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A Homotopy of Quiver Morphisms with Applications to Representations

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Abstract. It is shown that a morphism of quivers having a certain path lifting property has a decomposition that mimics the decomposition of maps of topological spaces into homotopy equivalences composed with fibrations. Such a decomposition enables one to describe the right adjoint of the restriction of the representation functor along a morphism of quivers having this path lifting property. These right adjoint functors are used to construct injective representations of quivers. As an application, the injective representations of the cyclic quivers are classified when the base ring is left noetherian. In particular, the indecomposable injective representations are described in terms of the injective indecomposable R-modules and the injective indecomposable $R[x, x^{-1}]$-modules.

Let Q be a quiver and R a ring. In this paper, we shall study the category (Q, R-Mod) of representations of Q by left R-modules. As in the work of Riedtmann [7] and Bongartz and Gabriel [1], we will be interested in representations induced by morphisms of quivers. More precisely, we shall refine an argument of Jensen [4] to construct, using adjoint pairs of functors, injective objects of (Q, R-Mod) with specific features. Our first result is reminiscent of the decomposition theorem [8, Theorem II.8.9] for maps of topological spaces which asserts that every continuous function is a homotopy equivalence composed with a fibration. It relates the following two properties of quiver morphisms:

- A morphism $f: Q \to Q'$ of quivers is said to have the *(right) unique path lifting property* if for every vertex v of Q and path p' of Q' such that $t(p') = f(v)$, there is at most one path p of Q such that $f(p) = p'$ and $t(p) = v$.
- A morphism $f: Q \to Q'$ of quivers is said to be a *(right) covering* if for every vertex v of Q and path p' of Q' such that $t(p') = f(v)$, there is a unique path p of Q such that $f(p) = p'$ and $t(p) = v$.

To state this result recall that a quiver T is called a tree if there exists a vertex v of T, called the *terminal vertex* of T, with the property that for every vertex w of T, there exists a unique path from w to v. An inclusion $Q \subset W$ of quivers is said to be a *forest* over Q if W is gotten by amalgamating to Q some set of trees along their terminal vertices.

Theorem 2.1 *A morphism $f: Q \to Q'$ of quivers has the unique path lifting property if and only if there is a forest W over Q and an extension $\bar{f}: W \to Q'$ of f which is a covering morphism.*

If R is a ring and $f: Q \to Q'$ is a morphism of quivers, then a restriction functor $f^*: (Q', R\text{-Mod}) \to (Q, R\text{-Mod})$ is induced on the respective categories of R-representations.

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Because this functor is exact, its right adjoint $f_*\colon (Q, R\text{-Mod}) \to (Q', R\text{-Mod})$ preserves injective objects. Our main result (Theorem 4.1) uses the decomposition theorem above to explicitly describe the right adjoint f_* for a morphism $f\colon Q \to Q'$ with the unique path lifting property. Let Q be the cyclic quiver $\tilde{A}_n$ with $n+1$ vertices $v_0, \ldots, v_n$ and $n+1$ arrows $a_i\colon v_i \to v_{i+1}$ for $0 \leq i < n$ and $a_n\colon v_n \to v_0$. When the ring R is left noetherian, our methods suffice to give a complete description of the injective representations of Q over R. To state the result we need the following notation: Given a left $R[x]$-module ${}_{R[x]}M$ and an arrow a of $\tilde{A}_n$, denote by $F_a(M)$ the representation of that assigns to every vertex the restricted R-module ${}_RM$ and where

$$F_a(M)(a_i) = \begin{cases} x & \text{if } a_i = a \\ 1_M & \text{otherwise.} \end{cases}$$

Theorem 6.5 *Let R be a left noetherian ring. Every injective representation of $\tilde{A}_n$ over R has a decomposition, unique up to isomorphism, of the form*

$$\bigoplus_{a \in \tilde{A}_n} F_a(E_a[x^{-1}]) \oplus F_{a_0}(\bar{E})$$

where each E_a is an injective left R-module and $\bar{E}$ is an injective left $R[x, x^{-1}]$-module.

