpha)/u$ and $x \geq q \geq 1$. Certainly $F\colon L^{p,x} \to L^{p,x}$ and hence by interpolating it follows that F maps $L^{p,q}$ to $L^{p,r}$.

Otherwise $q < v$. But then $r < u$ and consequently, $r' > u'$. Similar arguments show we can choose $y \geq 1$ such that $1/r' = \alpha/y + (1 - \alpha)/u'$ and $1/q' = \alpha/y + (1 - \alpha)/v'$. By duality $F \in M(p', u'; p', v')$, and again an interpolation argument shows that

$$F \in M(p', r'; p', q') = M(p, q; p, r). \qquad \blacksquare$$

Corollaries 3.13 and 4.6 also generalize to the case of an arbitrary compact abelian group.

Theorem 5.8 *Let G be an infinite, compact, abelian group. Suppose* $r, p \in (1, \infty)$ *and* $s, t, u, v \in [1, \infty]$ *with* $s \leq t$ *and* $u \leq v$. *If* $r \neq p, p'$ *then*

$$M(p, t; p, s) \neq M(r, v; r, u).$$

Proof We argue in a similar way to Theorem 5.4: either T is a homomorphic image of G, or $\widehat{G}$ contains infinitely many elements of finite order. In the second case the result is immediate from Corollary 4.6.

In the first case we consider the functions ρ_N and F used in the proof of Theorem 3.12 and let V_N denote the de la Vallée Poussin kernel of degree $2\lambda^{8N} + 1$. Let $\pi\colon G \to T$ be the canonical map and define $\widetilde{\rho_N} = \rho_N \circ \pi$, $\widetilde{V_N} = V_N \circ \pi$ and

$\widetilde{F} = F \circ \pi$. As the Lorentz norms of V_N and $\widetilde{V_N}$ are the same, Proposition 3.1 implies that $\|\widetilde{V_N}\|_{M(p,t;p,s)} \leq cN^{1/s-1/t}$. Similar arguments to those used in the proofs of Lemma 3.9 and Proposition 3.10 now show that if $1 < p \leq 2$ and $1 \leq s \leq t \leq \infty$ then

$$\|\widetilde{\rho_N}\|_{M(p,t;p,s)} \leq c2^{N/p}N^{\alpha} \quad \text{for } \alpha = \max\{1/s - 1/t, 1/s - 1/2, 1/2 - 1/t\}.$$

Thus $\|\widetilde{F}\|_{M(p,t;p,s)} < \infty$. Since $F \notin M(r, v; r, u)$ the same is true for $\widetilde{F}$. ■

When a measure μ is a Lorentz-improving multiplier then it follows by a standard interpolation argument that μ improves all Lorentz spaces, in the sense that for every $1 < q < \infty$ and $1 \leq s \leq \infty$ there is some $t = t(p, q, s) > s$ such that $\mu \in M(q, t; q, s)$. Our last result illustrates that this fails to be true, in a very strong way, for arbitrary Lorentz-improving multipliers.

Corollary 5.9 *If* $1 < p \leq 2$ *then*

$$M(p, \infty; p, 1) \not\subseteq \bigcup_{1<r<p} M(r, 1; r, \infty).$$

Proof This is part of the content of Theorem 3.12 for the circle group. For groups which contain infinitely many elements of finite order, or arbitrary compact, abelian groups, the argument is similar. ■

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