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IDEMPOTENTS IN NOETHERIAN GROUP RINGS

EDWARD FORMANEK

If G is a torsion-free group and F is a field, is the group ring $F[G]$ a ring without zero divisors? This is true if G is an ordered group or various generalizations thereof – beyond this the question remains untouched. This paper proves a related result.

THEOREM 1. *If G is a torsion-free Noetherian group and F is a field of characteristic* 0, *then $F[G]$ has no idempotents except* 0 *and* 1.

This theorem is far from an answer to the zero-divisor question for Noetherian groups since the conclusion that $F[G]$ has no nontrivial idempotents is much weaker (however, see the remark at the end of the paper). But the step to the class of torsion-free Noetherian groups is a real one since such groups cannot in general be ordered and the proof of Theorem 1 is completely different from the usual "order" argument. It is related to and based in part on the following, where the trace of $r = \sum r_g g \in F[G]$ is

$$\operatorname{tr}(r) = r_1 = \text{coefficient of the unit of } G \text{ in } r.$$

THEOREM 2 (Kaplansky [**2**, p. 123]; cf. Montgomery [**5**], or Passman [**6**]). *If F is a field of characteristic* 0, *G is any group, and $e \neq 0, 1$ is an idempotent in $F[G]$, then $\operatorname{tr}(e)$ is a real number strictly between* 0 *and* 1.

The situation for arbitrary characteristic has recently been settled by the next result, which answers a question posed by Kaplansky [**4**].

THEOREM 3 (Zalesskii [**7**]). *If F is a field (any characteristic) and e is an idempotent in $F[G]$, then $\operatorname{tr}(e)$ lies in the prime subfield of F.*

While we do not use Theorem 3, we mention it because the arguments used here are derived from Zalesskii's elegant proof. He first handles the characteristic $p \neq 0$ case with a generalized trace and then deduces the characteristic 0 case by a kind of specialization.

1. Noetherian groups. A group G is Noetherian if every properly ascending chain of subgroups of G is finite. Finite extensions of polycyclic groups are Noetherian and these are the only known examples of Noetherian groups.

LEMMA 4. *Let G be a Noetherian group and g an element of G of infinite order. Then for any $n \geqq 2$, g is not conjugate to g^n.*

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Proof. Suppose conversely that $xgx^{-1} = g^n$ where $n \geqq 2$. For $i \geqq 0$ let $H_i = gp\langle x^{-i}gx^i\rangle$.

$$(x^{-i-1}gx^{i+1})^n = x^{-i-1}g^n x^{i+1} = x^{-i}gx^i,$$

so $H_0 \subset H_1 \subset H_2 \ldots$ is a properly ascending chain of infinite cyclic subgroups of G.

2. A modification of Zalesskii's trace argument. For this section, the following data is fixed:

K is a field of characteristic $p \neq 0$;
G is a torsion-free Noetherian group;
$\{x\}$ is a conjugacy class of G, with $x \neq 1$.

For $g \in G$, let

$$\theta(g) = \begin{cases} p^i & \text{if } i \text{ is the least integer} \geqq 0 \text{ for which } g^{p^i} \in \{x\}, \\ 0 & \text{if } g^{p^i} \notin \{x\} \text{ for all } i \geqq 0. \end{cases}$$

For $i = 0, 1, 2 \ldots$ define K-linear maps $T_i : K[G] \to K$ by

$$T_i(\sum r_g g) = \sum \{r_g : \theta(g) = p^i\}.$$

LEMMA 5. *Let L denote the K-submodule of $K[G]$ generated by all differences $ab - ba$ for $a, b \in K[G]$. Then $T_i(z) = 0$ for all $z \in L$.*

Proof. This follows from the K-linearity of T_i and the fact that $\theta(gh) = \theta(hg)$ for all $g, h \in G$.