In the final section, a torsion theory is developed for the representations of a cyclic quiver $\tilde{A}_n$ over a ring R. According to this theory, the last summand of the typical injective representation displayed in Theorem 6.5 is torsion free, while the first $n+1$ summands are torsion. Theorem 6.5 also allows us to list (Theorem 6.6) completely and without repetition the indecomposable injective representations of $\tilde{A}_n$ over R as follows:

Torsion Given an indecomposable injective left R-module E and an arrow a of $\tilde{A}_n$, the representation $F_a(E[x^{-1}])$ is an indecomposable injective.

Torsion free Given an indecomposable injective left $R[x, x^{-1}]$-module $\bar{E}$, the representation $F_{a_0}(\bar{E})$ is an indecomposable injective.

This says that the spectrum of $(\tilde{A}_n, R\text{-Mod})$ consists of $n+1$ copies of the left spectrum of R and one copy of the left spectrum of $R[x, x^{-1}]$.

1 Preliminaries

By a *quiver* Q we mean a directed graph. The directed edges are called *arrows*. We let $a\colon v_1 \to v_2$ indicate that a is an arrow from the vertex v_1 to the vertex v_2. In this case we also write $i(a) = v_1$ and $t(a) = v_2$. By a *path* p of Q we mean a sequence $a_n \cdots a_2 a_1$ of arrows such that $i(a_2) = t(a_1), \ldots, i(a_n) = t(a_{n-1})$. We then write $i(p) = i(a_1)$ and $t(p) = t(a_n)$. If p and q are paths of Q such that $i(q) = t(p)$, we say qp is defined and let qp denote the obvious path. We extend the definition of a path and allow any vertex v of Q to be a (trivial) path with $i(v) = v = t(v)$. If $i(p) = v_1$, $t(p) = v_2$ for a path p of Q, we let $pv_1 = p$ and $v_2 p = p$. With these conventions any quiver Q is naturally a category. The objects are the vertices of Q and a path p is a morphism from $i(p)$ to $t(p)$. So if p is a path, $p\colon v_1 \to v_2$ just means $i(p) = v_1$ and $t(p) = v_2$. A quiver is said to be *discrete* if it has no arrows. Following [2] we let $\tilde{A}_n$ denote the *cyclic* quiver with $n+1$ vertices and $n+1$ arrows. A quiver T will be called a *tree* if T has a vertex v such that for every vertex w there is a unique path p with $p\colon w \to v$. Such a v is unique and will be called the *terminal* vertex of T. By a *subtree* of a quiver we mean a subquiver which is a tree. A quiver each of whose connected components is a tree will be called a *forest*. If $Q \subset W$ and W is the amalgamation of Q with a forest along the discrete quiver of terminal vertices of the trees of this forest, W is said to be a forest *over* Q. Since a trivial quiver $\{v\}$ is a tree, every quiver is a forest over itself. If T is a tree and $T \subset W$ is a forest over T, then W is a tree. If W is a forest over Q and W' is a forest over W then W' is a forest over Q. If Q is a quiver, by the *(right) path space* of Q we mean the quiver $P(Q)$ whose vertices are the paths p of Q and whose arrows are the pairs $(pa, p)\colon pa \to p$ where p is a path of Q and a an arrow of Q such that pa is defined. It is clear then that $P(Q)$ is a forest and that the terminal vertices of the connected components of $P(Q)$ are the vertices of Q. If v is a vertex of Q we let $P(Q)_v$ denote the subtree of $P(Q)$ containing all paths of Q with terminal vertex v. If p and q are paths of Q such that pq is defined, we extend the notation and let $(pq, p)\colon pq \to p$ denote a path of $P(Q)$. Note that there is also an obvious definition of a left path space of Q. However, we will not use this quiver and so say path space instead of right path space. We will observe a similar convention with other terminology in this paper. For example, if Q is a quiver and R a ring then the path ring of Q over R, denoted RQ, is the free (left) R-module with base the paths p of Q and multiplication qp as usual if $i(q) = t(p)$ and with $qp = 0$ if $i(q) \neq t(p)$. A *morphism* $f\colon Q \to Q'$ of quivers is usually defined to be a map of the respective sets of vertices and arrows such that $a\colon v_1 \to v_2$ implies $f(a)\colon f(v_1) \to f(v_2)$. We deviate slightly from this standard by allowing $f(a)$ to be the vertex $f(v)$ if $a\colon v \to v$. For p a path of Q, we extend the notation and define $f(p)$ so that f is a functor between categories. Hence if $p\colon v_1 \to v_2$ in Q then $f(p)\colon f(v_1) \to f(v_2)$ in Q'. If $f\colon Q \to Q'$ is a morphism and $f^{-1}(v')$ is a discrete quiver for each vertex v' of Q', then $f(a)$ must be an arrow of Q' for each arrow a of Q. If $f\colon Q \to Q'$ is a morphism of quivers, $P(f)\colon P(Q) \to P(Q')$ will denote the obvious morphism of the associated path quivers. For any quiver Q there is a unique morphism $i\colon P(Q) \to Q$ which maps any vertex p of $P(Q)$, *i.e.*, any path p of Q, to $i(p)$ and which maps the arrow (pa, p) to a.

