RAMANUJAN CONGRUENCES FOR p_{-k} (mod 11^r)

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1. Introduction. Denote by

$$E(x) = \prod_{m=1}^{\infty} (1 - x^m)$$

the Euler product, and by

$$P(x) = E(x)^{-1} = \sum_{n=0}^{\infty} p(n)x^n$$

the partition generating function. More generally, if k is any integer, put

$$P(x)^k = \sum_{n=0}^{\infty} p_{-k}(n)x^n,$$

so that $p(n) = p_{-1}(n)$. In [**3**], Atkin proved the following theorem.

THEOREM 1. *Suppose* $k > 0$ *and* $q = 2, 3, 5, 7$ *or* 13. *If* $24n \equiv k \pmod{q^r}$ *then* $p_{-k}(n) \equiv 0 \pmod{q^{\frac{1}{2}\alpha r + \varepsilon}}$, *where* $\varepsilon = \varepsilon(q, k) = O(\log k)$, *and where* α *depends on* q *and the residue of* k (mod 24) *according to the following table.*

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$q = 2$								3								3								0
$q = 3$			3			2			3			2			1			2			1			0
$q = 5$	2	1	1	1	2	2	1	1	1	1	1	0	0	0	1	1	0	0	0	1	1	0	0	0
$q = 7$	1	1	1	2	1	1	1	0	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0
$q = 13$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(An apparent misprint in the last column has been corrected here.)

In this table the non-blank entries are best possible in the sense that each residue class (mod 24) contains an integer k for which the exponent $\alpha(k, q)$ given in the table cannot be improved. The blank entries are only known to be non-negative; they are probably 0.

Atkin remarks that a similar theory undoubtedly exists for $q = 11$, and in [**2**] he shows that $\alpha(1, 11) = 2$ and $\varepsilon(1, 11) = 0$, thereby proving Ramanujan's conjectured congruences for $p(n)$ modulo powers of 11. He says, however, that his method does not yield the analogue of Theorem 1 for $q = 11$ and all $k > 0$. In this paper we show how the method can be modified to dispose of this general case. Specifically, we prove the following result.

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THEOREM 2. *If* $24n \equiv k \pmod{11^r}$, *then* $p_{-k}(n) \equiv 0 \pmod{11^{\frac{1}{2}\alpha r+\varepsilon}}$, *where* $\varepsilon = \varepsilon(k) = O(\log|k|)$ *and where, if* $k \geq 0$, α *depends on the residue of* $k \pmod{120}$ *as shown in the following table.*

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	2	1	2	1	1	1	2	2	1	1	2	2	1	2	1	0	0	1	1	0	0	1	1	0
24	1	1	1	1	2	2	1	1	2	2	1	0	0	0	0	1	1	0	0	1	1	1	0	0
48	1	1	2	2	1	1	1	0	1	0	1	0	0	1	1	0	0	1	0	1	0	1	0	0
72	2	1	1	1	2	1	2	1	2	1	2	2	1	1	1	2	1	2	1	2	1	1	1	0
96	0	0	1	0	1	0	1	0	1	1	0	0	0	1	0	1	0	1	0	1	1	0	0	0

Here the entry in the row labelled $24i$ *and the column labelled* j *is* $\alpha(24i+j)$. *For* $k<0$, *the dependence of* α *on* k *is the same as in the above table, except that the numbers in the last column must be changed to* 2, 2, 2, 0, 2.

Theorem 2 is best possible in the sense that every residue class (mod 120) contains both positive and negative values of k for which the above constants $\alpha(k)$ cannot be improved.

It will be noted that Theorem 2 deals with both positive and negative values of k, in contrast to Theorem 1. It is easy to see, however, that Theorem 1 can also be extended to negative k by using the method of proof outlined below. For $k<0$ the numbers in the last column of Theorem 1 must be changed to 6, 4, 2, 2, 0.

The case $k=-24$ is of particular interest, since $p_{24}(n) = \tau(n+1)$, where $\tau(n)$ is Ramanujan's τ-function. The proof of the extension of Theorem 1 to negative k yields the following congruences for $\tau(n)$:

$$\tau(n) \equiv 0 \pmod{2^{3r}} \quad \text{if} \quad n \equiv 0 \pmod{2^r};$$
$$\tau(n) \equiv 0 \pmod{3^{2r}} \quad \text{if} \quad n \equiv 0 \pmod{3^r};$$
$$\tau(n) \equiv 0 \pmod{5^r} \quad \text{if} \quad n \equiv 0 \pmod{5^r};$$
$$\tau(n) \equiv 0 \pmod{7^r} \quad \text{if} \quad n \equiv 0 \pmod{7^r}.$$

All of these congruences are already known from the multiplicative properties of $\tau(n)$. But for higher powers of the cusp form $\Delta(x) = \sum_{n=1}^{\infty} \tau(n)x^n$, Theorems 1 and 2 (with $k<0$) yield congruences for the coefficients modulo powers of 2, 3, 5, 7 and 11 which appear to be new.

2. Notation and preliminaries. We will use a mixture of the notation in Atkin's papers [**2**] and [**3**], with some modifications which turn out to be convenient. Let $\mathscr{L}$ be the complex vector space of all meromorphic Laurent series $f(x) = \sum_{n \geq n_0} a_n x^n$ convergent in some neighborhood of $x=0$, and let U denote the linear transformation of $\mathscr{L}$ into itself defined by

$$U\Big(\sum_{n\geq n_0} a_n x^n\Big) = \sum_{11n\geq n_0} a_{11n} x^n.$$

As above, let

$$P(x) = E(x)^{-1} = \prod_{m=1}^{\infty} (1-x^m)^{-1},$$

and let k be an integer fixed throughout the discussion. Define the function $p_{-k}(n)$ by the equation

$$P(x)^k = \sum_{n=0}^{\infty} p_{-k}(n) x^n \qquad (|x|<1).$$

We define a recursive sequence of "11-dissections" of $P(x)^k$ as follows:

$$\left.\begin{aligned} D_0(x) &= P(x)^k; \\ D_{2s-1}(x) &= U(x^{5k} D_{2s-2}(x)); \\ D_{2s}(x) &= U(D_{2s-1}(x)) \qquad (s\geq 1). \end{aligned}\right\} \tag{1}$$

To write (1) in a more uniform way, we define

$$\lambda_r = \begin{cases} 0 & \text{if } r \text{ is odd,} \\ k & \text{if } r \text{ is even.} \end{cases} \tag{2}$$

Then (1) becomes:

$$\left.\begin{aligned} D_0(x) &= P(x)^k; \\ D_r(x) &= U(x^{5\lambda_{r-1}} D_{r-1}(x)) \qquad (r\geq 1). \end{aligned}\right\} \tag{3}$$

It should be clear that the functions $D_r(x)$ are power series of the form

$$D_r(x) = \sum_{m\geq \mu_r} p_{-k}(11^r m + n_r) x^m, \tag{4}$$

where n_r is an integer depending only on r, and μ_r is the least integer m such that $11^r m + n_r \geqslant 0$. Substituting (2) into (1), we obtain a recurrence for n_r, viz $n_0=0$ and

$$n_{2s-1} = -5k - 11^2 n_{2s-2}, \qquad n_{2s} = n_{2s-1} \qquad (s\geq 1).$$

The solution to this recurrence is easily found to be

$$n_{2s-1} = n_{2s} = -k(11^{2s}-1)/24, \tag{5}$$

from which we see that

$$24 n_r \equiv k \pmod{11^r} \quad \text{for all } r.$$

Hence the coefficients of $D_r(x)$ are the numbers $p_{-k}(n)$, where $24n \equiv k \pmod{11^r}$.

