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LATTICE POINTS IN A RANDOM PARALLELOGRAM

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Abstract

A Fourier series is obtained for the variance of the number of points lying in a parallelogram thrown randomly on to a square lattice at a fixed angle.

GEOMETRICAL PROBABILITY

Suppose that a parallelogram C of area A is thrown at random on to a square lattice of unit side. Let N be the number of lattice points that fall within it. Then $E(N) = A$. For a rectangle with sides parallel to the axes, the variance of N is given by Kendall and Moran (1963), p. 104. Rosen (1989) recently obtained $\operatorname{var} N$ for a rectangle with fixed orientation to the lattice. He reduced the rectangle by subtracting the maximal area for which the variance is 0, then dissected the remaining area into parts whose variances and covariances could be evaluated conveniently. This letter describes a different method, leading to a Fourier series.

For every lattice point (a_0, b_0), and pair of coprime integers a, b, the line passing through precisely the lattice points $(a_0 + na, b_0 + nb)$, $n = 0, \pm 1, \pm 2, \dots$, with $\theta = \tan^{-1}(b/a)$ will be called a θ-line. The distance between consecutive points on a θ-line, $(a^2 + b^2)^{\frac{1}{2}}$, will be denoted by d. The perpendicular distance between adjacent θ-lines is therefore $(a^2 + b^2)^{-\frac{1}{2}} = \cos\theta / a$ and it will be denoted by u. Consider a parallelogram in which one pair of parallel sides has length l, the perpendicular distance between them is w, and the internal acute angle is φ.

We use $[x]$, $\{x\}$ for the integral and fractional parts of x. Suppose that the parallelogram falls so that its sides of length l make an angle θ with the x-axis, with $\tan\theta$ rational, and that a fixed vertex is distributed randomly over the lattice. Let M be the random number of θ-lines that intersect the parallelogram. Then $\Pr(M = [w/u]) = 1 - \{w/u\}$, $\Pr(M = [w/u] + 11) = \{w/u\}$. Hence $E(M) = (w/u)$, $\operatorname{var} M = \{w/u\}(1 - \{w/u\})$. Now $N = N_1 + N_2 + \cdots + N_M$, where N_i is the number of lattice points contained within the parallelogram, on the ith θ-line intersecting it, counted from one side. We have $\Pr(N_i = [l/d]) = 1 - \{l/d\}$, $\Pr(N_i = [l/d] + 1) = \{l/d\}$, $E(n_i) = (l/d)$ and $v = \operatorname{var} N_i = \{l/d\}(1 - \{l/d\})$. Now $\operatorname{var}(N \mid M) = M \operatorname{var} N_i + 2\sum_{1 \le i < j \le M} \operatorname{cov}(N_i, N_j)$. To obtain the covariances, consider first two one-dimensional intervals I_1, I_2, each of length x. The first is placed randomly on the horizontal axis of the lattice, and I_2 is placed on a lattice line parallel to it. They are rigidly joined so that when the left end of I_1 is at $(0, 0)$, the left end of I_2 is at a distance y, called the offset, from the nearest lattice point on its left. If the left end of I_1 is uniformly distributed on $(0, 1)$, the covariance $c(x, y)$ between the numbers of lattice points in the two intervals is found to be:

if $\{x\} \leqq \frac{1}{2}$,

$$
\begin{aligned}
c(x,y) &= \{x\}(1-\{x\}) - \{y\} && \text{if } \{y\} \leqq \{x\} \\
&= \{x\}(1-\{x\}) - \{x\} && \text{if } \{x\} \leqq \{y\} \leqq 1-\{x\} \qquad (1)\\
&= \{x\}(1-\{x\}) - (1-\{y\}) && \text{if } \{y\} \geqq 1-\{x\}
\end{aligned}
$$

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if $\{x\} \geqq \frac{1}{2}$,

$$c(x, y) = c(1 - \{x\}, y); \quad \text{and} \quad c(x + n, y) = c(x, y), \quad n = \pm 1, \pm 2, \ldots .$$

The expression (1) satisfies $c(x, y) = c(x, 1 - y)$ and has a convergent Fourier series:

$$c(x, y) = \sum_{t=1}^{\infty} \{(1 - \cos 2\pi tx)(\cos 2\pi ty)\}/(4\pi^2 t^2). \tag{2}$$

The parallelogram cuts off intervals of length l on the θ-lines that intersect it. We need the offset between each pair. Let one corner of the parallelogram be at the origin, and one of the sides of length l lie along a θ-line. Consider the jth next θ-line as we move right along the x-axis. The first lattice point that we come to on moving left along this θ-line has a y coordinate of $-s_j$, where s_j is the smallest positive solution of the linear congruence $j - s_j a \equiv 0 \pmod{b}$. The offset is then calculated by elementary geometry to be $y_j = s_j \sec \theta + ju \cot \varphi - ju \cot \theta$. This, and the side l of the parallelogram, have to be divided by d, to correspond with unit spacing of the points. Then

$$\operatorname{var}(N \mid M) = Mv + 2 \sum_{j=1}^{M-1} (M - j)c(l/d, y_j/d).$$

Using the formula for unconditional variance in terms of conditional variance, and writing $m = [w/u]$, we obtain

$$\begin{aligned} \operatorname{var}(N) = {} & (l^2/d^2)\{w/u\}(1 - \{w/u\}) + (w/u)\{l/d\}(1 - \{l/d\}) \\ & + 2 \sum_{j=1}^{m-1} (M - j)c(l/d, y_j/d) + 2\{w/u\} \sum_{j=1}^{m} c(l/d, y_j/d). \end{aligned} \tag{3}$$

The first term is the variance with respect to M of the conditional expected number of points. The other terms are the expectation over M of the above conditional variance. The fourth term contains the covariances between numbers of points on the $([w/u] + 1)$th line, which is only present with probability $\{w/u\}$, with each of the rest. Finally, var N can be expressed as the sum of $[w/u] + 1$ Fourier series, by substituting (1) in (3).

References

KENDALL, M. G. AND MORAN, P. A. P. (1963) *Geometrical Probability*. Griffin, London.

ROSEN, D. (1989) Coverage of a square lattice by an inclined rectangle. *Adv. Appl. Prob.* **21**, 705–707.