2 Factoring Morphisms of Quivers

A morphism $f\colon Q \to Q'$ of quivers is said to have the *(right) unique path lifting property* if for every vertex v of Q and path p' of Q' such that $t(p') = f(v)$, there is at most one path p of Q such that $f(p) = p'$ and $t(p) = v$ (see [8, p. 68]) for the topological version of this property). If $f\colon Q \to Q'$ has the unique path lifting property then for any arrow a of Q, $f(a)$ is an arrow of Q' (*i.e.*, cannot be a vertex). Any embedding $e\colon Q \subset Q'$ of quivers satisfies the unique path lifting property. A morphism $f\colon Q \to Q'$ is called a *(right) covering morphism* if for every vertex v of Q and path p' of Q' such that $t(p') = f(v)$, there is a unique path p of Q such that $f(p) = p'$ and $t(p) = v$ (see [8, p. 62]).

Theorem 2.1 *A morphism $f\colon Q \to Q'$ of quivers has the unique path lifting property if and only if there is a forest W over Q and an extension $\bar{f}\colon W \to Q'$ of f which is a covering morphism.*

Proof If $\bar{f}\colon W \to Q'$ is such an extension, then $f = \bar{f} \circ e$ where $e\colon Q \subset W$ is the embedding morphism. Since both e and $\bar{f}$ have the unique path lifting property, so does f. Now suppose $f\colon Q \to Q'$ has the unique path lifting property. For each vertex v of Q, consider the unique subtree T_v of $P(Q')$ whose vertices are the trivial paths $f(v)$ and the paths of the form $a'p'$ where a' is an arrow of Q' such that $t(a') = f(v)$ and such that there is no arrow a of Q with $t(a) = v$ and $f(a) = a'$. We will say such a path $a'p'$ is a path that we cannot begin lifting to a path terminating at v. We amalgamate each T_v with Q by identifying $v \in Q$ with $f(v) \in T_v$. Let W be the resulting forest over Q and $\bar{f}\colon W \to Q'$ the unique morphism such that $\bar{f}|Q = f$ and such that $\bar{f}|T_v$ agrees with $i\colon P(Q') \to Q'$. Since $i\big(f(v)\big) = f(v)$ there is no problem with compatibility. Next we verify that the extension $\bar{f}\colon W \to Q'$ is a covering. It is enough to prove that for every vertex $x \in W$, the morphism of trees