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Therefore Theorem 2 can be written in the form

$$D_r(x) \equiv 0 \pmod{11^{\frac{1}{2}\alpha r+\varepsilon}}, \tag{6}$$

where as usual the congruence

$$\sum a_n x^n \equiv \sum b_n x^n \pmod{M}$$

between Laurent series with integer coefficients means that $a_n \equiv b_n \pmod{M}$ for all n.

From (3) we see that the integer μ_r appearing in (4) satisfies the equations

$$\mu_0 = 1,$$
$$\mu_r = \lceil (5\lambda_{r-1} + \mu_{r-1})/11 \rceil \qquad (r \geq 1),$$

where $\lceil \theta \rceil$ denotes the ceiling of θ, i.e. the least integral upper bound for θ. To obtain an explicit formula for μ_r we use the fact, already noted above, that μ_r is the least integer m such that $11^r m + n_r \geq 0$. If $r = 2s-1$, we see from (5) that this is equivalent to

$$11^{2s-1} m \geq k(11^{2s}-1)/24,$$

i.e.

$$m \geq 11k/24 - k/24 \,.\, 11^{2s-1}.$$

Therefore

$$\mu_{2s-1} = \lceil 11k/24 - k/24 \,.\, 11^{2s-1} \rceil.$$

It is easily seen from this formula that

$$\mu_{2s-1} = \lceil 11k/24 \rceil + \omega(k) \quad \text{if} \quad 11^{2s-1} > |k|, \tag{7}$$

where

$$\omega(k) = \begin{cases} 1 & \text{if } k < 0 \text{ and } 24 \mid k, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, if $r = 2s$ the inequality $11^r m + n_r \geq 0$ is equivalent to $11^{2s} m \geq k(11^{2s}-1)/24$, and hence

$$\mu_{2s} = \lceil k/24 - k/24 \,.\, 11^{2s} \rceil.$$

In particular,

$$\mu_{2s} = \lceil k/24 \rceil + \omega(k) \quad \text{if} \quad 11^{2s} > |k|. \tag{8}$$

In order to prove (6) analytically, we replace the functions $D_r(x)$ by a related sequence $L_r(x)$ of automorphic functions on the group $\Gamma_0(11)$. These are defined as follows. Put

$$\left.\begin{aligned} &\phi(x) = x^5 P(x)/P(x^{121}), \\ &L_0(x) = 1, \\ &L_r(x) = U(\phi(x)^{\lambda_{r-1}} L_{r-1}(x)) \qquad (r \geq 1). \end{aligned}\right\} \tag{9}$$

It is easy to verify that

$$L_{2s-1}(x) = D_{2s-1}(x)/P(x^{11})^k,$$
$$L_{2s}(x) = D_{2s}(x)/P(x)^k \qquad (s \geq 1).$$

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Since the denominators $P(x^{11})^k$ and $P(x)^k$ here are power series with integer coefficients and constant term 1, any congruence of the form $D_r(x) \equiv 0 \pmod M$ holds if and only if $L_r(x) \equiv 0 \pmod M$. Hence (6) is equivalent to

$$L_r(x) \equiv 0 \pmod{11^{\frac{1}{2}\alpha r + \varepsilon}}; \tag{10}$$

it is in this form that we will prove Theorem 2.

Now let $x = e^{2\pi i \tau}$, and recall the definition of the Dedekind η-function:

$$\eta(\tau) = x^{1/24} \prod_{m=1}^{\infty} (1 - x^m) = x^{1/24} E(x).$$

Since $E(x) = P(x)^{-1}$, we have

$$\frac{\eta(121\tau)}{\eta(\tau)} = \frac{x^{121/24} E(x^{121})}{x^{1/24} E(x)} = x^5 \frac{P(x)}{P(x^{121})} = \phi(x).$$

Hence [**6**, Theorem 1] $\phi(x)$ is a function on $\Gamma_0(121)$.

For any positive integer N, we denote by $R_0(N)$ the Riemann surface of the group $\Gamma_0(N)$, and by $K_0(N)$ the field of meromorphic functions on $R_0(N)$. For any N, $R_0(N)$ has a cusp at $\tau = i\infty$, and $x = e^{2\pi i \tau}$ is a uniformizing parameter there. If $f(\tau) \in K_0(N)$, its Laurent expansion about $\tau = i\infty$ therefore has the form $f(\tau) = \sum_{n \geq n_0} a_n x^n$. We call this the Fourier series of $f(\tau)$, and commit an abuse of language by also denoting it by $f(x)$. Clearly

$$\begin{aligned} Uf(x) &= \tfrac{1}{11} \sum_{h=0}^{10} f(e^{2\pi i h/11} x^{1/11}) \\ &= \tfrac{1}{11} \sum_{h=0}^{10} f\left(\frac{\tau + h}{11}\right). \end{aligned} \tag{11}$$

Hence (see for example [**1**, pp. 80–82]), if $f(x) \in K_0(121)$ then $Uf(x) \in K_0(11)$. It follows from this that all the functions $L_r(x)$ defined above are in $K_0(11)$. We need to know something about their orders at the cusps 0 and $i\infty$ of $R_0(11)$; to describe the relevant function theory we use the following notation. If $f(\tau) \in K_0(121)$ and P is a point of $R_0(121)$, we denote by $\mathrm{Ord}_P f(\tau)$ the order of $f(\tau)$ at P. If $g(\tau) \in K_0(11)$ and p is a point of $R_0(11)$, we denote by $\mathrm{ord}_p\, g(\tau)$ the order of $g(\tau)$ at p. We may think of a point of $R_0(N)$ as a place of $K_0(N)$, and use the language of valuation theory. In this terminology, the cusp ∞ of $R_0(11)$ splits completely into the cusps ∞ and $h/11$ $(1 \leq h \leq 10)$ of $R_0(121)$. On the other hand the cusp 0 of $R_0(11)$ ramifies completely in the covering $R_0(121)$, with ramification number $[K_0(121) : K_0(11)] = [\Gamma_0(11) : \Gamma_0(121)] = 11$. (These facts follow easily from an examination of the stabilizers in $\Gamma_0(11)$ and $\Gamma_0(121)$ of the relevant cusps.) Hence we have the following result.