LEMMA 6. *For any $r \in K[G]$ and any $i \geqq 0$, $T_i(r^p) = [T_{i+1}(r)]^p$.*

Proof. Let $r = \sum r_g g$. Two observations are needed. First, because we are in characteristic p,

$$r^p = (\sum r_g g)^p = \sum r_g{}^p g^p + z,$$

where $z \in L$. Second, because Lemma 4 precludes the possibility that both $g \in \{x\}$ and $g^{p^{i+1}} \in \{x\}$,

$$\theta(g^p) = p^i \text{ if and only if } \theta(g) = p^{i+1}.$$

Therefore,

$$\begin{aligned} T_i(r^p) &= T_i(\sum r_g{}^p g^p + z) \\ &= \sum \{r_g{}^p : \theta(g^p) = p^i\} \\ &= [\sum \{r_g : \theta(g) = p^{i+1}\}]^p \\ &= [T_{i+1}(r)]^p. \end{aligned}$$

LEMMA 7. *If e is an idempotent of $K[G]$, then $T_0(e) = 0$.*

Proof. Only finitely many group elements occur in e, so for a large enough i there are no group elements in e for which $\theta(g) = p^i$, and then $T_i(e) = 0$.

But $e^p = e$, so Lemma 6 says that

$$T_0(e) = [T_1(e)]^p, T_1(e) = [T_2(e)]^p, \ldots, T_{i-1}(e) = [T_i(e)]^p.$$

Therefore,

$$0 = T_i(e) = T_{i-1}(e) = \ldots = T_0(e).$$

3. Proof of Theorem 1. We need a result from commutative algebra.

LEMMA 8. *Let R be a commutative domain with unit, finitely generated as a ring. Then the Jacobson radical of R is* 0 [**3**, p. 18, Theorem 30–31] *and for each maximal ideal M of R, R/M is a finite field* [**1**, p. 84, Problem 6].

Now suppose that $e = \sum e_g g$ is an idempotent of $F[G]$, where F is a field of characteristic 0 and G is a torsion-free Noetherian group. Let R be the subring (with unit) of F generated by the set of e_g. For any $x \in G$, $x \neq 1$, let

$$T_x(e) = \sum [e_g : g \in \{x\}].$$

Then $T_x(e) = 0$. If not, since R has Jacobson radical 0 (Lemma 8) there is a maximal ideal M of R with $T_x(e) \notin M$. R/M is a finite field (Lemma 8) of some characteristic $p \neq 0$, and the image of e in $R/M[G]$ is an idempotent $\bar{e}$. Further, the image of $T_x(e)$ in R/M is just $T_0(\bar{e})$ where $T_0 : R/M[G] \to R/M$ is defined (relative to x) as in § 2. Since $T_0(\bar{e}) = 0$ in R/M (Lemma 7) but $T_x(e) \notin M$, we have a contradiction.

Now consider $e = \sum e_g g$. Since e is an idempotent, $\sum e_g = 0$ or 1. But

$$\sum \{e_g : g \neq 1\} = 0$$

since $T_x(e) = 0$ for all $x \in G$, $x \neq 1$.
Hence,

$$e_1 = \mathrm{tr}(e) = 0 \text{ or } 1,$$

and so $e = 0$ or 1, by Kaplansky's theorem.

4. A generalization of Theorem 1. The alert reader may have noticed that the hypothesis that G be Noetherian in Theorem 1 is stronger than necessary. For example, it suffices that G satisfy the ascending chain condition on cyclic subgroups. However, one can do still better, and for the sake of completeness we state the most general result which follows from the proof of Theorem 1.

THEOREM 9. *Suppose F is a field of characteristic* 0 *and G is a group which satisfies*

(*) *For each $g \in G$, $g \neq 1$, there are infinitely many primes p such that g is not conjugate to any of $g^p, g^{p^2}, g^{p^3}, \ldots$.*

Then $F[G]$ has no idempotents except 0 *and* 1.

The proof is almost identical to the proof of Theorem 1. The following slightly sharper version of Lemma 8 is needed.

LEMMA 10. *Let R be a commutative domain of characteristic* 0, *finitely generated as a ring, and r a non-zero element of R. Then, for all primes p except a finite number (depending on r), there exists a maximal ideal M of R such that $r \notin M$ and R/M is a finite field of characteristic p.*

5. Polycyclic group rings. If G is a torsion-free polycyclic group and F is a field, then $F[G]$ is a Noetherian ring which has a ring of fractions A which is a simple Artinian ring. Thus A is a complete matrix ring over a division ring. If one could show that A had no non-trivial idempotents it would follow that A was a division ring and that $F[G]$ had no zero divisors.

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Carleton University,
Ottawa, Ontario

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