$$P(\bar{f})|P(W)_x\colon P(W)_x \to P(Q')_{\bar{f}(x)}$$

is an isomorphism. But a morphism of trees is an isomorphism if it is a bijection on the vertices. Suppose first that $x \in T_v \setminus \{v\}$ for some vertex $v \in Q$. Then $x = p'$ where p' is a path in Q' terminating at $f(v)$ which cannot begin to be lifted to a path terminating at v. We have that $\bar{f}(x) = i(p')$. Now it is clear that the paths in Q' terminating at $i(p')$ are in bijective correspondence with paths of W, that is of T_v, terminating at p'. In fact the bijection is given by $q' \mapsto (p'q', p')$. Now consider the case $x = v \in Q$. A path p in W which terminates at v may be factored $p = p_1p_2$ where p_1 is the maximal end segment of p still in Q. Since W is a forest over Q, p_2 is a path in $T_{i(p_1)}$ and so is of the form $\big(q_1', fi(p_1)\big)$ where q_1' is a path of Q' terminating at $fi(p_1)$ which cannot begin to be lifted to a path terminating at $i(p_1)$. Also $\bar{f}(p) = \bar{f}(p_1)\bar{f}(p_2) = f(p_1)i\big(q_1', fi(p_1)\big) = f(p_1)q_1'$. Let s' be a path in Q' terminating at $f(v)$. Factor $s' = s_1s_2$ where s_1 is the maximal end segment of s' which lifts to a path p_1 terminating at v, $s_1 = f(p_1)$. By maximality, s_2 cannot begin to be lifted to a path terminating at $i(p_1)$. Thus $s_2 \in T_{i(p_1)}$ and $\bar{f}\Big(p_1\big(s_2, fi(p_1)\big)\Big) = f(p_1)i\big(s_2, fi(p_1)\big) = f(p_1)s_2 = s$ showing that $P(\bar{f})|P(W)_v$ is onto $P(Q')_{f(v)}$. It remains to be shown that $P(\bar{f})|P(W)_v$ is an injection, that is, that $p_1\big(s_2, fi(p_1)\big)$ above is the unique lifting of s' to a path terminating at v. So let q be another such lifting. We have $\bar{f}(q) = s'$ and $t(q) = v$. Write $q = q_1\big(q_2, fi(q_1)\big)$ where q_1 is a maximal end segment still in Q and q_2 is a path in Q' terminating at $fi(q_1)$ which cannot begin to be lifted to a path terminating at $i(q_1)$. Now $s' = \bar{f}(q) = f(q_1)q_2 = s_1s_2$. By the definition of s_1, $f(q_1)$ is an end segment of $s_1 = f(p_1)$ and by the unique path lifting property of f, q_1 is an end segment of p_1. Now q_2 cannot begin to be lifted so it must be that $f(q_1) = s_1$ and $q_2 = s_2$. But then $q_1 = p_1$ and so we are done. ∎

3 Representation of Quivers

Let Q be a quiver and R a ring. Let R-Mod denote the category of left R-modules. By a *representation* X of Q over R we mean a functor $X\colon Q \to R$-Mod. A representation X is specified by giving a module $X(v)$ for each vertex v of Q and a linear map $X(a)\colon X(v_1) \to X(v_2)$ for each arrow $a\colon v_1 \to v_2$ of Q. If $X, Y\colon Q \to R$-Mod are two representations of Q over R, by a morphism $\sigma\colon X \to Y$ we mean a natural transformation. Such a σ is determined by an R-morphism $\sigma(v)\colon X(v) \to Y(v)$ for each vertex v of Q such that for each arrow $a\colon v_1 \to v_2$ of Q the diagram

$$\begin{array}{ccc} X(v_1) & \xrightarrow{X(a)} & X(v_2) \\ \Big\downarrow{\scriptstyle \sigma(v_1)} & & \Big\downarrow{\scriptstyle \sigma(v_2)} \\ Y(v_1) & \xrightarrow{Y(a)} & Y(v_2) \end{array}$$