Lemma 1. *Suppose $g(\tau) \in K_0(11)$. Then if $g(\tau)$ is regarded as an element of the extension field $K_0(121)$, we have*

$$\mathrm{Ord}_{h/11}\, g(\tau) = \mathrm{Ord}_\infty\, g(\tau) = \mathrm{ord}_\infty\, g(\tau) \qquad (1 \leq h \leq 10),$$

$$\mathrm{Ord}_0\, g(\tau) = 11\, \mathrm{ord}_0\, g(\tau).$$

The next result is a special case of Corollary 1.10 of [**5**].

LEMMA 2. *Suppose* $f(\tau)\in K_0(121)$. *Then* $g(\tau)=U(f(x))\in K_0(11)$, *and*

$$\mathrm{ord}_\infty g(\tau)\geq \tfrac{1}{11}\,\mathrm{Ord}_\infty f(\tau),$$

$$\mathrm{ord}_0 g(\tau)\geq \min_{0\leq h\leq 10} \mathrm{Ord}_{h/11} f(\tau).$$

If $f(\tau)$ *is holomorphic in the interior of the upper half plane, so is* $g(\tau)$.

Now (cf. [**6**]) the function $\phi(\tau)=\eta(121\tau)/\eta(\tau)\in K_0(121)$ has a zero of order 5 at ∞, a pole of multiplicity 5 at 0, and is holomorphic and non-zero elsewhere on $R_0(121)$. From this it follows by repeated application of Lemmas 1 and 2 that the functions $L_r(x)$ defined in (9) are holomorphic everywhere on $R_0(11)$ except at 0 and ∞. Let $\mathcal{V}$ be the vector space of functions $g(\tau)\in K_0(11)$ which are holomorphic except possibly at 0 and ∞; thus $L_r(x)\in\mathcal{V}$.

Atkin [**2**] has constructed a basis for $\mathcal{V}$ which we shall use with one small modification; however we find it convenient to use a different notation for the basis elements. Specifically, if $\nu\neq 0$ or -1 we denote by $J_\nu(\tau)$ the element of Atkin's basis whose order at ∞ is ν. We define $J_0(\tau)=1$ and $J_{-1}(\tau)=J_{-6}(\tau)J_5(\tau)$. In terms of Atkin's notation, $J_{-1}(\tau)=B(\tau)+12$. The following lemma lists some of the essential properties of the functions $J_\nu(\tau)$; these properties are all proved in [**2**].

LEMMA 3. *For all* $\nu\in\mathbb{Z}$, *we have:*

(i) $J_{\nu+5}(\tau)=J_\nu(\tau)J_5(\tau)$,

(ii) $\{J_\nu(\tau)\mid -\infty<\nu<\infty\}$ *is a basis of* $\mathcal{V}$,

(iii) $\mathrm{ord}_\infty J_\nu(\tau)=\nu$,

(iv) $$\mathrm{ord}_0 J_\nu(\tau)=\begin{cases}-\nu & \text{if } \nu\equiv 0 \pmod 5,\\ -\nu-1 & \text{if } \nu\equiv 1,\ 2 \text{ or } 3 \pmod 5,\\ -\nu-2 & \text{if } \nu\equiv 4 \pmod 5,\end{cases}$$

(v) *the Fourier series of* $J_\nu(\tau)$ *has integer coefficients, and is of the form* $J_\nu(x)=x^\nu+\dots$.

The space $\mathcal{V}$ is an algebra; in view of Lemma 3(i), the structure constants of the basis $\{J_\nu\}$ are completely determined by the following multiplication table.

	J_1	J_2	J_3	J_4
J_1	J_2+J_3	J_3+11J_5	J_4	$J_5+12J_6+11J_7$
J_2		J_4-J_5	J_5+11J_6	$J_6+11J_7+11J_8$
J_3			J_6+11J_7	$J_7+12J_8+11^2J_{10}$
J_4				$J_8+12J_9+11J_{10}+11^2J_{11}$

This is equivalent to Table 5 of [**2**].

We will also use the fact that

$$J_5(\tau)=(\eta(11\tau)/\eta(\tau))^{12}. \tag{12}$$

Finally, we denote the 11-adic order of an integer n by $\pi(n)$.

3. The key inequality. From the discussion of the preceding section, we know that the functions L_r are in the space $\mathcal{V}$, and that $\operatorname{ord}_\infty L_r \geq \mu_r$. Since $\operatorname{ord}_\infty J_\nu = \nu$, we can accordingly write

$$L_r = \sum_{\nu=\mu_r}^{\infty} a_{r,\nu} J_\nu. \tag{13}$$

From Lemmas 1 and 2, we see that $\mathcal{V}$ is mapped into itself by the linear transformation

$$T_\lambda : g(x) \to U(\phi(x)^\lambda g(x))$$

for any integer λ. Let $C^{(\lambda)} = (c_{\mu\nu}^{(\lambda)})$ be the matrix of T_λ with respect to the basis $\{J_\nu\}$ of $\mathcal{V}$ (where, following Atkin, we write the elements of $\mathcal{V}$ as row vectors and let matrices act on the right). Thus

$$U(\phi(x)^\lambda J_\mu(x)) = \sum_\nu c_{\mu\nu}^{(\lambda)} J_\nu(x). \tag{14}$$

Here and in the sequel, $\sum_\nu$ denotes a sum from $\nu=-\infty$ to $\nu=\infty$, in which only a finite number of terms are non-zero. By (9), the components $a_{r,\nu}$ satisfy the recurrence

$$a_{r,\nu} = \sum_{\mu=\mu_{r-1}}^{\infty} a_{r-1,\mu} c_{\mu,\nu}^{(\lambda_{r-1})}, \tag{15}$$

together with the initial conditions

$$\left.\begin{aligned} &a_{0,0}=1, \\ &a_{0,\nu}=0 \quad \text{if} \quad \nu\neq 0. \end{aligned}\right\} \tag{16}$$

Our plan is to show that the $a_{r,\nu}$ are integers, and that

$$a_{r,\nu} \equiv 0 \pmod{11^{\frac{1}{2}\alpha r}} \quad \text{for all } \nu.$$

In order to do this, we will show that the numbers $c_{\mu,\nu}^{(\lambda)}$ are integers, and obtain a lower bound for their 11-adic orders. Specifically, we will prove that

$$\pi(c_{\mu\nu}^{(\lambda)}) \geq [(11\nu-\mu-5\lambda+\delta)/10], \tag{17}$$

where $\delta=\delta(\mu,\nu)$ depends on the residues of μ and ν (mod 5) according to the following

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table.