is commutative. The representations of Q over R then form a category denoted $(Q, R\text{-Mod})$. Clearly $(Q, R\text{-Mod})$ is an abelian category. If Q is a quiver with a finite number of vertices, then the path ring RQ has an identity $1 = v_1 + \cdots + v_n$ where $v_1, \ldots, v_n$ are the distinct vertices of Q. The categories $(Q, R\text{-Mod})$ and RQ-Mod are in that case equivalent. In fact, if M is a left RQ-module, construct a representation X so that $X(v) = vM$ for any vertex v and for any arrow $a\colon v_1 \to v_2$, $X(a)\colon v_1M \to v_2M$ is just scalar multiplication by a. This construction gives an equivalence. Let $f\colon Q \to Q'$ be a morphism of quivers. Associated to the morphism f is the restriction functor $f^*\colon (Q', R\text{-Mod}) \to (Q, R\text{-Mod})$ which assigns to an object X the composition of functors $X \circ f$. Hence $f^*(X)(v) = X\big(f(v)\big)$ for each vertex v of Q and $f^*(X)(a) = X\big(f(a)\big)$ for each arrow a of Q. The representation $f^*(X)$ is called the *restriction* of X along f. Our concern is to describe, for certain $f\colon Q \to Q'$, the right adjoint $f_*\colon (Q, R\text{-Mod}) \to (Q', R\text{-Mod})$ of the restriction functor f^*. This means that for every $Y \in (Q', R\text{-Mod})$ and $X \in (Q, R\text{-Mod})$ we have a natural isomorphism

$$\operatorname{Hom}\big(f^*(Y), X\big) \cong \operatorname{Hom}\big(Y, f_*(X)\big).$$

Recall that if $f\colon Q \to Q'$ and $g\colon Q' \to Q''$ are quiver morphisms and if f^* and g^* have right adjoints f_* and g_*, then the right adjoint of $f^* \circ g^* = (g \circ f)^*$ is $g_* \circ f_*$, *i.e.*, $(g \circ f)_* = g_* \circ f_*$. We will use this observation to describe the right-adjoint of f^* for a morphism $f\colon Q \to Q'$ having the unique path lifting property. Using Theorem 2.1 and its notation, we only need find the right adjoints of g^* for $g\colon Q \subset W$ where W is a forest over Q and of $(\bar{f})^*$ where $\bar{f}\colon W \to Q'$ is a covering morphism.

Example 3.1 Let Q be a quiver and $c\colon Q \to \{v\}$ the unique morphism from a quiver Q to the trivial discrete quiver $\{v\}$. Then $(\{v\}, R\text{-Mod}) \cong R\text{-Mod}$, so we identify these categories. The restriction functor

$$c^*\colon R\text{-Mod} \to (Q, R\text{-Mod})$$

is such that $c^*(M)$ is a constant functor for every left R-module M. We have $c^*(M)(v) = M$ for each vertex v and $c^*(M)(a) = 1_M$ for each arrow a. By the definition of inverse limit (see [9, p. 99]), there is a natural isomorphism

$$\operatorname{Hom}_R(M, \varprojlim X) \cong \operatorname{Hom}\big(c^*(M), X\big).$$

Hence the right adjoint $c_*\colon (Q, R\text{-Mod}) \to R\text{-Mod}$ of c^* is given by

$$c_*(X) = \varprojlim X.$$

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Similarly, the left adjoint of c^* is the functor $X \mapsto \varinjlim X$. If T is a tree with terminal vertex v and if $e\colon \{v\} \subset T$ is the embedding morphism, then clearly

$$e^*(X) = X(v) \cong \varinjlim X$$

for any representation X of T. But by the above, the right adjoint e_* of e^* is then the restriction functor c^* along $c\colon T \to \{v\}$. We admit this is easy to see directly.