$\mu \backslash \nu$	0	1	2	3	4
0	−1	8	7	6	15
1	0	9	8	2	11
2	1	10	4	3	12
3	2	6	5	4	13
4	3	7	6	5	9

In the case $\lambda = 0$, the existence of an inequality of the form (17) with δ "small and of irregular behavior" was conjectured by Atkin. From the table we see that $\delta \geq -1$ in all cases, and therefore

$$\pi(c_{\mu,\nu}^{(\lambda)}) \geq [(11\nu - \mu - 5\lambda - 1)/10]. \tag{18}$$

To work towards the proof of (17), we note first that by (12) and property (i), we have

$$\begin{aligned} J_{\mu+5}(x) &= J_5(x)J_\mu(x) \\ &= (\eta(11\tau)/\eta(\tau))^{12}J_\mu(x) \\ &= (\eta(11\tau)/\eta(121\tau))^{12}(\eta(121\tau)/\eta(\tau))^{12}J_\mu(x) \\ &= J_5(x^{11})^{-1}\phi(x)^{12}J_\mu(x). \end{aligned}$$

Hence

$$\begin{aligned} U(J_{\mu+5}(x)\phi(x)^\lambda) &= U(J_5(x^{11})^{-1}\phi(x)^{\lambda+12}J_\mu(x)) \\ &= J_5(x)^{-1}U(\phi(x)^{\lambda+12}J_\mu(x)), \end{aligned}$$

where we have used the obvious fact that $U(f(x^{11})g(x)) = f(x)U(g(x))$. By virtue of (14), this implies that

$$\begin{aligned} \sum_\nu c_{\mu+5,\nu}^{(\lambda)} J_\nu &= \sum_\nu c_{\mu,\nu}^{(\lambda+12)} J_\nu J_5^{-1} \\ &= \sum_\nu c_{\mu,\nu}^{(\lambda+12)} J_{\nu-5} \\ &= \sum_\nu c_{\mu,\nu+5}^{(\lambda+12)} J_\nu. \end{aligned}$$

Since the J_ν form a basis of $\mathcal{V}$, we can equate coefficients and obtain $c_{\mu+5,\nu}^{(\lambda)} = c_{\mu,\nu+5}^{(\lambda+12)}$, or equivalently

$$c_{\mu-5,\nu+5}^{(\lambda+12)} = c_{\mu,\nu}^{(\lambda)}. \tag{19}$$

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The next step is to obtain another relation between the numbers $c_{\mu,\nu}^{(\lambda)}$; in this one μ will be fixed. Let $t=\phi(x^{1/11})$; then as shown in [4], t satisfies an algebraic equation of degree 11 with coefficients in $\mathcal{V}$. In the present notation this equation is

$$t^{11}-\sigma_1 t^{10}+\sigma_2 t^9+\sigma_3 t^8+\sigma_4 t^7-\sigma_5 t^6+\sigma_6 t^5-\sigma_7 t^4+\sigma_8 t^3-\sigma_9 t^2+\sigma_{10} t-\sigma_{11}=0, \quad (20)$$

where

$$\begin{aligned}
\sigma_1 &= 11^2 J_1+2\,.\,11^3 J_2+11^4 J_3+11^5 J_5,\\
\sigma_2 &= -11\,J_1+8\,.\,11^2 J_2+9\,.\,11^3 J_3+11^5 J_5,\\
\sigma_3 &= 2\,.\,11^2 J_2+4\,.\,11^3 J_3+5\,.\,11^4 J_5,\\
\sigma_4 &= 2\,.\,11\,J_2+12\,.\,11^2 J_3+11^4 J_5,\\
\sigma_5 &= 2\,.\,11^2 J_3-11^3 J_5,\\
\sigma_6 &= 2\,.\,11\,J_3-11^3 J_5,\\
\sigma_7 &= -11^2 J_5,\\
\sigma_8 &= 11^2 J_5,\\
\sigma_9 &= 5\,.\,11 J_5,\\
\sigma_{10} &= 11 J_5,\\
\sigma_{11} &= J_5.
\end{aligned} \quad (21)$$

The roots of (20) are $\phi(e^{2\pi ih/11}x^{1/11})$, $0\leq h\leq 10$. In (20) we substitute $t=\phi(e^{2\pi ih/11}x^{1/11})$ and multiply through by

$$J_\mu(e^{2\pi ih/11}x^{1/11})\phi(e^{2\pi ih/11}x^{1/11})^{\lambda-11} \qquad (0\leq h\leq 10).$$

We then sum over all these h; in view of (11) this yields

$$U(J_\mu(x)\phi(x)^\lambda)=\sum_{i=1}^{11}(-1)^{i-1}\sigma_i(x)U(J_\mu(x)\phi(x)^{\lambda-i}).$$

Hence from (14),

$$\sum_\nu c_{\mu\nu}^{(\lambda)}J_\nu=\sum_{i=1}^{11}(-1)^{i-1}\sigma_i\sum_\nu c_{\mu\nu}^{(\lambda-i)}J_\nu. \quad (22)$$

We next replace the σ_i by their expressions (21) in terms of the J_ν. Using the multiplication table for the J_ν displayed in Section 2, we can then express the right side of (22) as a linear combination $\sum_\nu d_\nu J_\nu$, after which we can equate $c_{\mu\nu}^{(\lambda)}$ with d_ν. The details of this are rather tedious; the final recurrence obtained depends on the residue class of ν (mod 5). To

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save space we give only the case $\nu \equiv 0 \pmod 5$ in full:

$$\begin{aligned} c_{\mu,\nu}^{(\lambda)} = {} & 11^2 c_{\mu,\nu-1}^{(\lambda-1)} + 2 \cdot 11^3 c_{\mu,\nu-2}^{(\lambda-1)} + 10 \cdot 11^3 c_{\mu,\nu-3}^{(\lambda-1)} + 2 \cdot 11^4 c_{\mu,\nu-4}^{(\lambda-1)} \\ & + 11^5 c_{\mu,\nu-5}^{(\lambda-1)} + 11^6 c_{\mu,\nu-6}^{(\lambda-1)} + 11 c_{\mu,\nu-1}^{(\lambda-2)} - 8 \cdot 11^2 c_{\mu,\nu-2}^{(\lambda-2)} \\ & - 90 \cdot 11^2 c_{\mu,\nu-3}^{(\lambda-2)} - 8 \cdot 11^3 c_{\mu,\nu-4}^{(\lambda-2)} - 11^5 c_{\mu,\nu-5}^{(\lambda-2)} - 9 \cdot 11^5 c_{\mu,\nu-6}^{(\lambda-2)} \\ & + 2 \cdot 11^2 c_{\mu,\nu-2}^{(\lambda-3)} + 42 \cdot 11^2 c_{\mu,\nu-3}^{(\lambda-3)} + 2 \cdot 11^3 c_{\mu,\nu-4}^{(\lambda-3)} + 5 \cdot 11^4 c_{\mu,\nu-5}^{(\lambda-3)} \\ & + 4 \cdot 11^5 c_{\mu,\nu-6}^{(\lambda-3)} - 2 \cdot 11 c_{\mu,\nu-2}^{(\lambda-4)} - 130 \cdot 11 c_{\mu,\nu-3}^{(\lambda-4)} - 2 \cdot 11^2 c_{\mu,\nu-4}^{(\lambda-4)} \\ & - 11^4 c_{\mu,\nu-5}^{(\lambda-4)} - 12 \cdot 11^4 c_{\mu,\nu-6}^{(\lambda-4)} + 2 \cdot 11^2 c_{\mu,\nu-3}^{(\lambda-5)} - 11^3 c_{\mu,\nu-5}^{(\lambda-5)} \\ & + 2 \cdot 11^4 c_{\mu,\nu-6}^{(\lambda-5)} - 2 \cdot 11 c_{\mu,\nu-3}^{(\lambda-6)} + 11^3 c_{\mu,\nu-5}^{(\lambda-6)} - 2 \cdot 11^3 c_{\mu,\nu-6}^{(\lambda-6)} \\ & - 11^2 c_{\mu,\nu-5}^{(\lambda-7)} - 11^2 c_{\mu,\nu-5}^{(\lambda-8)} + 5 \cdot 11 c_{\mu,\nu-5}^{(\lambda-9)} - 11 c_{\mu,\nu-5}^{(\lambda-10)} + c_{\mu,\nu-5}^{(\lambda-11)}. \end{aligned} \tag{23}$$

Suppose we can prove that for some fixed μ, the numbers $c_{\mu\nu}^{(\lambda)}$ $(-10 \le \lambda \le 0)$ are integers. Then from (23) it follows that for $\nu \equiv 0 \pmod 5$, the $c_{\mu\nu}^{(\lambda)}$ are integers for all $\lambda > 0$ as well. Moreover they satisfy the inequality

$$\pi(c_{\mu\nu}^{(\lambda)}) \ge \min_{\substack{1 \le \rho \le 11 \\ 1 \le \sigma \le 6}} (\pi(c_{\mu,\nu-6}^{(\lambda-\rho)}) + e_\sigma^{(\rho)}), \tag{24}$$

where the numbers $e_\sigma^{(\rho)}$ are given by the following table, in which the blank entries are ∞.

$\sigma \backslash \rho$	1	2	3	4	5	6	7	8	9	10	11
1	2	1									
2	3	2	2	1							
3	3	2	2	1	2	1					
4	4	3	3	2							
5	5	5	4	4	3	3	2	2	1	1	0
6	6	5	5	4	4	3					

$\nu \equiv 0 \pmod 5$

For completeness we give the numbers $e_\sigma^{(\rho)}$ appearing in the analogues of (23) for the other residue classes of $\nu \pmod 5$.

$\sigma \backslash \rho$	1	2	3	4	5	6	7	8	9	10	11
1	2	1									
2	2	1	2	1							
3	4	3	3	2	2	1					
4	5	4	4	3	3	2					
5	5	5	4	4	3	3	2	2	1	1	0

$\nu \equiv 1 \pmod 5$

$\sigma \backslash \rho$	1	2	3	4	5	6	7	8	9	10	11
1	2	1									
2	3	2	2	1							
3	3	2	3	2	2	1					
4	5	4	4	3	3	2					
5	5	5	4	4	3	3	2	2	1	1	0

$\nu \equiv 2 \pmod 5$

$\sigma \backslash \rho$	1	2	3	4	5	6	7	8	9	10	11
1	2	2									
2	2	1	2	1							
3	4	3	3	2	2	1					
4	4	3	3	2	2	1					
5	5	5	4	4	3	3	2	2	1	1	0

$\nu \equiv 3 \pmod 5$

$\sigma \backslash \rho$	1	2	3	4	5	6	7	8	9	10	11
1	2	1									
2	3	2	2	1							
3	4	3	3	2	2	1					
4											
5	5	5	4	4	3	3	2	2	1	1	0

$\nu \equiv 4 \pmod 5$

It is now a straightforward though tiresome exercise to verify by induction on λ that if (17) holds for some fixed μ and all λ with $-10 \le \lambda \le 0$, it also holds for that μ and all $\lambda > 0$. In the same way we can solve (23) and its analogues with $\nu \equiv 1, 2, 3, 4 \pmod 5$ for $c^{(\lambda-11)}_{\mu,\nu-5}$, and show that (17) holds for $\lambda < -10$.