Now let W be a forest over Q and let $e\colon Q \subset W$ be the inclusion morphism. Considering e as a functor, we see that its left adjoint $d\colon W \to Q$ is the retraction of e, *i.e.*, $d \circ e = 1_Q$, that collapses each amalgamated tree to its terminal vertex. The next proposition then follows from [9, p. 112, Exercise 29].

Proposition 3.2 *Let W be a forest over Q and $e\colon Q \subset W$ the inclusion morphism. The right adjoint of the restriction functor e^* is the restriction functor d^* (with d as above).*

This proposition implies that if X is a representation of Q over R and T is one of the trees of W amalgamated to Q along its terminal vertex v, then $e_*(X)$ restricted to T is the constant representation associated with the module $X(v)$.

4 The Right Adjoint

In order to facilitate the proof of the next theorem we recall the notion of a coordinate-wise function: Let $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ be indexed families of sets. A function

$$h\colon \prod_{i\in I} X_i \to \prod_{j\in J} Y_j$$

is said to be *coordinate-wise* if there is a function $c\colon J \to I$ of the index sets and for each $j \in J$ a function $h_{c(j)}\colon X_{c(j)} \to Y_j$ such that $h\big((x_i)_{i\in I}\big) = \big(h_{c(j)}(x_{c(j)})\big)_{j\in J}$. Such a coordinate-wise function is denoted $h = \prod_{j\in J} h_{c(j)}$. The following diagram is then commutative:

$$\begin{array}{ccc} \prod_{i\in I} X_i & \xrightarrow{\prod_{j\in J} h_{c(j)}} & \prod_{j\in J} Y_j \\ \Big\downarrow{\scriptstyle \pi(c(j))} & & \Big\downarrow{\scriptstyle \pi(j)} \\ X_{c(j)} & \xrightarrow{h_{c(j)}} & Y_j \end{array}$$

where $\pi(v)$ denotes projection onto the v-component. If c is onto, then h is the unique function for which this diagram commutes. Let $c\colon J \to I$ be a function of index sets. The coordinate-wise function $h = \prod_{j\in J} h_{c(j)}$ has the property that if $g\colon X \to Y$ is a function and $p_i\colon X \to X_i$ and $q_j\colon Y \to Y_j$ are families of functions such that the diagrams

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \Big\downarrow{\scriptstyle p_{c(j)}} & & \Big\downarrow{\scriptstyle q_j} \\ X_{c(j)} & \xrightarrow{h_{c(j)}} & Y_j \end{array}$$

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commute for every $j \in J$, then, thinking of X and Y as trivial cartesian products, the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \big\downarrow{\textstyle\prod p_i} & & \big\downarrow{\textstyle\prod q_j} \\ \prod_{i\in I} X_i & \xrightarrow{h} & \prod_{j\in J} Y_j \end{array}$$

also commutes.

Let $f\colon Q \to Q'$ be a covering morphism of quivers. For a ring R, define a functor

$$f_*\colon (Q, R\text{-Mod}) \to (Q', R\text{-Mod})$$

as follows (we shall verify presently that this is the adjoint of the restriction functor f^*): Let $X \in (Q, R\text{-Mod})$.

Vertices If $v' \in Q'$ is a vertex, then

$$f_*(X)(v') := \prod_{f(v)=v'} X(v).$$

If v' is not in the image of f, this means that $f_*(X)(v') = 0$.

Arrows If $a'\colon v' \to w'$ is an arrow of Q', let

$$f_*(X)(a')\colon \prod_{f(v)=v'} X(v) \to \prod_{f(w)=w'} X(w)$$

be the coordinate-wise function $\prod_{f(w)=w'} h_{c(w)}$ where

$$c\colon \{w \mid f(w) = w'\} \to \{v \mid f(v) = v'\}$$

is defined by the rule $w \mapsto i(a)$ where $f(a) = a'$ and $t(a) = w$, and $h_{c(w)}\colon X\big(i(a)\big) \to X(w)$ is just $X(a)$. In short, $f_*(X)(a') := \prod_{f(a)=a'} X(a)$.