From (19) we see that $c^{(\lambda)}_{\mu,\nu}$ is invariant under the map $\lambda \to \lambda + 12$, $\mu \to \mu - 5$, $\nu \to \nu + 5$. The right side of (17) is also invariant under this map. From this fact, together with the above remarks, we see that it suffices to prove that the $c^{(\lambda)}_{\mu\nu}$ are integers satisfying (17) for $-10 \le \lambda \le 0$ and five values of μ, one in each residue class (mod 5). For ease of computation we do this for $-4 \le \mu \le 0$. The case $\mu = 0$ is somewhat special; evidently it involves computing $U(\phi^\lambda)$ for $-10 \le \lambda \le 0$. This can be done by applying Newton's identities for power sums to the reciprocal equation of the modular equation (20). Alternatively, it can also be done by the method described below, and then the Newton identities can be used to give a new derivation of (20). We will discuss only the case $\mu = -2$ in detail, as the other cases are quite analogous. Consider then the function $\phi^\lambda J_{-2}$ on the Riemann surface $R_0(121)$. As already noted, $\mathrm{Ord}_0\, \phi = -5$, $\mathrm{Ord}_\infty\, \phi = 5$, and $\mathrm{Ord}_P\, \phi = 0$ for all other points P of $R_0(121)$. By Lemma 3, we have $\mathrm{ord}_0\, J_{-2} = 1$, $\mathrm{ord}_\infty\, J_{-2} = -2$, and $\mathrm{ord}_p\, J_{-2} \ge 0$ for all other points p of $R_0(11)$. Hence by Lemma 1, on the surface $R_0(121)$ we have $\mathrm{Ord}_0\, J_{-2} = 11$, $\mathrm{Ord}_\infty\, J_{-2} = \mathrm{Ord}_{h/11}\, J_{-2} = -2$ $(1 \le h \le 10)$, and $\mathrm{Ord}_P\, J_{-2} \ge 0$ for all other points P. Since $\mathrm{Ord}_P(fg) = \mathrm{Ord}_P\, f + \mathrm{Ord}_P\, g$, it follows that $\mathrm{Ord}_0\, \phi^\lambda J_{-2} = -5\lambda + 11$, $\mathrm{Ord}_\infty\, \phi^\lambda J_{-2} = 5\lambda - 2$, $\mathrm{Ord}_{h/11}\phi^\lambda J_{-2} = -2$ $(1 \le h \le 10)$, and $\mathrm{Ord}_P(\phi^\lambda J_{-2}) \ge 0$ for all other $P \in R_0(121)$. By Lemma 2, this implies that $\mathrm{ord}_0\, U(\phi^\lambda J_{-2}) \ge \min(-5\lambda + 11, -2)$, $\mathrm{ord}_\infty\, U(\phi^\lambda J_{-2}) \ge \lceil (5\lambda - 2)/11 \rceil$, and $\mathrm{ord}_p\, U(\phi^\lambda J_{-2}) \ge 0$ for all other points $p \in R_0(11)$. In particular, if $\lambda \le 0$, we have $\mathrm{ord}_0\, U(\phi^\lambda J_{-2}) \ge -2$, $\mathrm{ord}_\infty\, U(\phi^\lambda J^{-2}) \ge \lceil (5\lambda - 2)/11 \rceil$, and $\mathrm{ord}_p\, U(\phi^\lambda J_{-2}) \ge 0$ for all other points $p \in R_0(11)$. By Lemma 3(iv), J_1 has a double pole at 0, while for all $\nu \ge 2$, J_ν has a pole of multiplicity at least 3 at 0. Therefore if $\lambda \le 0$, all the coefficients $c^{(\lambda)}_{-2,\nu}$ in the equation

$$U(\phi^\lambda J_{-2}) = \sum_\nu c^{(\lambda)}_{-2,\nu} J_\nu, \tag{25}$$

vanish if either $\nu < \lceil (5\lambda - 2)/11 \rceil$ or $\nu \ge 2$. Hence (25) has the following form for $\lambda \le 0$:

$$U(\phi^\lambda J_{-2}) = \sum_{\nu = \lceil (5\lambda-2)/11 \rceil}^{1} c^{(\lambda)}_{-2,\nu} J_\nu. \tag{26}$$

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Thus the number of non-zero coefficients $c^{(\lambda)}_{-2,\nu}$ for $\lambda \le 0$ is at most $2-\lceil(5\lambda-2)/11\rceil$, which in the range $-10 \le \lambda \le 0$ under consideration is at most 6. It is accordingly quite easy to find $c^{(\lambda)}_{-2,\nu}$ for any fixed λ in this range by equating the first $2-\lceil(5\lambda-2)/11\rceil$ coefficients in the Fourier series of the two sides of (26). This leads to a system of linear equations for $c^{(\lambda)}_{-2,\nu}$ whose matrix is triangular, since $\mathrm{ord}_\infty J_\nu = \nu$. In this way the following expressions are obtained:

$$\begin{aligned}
U(\phi^{-10}J_{-2}) &= 11^6 J_1,\\
U(\phi^{-9}J_{-2}) &= 155J_{-4} + 666\,.\,11J_{-3} + 3\,.\,11^4J_{-2} + 14\,.\,11^4J_{-1} + 28\,.\,11^4J_0 + 11^6J_1,\\
U(\phi^{-8}J_{-2}) &= -11^5J_1,\\
U(\phi^{-7}J_{-2}) &= -230J_{-3} - 168\,.\,11J_{-2} - 24\,.\,11^3J_0 - 11^5J_1,\\
U(\phi^{-6}J_{-2}) &= 11^4J_1,\\
U(\phi^{-5}J_{-2}) &= -170J_{-2} + 11^4J_1,\\
U(\phi^{-4}J_{-2}) &= J_{-2} + 2\,.\,11^2J_{-1} + 12\,.\,11^2J_0 + 10\,.\,11^3J_1,\\
U(\phi^{-3}J_{-2}) &= 2\,.\,11J_{-1} + 28\,.\,11J_0 + 11^3J_1,\\
U(\phi^{-2}J_{-2}) &= 11^2J_1,\\
U(\phi^{-1}J_{-2}) &= -30 - 11^2J_1,\\
U(J_{-2}) &= -12 - 11J_1.
\end{aligned} \tag{27}$$

It is now trivial to check that (17) holds if $\mu = -2$ and $-10 \le \lambda \le 0$. As explained above, this implies its validity for $\mu = -2$, λ arbitrary. The cases $\mu = 0, -1, -3, -4$ are dealt with similarly.

Some of the initial conditions (27) and their analogues for the other values of μ are rather interesting identities. We mention for example the complete results for $\lambda = -10$:

$$\begin{aligned}
U(\phi^{-10}) &= 11^4,\\
U(J_{-1}\phi^{-10}) &= 11^4J_{-1},\\
U(J_{-2}\phi^{-10}) &= 11^6J_1,\\
U(J_{-3}\phi^{-10}) &= 11^7J_2,\\
U(J_{-4}\phi^{-10}) &= 11^8J_3.
\end{aligned} \tag{28}$$

To return to the main problem, we recall from (14) that

$$U(\phi^\lambda J_\mu) = \sum_\nu c^{(\lambda)}_{\mu\nu}J_\nu.$$

It follows from Lemma 3(v) that the Fourier series of $U(\phi^\lambda J_\mu)$ has all coefficients divisible by 11 if and only if

$$c^{(\lambda)}_{\mu\nu} \equiv 0 \pmod{11} \quad \text{for all } \nu. \tag{29}$$

If (29) holds we put $\theta(\lambda, \mu) = 1$; otherwise we put $\theta(\lambda, \mu) = 0$. From the recurrence (23) we find that

$$c_{\mu,\nu}^{(\lambda)} \equiv c_{\mu,\nu-5}^{(\lambda-11)} \pmod{11} \quad \text{for} \quad \nu \equiv 0 \pmod 5.$$

The analogous recurrences for the other residue classes of ν (mod 5) yield this same congruence. As a result we have

$$\theta(\lambda, \mu) = \theta(\lambda - 11, \mu). \tag{30}$$

On the other hand, we see from (19) that

$$\theta(\lambda + 12, \mu - 5) = \theta(\lambda, \mu). \tag{31}$$

Equations (30) and (31) show that the function $\theta(\lambda, \mu)$ is completely determined by its values in the range $0 \leq \lambda \leq 10$, $0 \leq \mu \leq 4$. These values can be read off from (27) and its analogue for the other μ. They are listed below.

$\mu \backslash \lambda$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	0	1	0	1	0	1	1	0	0
1	1	1	0	1	0	0	0	1	1	0	0
2	1	1	1	0	0	0	0	1	1	0	0
3	1	0	1	0	1	0	0	1	1	0	0
4	1	0	1	0	1	0	1	1	0	0	0

4. Proof of Theorem 2. For any Laurent series

$$f(x) = \sum_{n \geq n_0} c_n x^n$$

with integer coefficients, define

$$\pi(f) = \min_{n \geq n_0} \pi(c_n).$$

Put $A_0 = 0$, and

$$A_r = \sum_{i=0}^{r-1} \theta(\lambda_i, \mu_i) \tag{32}$$

for $r \geq 1$. We will prove by induction on r that

$$\pi(L_r) \geq A_r. \tag{33}$$

In view of (13) and Lemma 3(v), it suffices to prove that

$$\pi(a_{r,\nu}) \geq A_r, \quad \text{for all} \quad \nu \geq \mu_r. \tag{34}$$

To facilitate the induction, it is convenient to prove an inequality stronger than (34), viz

$$\pi(a_{r,\nu}) \geq A_r + [(\nu - \mu_r)/2] \quad \text{for} \quad \nu \geq \mu_r. \tag{35}$$

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From (16) we see that (35) holds for $r=0$. Suppose $r\geq 1$, and that

$$\pi(a_{r-1,\nu})\geq A_{r-1}+[(\nu-\mu_{r-1})/2] \quad \text{for} \quad \nu\geq\mu_{r-1} \tag{36}$$

has been shown to hold. From the recurrence (15) we have

$$\pi(a_{r,\nu})\geq \min_{\mu\geq\mu_{r-1}}(\pi(a_{r-1,\mu})+\pi(c_{\mu\nu}^{(\lambda_{r-1})})).$$

Thus in order to complete the induction, it suffices to prove that

$$\pi(a_{r-1,\mu})+\pi(c_{\mu\nu}^{(\lambda_{r-1})})\geq A_r+[(\nu-\mu_r)/2] \quad \text{for all} \quad \mu\geq\mu_{r-1},\ \nu\geq\mu_r. \tag{37}$$

First suppose $\nu\geq\mu_r+2$. Then by (18) and the induction hypothesis (36), the left side of (37) is at least equal to

$$A_{r-1}+[(\mu-\mu_{r-1})/2]+[(11\nu-\mu-5\lambda_{r-1}-1)/10].$$

This expression cannot decrease if μ is increased by 2, so its minimum occurs when $\mu=\mu_{r-1}+1$, and it is therefore at least equal to

$$A_{r-1}+[(11\nu-\mu_{r-1}-5\lambda_{r-1}-2)/10]. \tag{38}$$

Since $\mu_r=\lceil(5\lambda_{r-1}+\mu_{r-1})/11\rceil$, we have $\mu_r\geq(5\lambda_{r-1}+\mu_{r-1})/11$; so (38) is at least equal to

$$A_{r-1}+[(11\nu-11\mu_r-2)/10]=A_{r-1}+1+[(11(\nu-\mu_r)-12)/10].$$

This in turn is at least $A_r+[(\nu-\mu_r)/2]$ in view of the obvious inequalities $A_{r-1}+1\geq A_r$ and $[(11n-12)/10]\geq[n/2]$ for all integers $n\geq 2$. Thus (37) is proved for $\nu\geq\mu_r+2$.

Next suppose $\nu=\mu_r$ or μ_r+1. Thus (37) reduces to

$$\pi(a_{r-1,\mu})+\pi(c_{\mu\nu}^{(\lambda_{r-1})})\geq A_r. \tag{39}$$

This inequality holds for $\mu=\mu_{r-1}$ since $\pi(a_{r-1,\mu_{r-1}})\geq A_{r-1}$ and $\pi(c_{\mu_{r-1},\nu}^{(\lambda_{r-1})})\geq\theta(\lambda_{r-1},\mu_{r-1})$. It also holds for $\mu\geq\mu_{r-1}+2$, since by the induction hypothesis (36), we have

$$\pi(a_{r-1,\mu})\geq A_{r-1}+[(\mu-\mu_r)/2]\geq A_{r-1}+1\geq A_r. \tag{40}$$

This leaves only the case $\mu=\mu_{r-1}+1$ to be disposed of. In this case we have

$$\pi(a_{r-1,\mu_{r-1}+1})+\pi(c_{\mu_{r-1}+1,\nu}^{(\lambda_{r-1})})\geq A_{r-1}+[(11\nu-(\mu_{r-1}+1)-5\lambda_{r-1}+\delta)/10],$$

where $\delta=\delta(\mu_{r-1}+1,\nu)$ is the function tabulated after (17). Since $\nu=\mu_r$ or μ_r+1, in order to prove (39) it suffices to show that

$$[(11\mu_r-\mu_{r-1}-5\lambda_{r-1}-1+\delta(\mu_{r-1}+1,\mu_r))/10]\geq 1 \tag{41}$$

and

$$[(11(\mu_r+1)-\mu_{r-1}-5\lambda_{r-1}-1+\delta(\mu_{r-1}+1,\mu_r+1))/10]\geq 1 \tag{42}$$

whenever $\theta(\lambda_{r-1},\mu_{r-1})=1$. By (30) and (31), the function $\theta(\lambda_{r-1},\mu_{r-1})$ is invariant under each of the maps

$$\lambda_{r-1}\to\lambda_{r-1}+11, \qquad \mu_{r-1}\to\mu_{r-1}$$

and

$$\lambda_{r-1}\to\lambda_{r-1}+12, \qquad \mu_{r-1}\to\mu_{r-1}-5.$$

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Under these maps, $\mu_r = \lceil (5\lambda_{r-1} + \mu_{r-1})/11 \rceil$ is increased by 5, and so the quantities on the left sides of (41) and (42) are unchanged. (Recall that $\delta(\mu, \nu)$ has period 5 in its arguments.) Hence it suffices to prove that (41) and (42) hold under the hypotheses that $0 \le \lambda_{r-1} \le 10$, $0 \le \mu_{r-1} \le 4$, and $\theta(\lambda_{r-1}, \mu_{r-1}) = 1$. There are 25 cases where these conditions are satisfied, and a straightforward check of the table of $\delta(\mu, \nu)$ shows that (39) holds in all of them. This completes the proof of (37) and hence of (35).

By (2), (7) and (8), we have

$$\begin{aligned} A_r &= \sum_{i=0}^{r-1} \theta(\lambda_i, \mu_i) \\ &= \sum_{0 \le i \le \log_{11}|k|} \theta(\lambda_i, \mu_i) + N_1 \theta\left(0, \left\lceil \frac{11k}{24} \right\rceil + \omega(k)\right) + N_2 \theta\left(k, \left\lceil \frac{k}{24} \right\rceil + \omega(k)\right), \end{aligned} \tag{42}$$

where N_1 is the number of odd integers i in the interval $\log_{11}|k| < i \le r-1$, and N_2 is the number of even integers in this interval. If $r \le \log_{11}|k| + 1$ then $N_1 = N_2 = 0$, and (42) yields

$$A_r \le \log_{11}|k| + 1. \tag{43}$$

If $r > \log_{11}|k| + 1$ then

$$|N_1 - \tfrac{1}{2}(r - 1 - \log_{11}|k|)| + |N_2 - \tfrac{1}{2}(r - 1 - \log_{11}|k|)| < 1.$$

Hence if we set

$$\alpha = \theta(0, \lceil 11k/24 \rceil + \omega(k)) + \theta(k, \lceil k/24 \rceil + \omega(k)), \tag{44}$$

we see from (42) that

$$|A_r - \tfrac{1}{2}\alpha(r - 1 - \log_{11}|k|)| < \log_{11}|k| + 2.$$

Therefore

$$|A_r - \tfrac{1}{2}\alpha r| < 2 + \tfrac{1}{2}\alpha + (1 + \tfrac{1}{2}\alpha)\log_{11}|k|. \tag{45}$$

Regardless of whether (43) or (45) holds, we have $A_r = \frac{1}{2}\alpha r + O(\log|k|)$. In view of (33), this proves that if $24n \equiv k \pmod{11^r}$ then $p_{-k}(n) \equiv 0 \pmod{11^{\frac{1}{2}\alpha r + \varepsilon}}$, where $\varepsilon = \varepsilon(k) = O(\log|k|)$, and where α is defined by (44). We now have to show that α is a periodic function of k with period 120. For definiteness we do this for the case where k is not a negative multiple of 24, so that $\omega(k) = 0$. The case $\omega(k) = 1$ is handled similarly. If k is increased by 120, $\lceil 11k/24 \rceil$ is increased by 55, and so by (30) and (31), $\theta(0, \lceil 11k/24 \rceil)$ does not change. On the other hand, $\theta(k, \lceil k/24 \rceil)$ is replaced by $\theta(k + 120, \lceil k/24 \rceil + 5)$ which by (31) is equal to $\theta(k + 132, \lceil k/24 \rceil)$. Since $132 \equiv 0 \pmod{11}$, this is in turn equal to $\theta(k, \lceil k/24 \rceil)$ by (30). Thus $\alpha(k + 120) = \alpha(k)$.

The final task is now to compute $\alpha(k)$ from (44) for $1 \le k \le 120$, and for $k = -24$, -48, -72, -96, -120. This yields the table described in the statement of Theorem 2, and the proof is complete.

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5. Best possible nature of Theorem 2. It is clear that if

$$\pi(L_r) = A_r \quad \text{for all } r \tag{46}$$

then the constant α defined by (44) cannot be improved. Since

$$L_r = \sum_{\nu=\mu_r} a_{r,\nu} J_\nu,$$

and the Fourier series of J_ν has the form $x^\nu + \dots$, we have

$$\pi(L_r) = \min_{\nu \geq \mu_r} a_{r,\nu}.$$

Now (46) certainly holds if

$$\left.\begin{array}{lll} \pi(c^{(\lambda_r)}_{\mu_r\nu}) = \theta(\lambda_r, \mu_r) & \text{for} & \nu = \lceil (5\lambda_r + \mu_r)/11 \rceil, \\ \pi(c^{(\lambda_r)}_{\mu_r\nu}) \geq \theta(\lambda_r, \mu_r) + 2 & \text{for} & \nu > \lceil (5\lambda_r + \mu_r)/11 \rceil \quad \text{for all } r. \end{array}\right\} \tag{47}$$

Indeed it is easily proved by induction on r that (47) implies

$$\pi(a_{r,\mu_r}) = A_r,$$
$$\pi(a_{r,\nu}) \geq A_r + 2 \quad \text{for all} \quad \nu > \mu_r.$$

To determine completely the set S of all pairs (λ, μ) for which

$$\pi(c^{(\lambda)}_{\mu\nu}) = \theta(\lambda, \theta) \qquad \text{for} \quad \nu = \lceil (5\lambda + \mu)/11 \rceil,$$
$$\pi(c^{(\lambda)}_{\mu\nu}) \geq \theta(\lambda, \mu) + 2 \quad \text{for} \quad \nu > \lceil (5\lambda + \mu)/11 \rceil$$

requires a study of $c^{(\lambda)}_{\mu\nu} \pmod{11^3}$. However, a study $\pmod{11^2}$ shows that S contains at least the following pairs, which are enough for our purposes:

$(\lambda, 0)$ for $\lambda \equiv 5, 6, 9, 10 \pmod{11}$ but $\lambda \not\equiv -6 \pmod{11^2}$,

$(\lambda, 1)$ for $\lambda \equiv 5, 6, 8, 9, 10 \pmod{11}$ but $\lambda \not\equiv -14 \pmod{11^2}$,

$(\lambda, 2)$ for $\lambda \equiv 5, 6, 9, 10 \pmod{11}$,

$(\lambda, 3)$ for $\lambda \equiv 4, 5, 6, 9 \pmod{11}$ but $\lambda \not\equiv -18 \pmod{11^2}$,

$(\lambda, 4)$ for $\lambda \equiv 5, 9 \pmod{11}$.

Moreover, by (19); S is invariant under the map $(\lambda, \mu) \to (\lambda + 12, \mu - 5)$. It is now a simple though lengthy matter to find in each residue class (mod 120) both positive and negative values of k for which all the pairs (λ_r, μ_r) belong to S, and for which the constant α in Theorem 2 can therefore not be improved.

6. Conclusion. Although our proof of Theorem 2 contains elements from both [**2**] and [**3**], it is mostly in the spirit of [**3**]. In particular we have totally avoided the canonical involution $\tau \to -1/11\tau$ of $R_0(11)$, and have made almost all our calculations at the point ∞ of $R_0(11)$, around which the U-operator is in a sense built. Nevertheless it is quite clear that the crucial inequality (17) is really a consequence of the 11-adic behavior of the

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coefficients of the Puiseux expansions about $\tau=0$ of the roots $\phi((\tau+h)/11)$ of the modular equation (20). A direct attack on these coefficients (which are algebraic numbers, but in general irrational for $1 \leq h \leq 10$) would be desirable, since it might obviate the numerical computations used in the above proof of (17), and ease the development of an analogous theory for larger primes.

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