Theorem 4.1 *Let $f\colon Q \to Q'$ be a covering morphism of quivers. The right adjoint of the restriction functor f^* (with respect to representations over a ring R) is the functor f_* defined above.*

Proof Given representations X of Q and Y of Q' we need to exhibit a natural isomorphism

$$\mathrm{Hom}\big(Y, f_*(X)\big) \cong \mathrm{Hom}\big(f^*(Y), X\big).$$

Let $v' \in Q'$ be a vertex. We have

$$\begin{aligned}\operatorname{Hom}_R\big(Y(v'), f_*(X)(v')\big) &= \operatorname{Hom}_R\Big(Y(v'), \prod_{f(v)=v'} X(v)\Big)[1] \\ &= \prod_{f(v)=v'} \operatorname{Hom}_R\big(Y(v'), X(v)\big) \\ &= \prod_{f(v)=v'} \operatorname{Hom}_R\big(f^*(Y)(v), X(v)\big).\end{aligned}$$

If $\sigma\colon Y \to f_*(X)$ is a morphism (and so $\sigma(v')\colon Y(v') \to f_*(X)(v')$) we may define $\bar\sigma\colon f^*(Y) \to X$ such that for each v with $f(v) = v'$, $\bar\sigma(v)$ is the v-component of $\sigma(v') \in \operatorname{Hom}_R\big(Y(v'), f_*(X)(v')\big)$ thought of as the product above. We will verify that this is a morphism of representations. Inversely, if $\tau\colon f^*(Y) \to X$ is a morphism of representations, then for each vertex $v \in Q$, $\tau(v)\colon f^*(Y)(v) \to X(v)$. So we define $\bar\tau\colon Y \to f_*(X)$ for a vertex $v' \in Q'$ as the element $\bar\tau(v') = \big(\tau(v)\big)_{f(v)=v'}$. We will also verify that this also gives a morphism of representations. Since the operations $\sigma \mapsto \bar\sigma$ and $\tau \mapsto \bar\tau$ are mutual inverses, we then will get the desired natural isomorphism. Let $\sigma\colon Y \to f_*(X)$ be a morphism and let $a_0\colon v_0 \to w_0$ be an arrow in Q. If $a' = f(a_0)$, $v' = f(v_0)$ and $w' = f(w_0)$, then by hypothesis

(1)
$$\begin{array}{ccc} Y(v') & \xrightarrow{Y(a')} & Y(w') \\ \big\downarrow{\scriptstyle \sigma(v')} & & \big\downarrow{\scriptstyle \sigma(w')} \\ f_*(X)(v') & \xrightarrow{f_*(X)(a')} & f_*(X)(w') \end{array}$$

is a commutative diagram. From the definitions, this is the same as

(1)
$$\begin{array}{ccc} f^*(Y)(v_0) & \xrightarrow{f^*(Y)(a_0)} & f^*(Y)(w_0) \\ \big\downarrow{\scriptstyle \sigma(v')} & & \big\downarrow{\scriptstyle \sigma(w')} \\ \prod_{f(v)=v'} X(v) & \xrightarrow{f_*(X)(a')} & \prod_{f(w)=w'} X(w). \end{array}$$

By the definition of $f_*(X)(a') = \prod_{f(a)=a'} X(a)$, we have the commutativity of the diagram

(2)
$$\begin{array}{ccc} \prod_{f(v)=v'} X(v) & \xrightarrow{f_*(X)(a')} & \prod_{f(w)=w'} X(w) \\ \big\downarrow{\scriptstyle \pi(v_0)} & & \big\downarrow{\scriptstyle \pi(w_0)} \\ X(v_0) & \xrightarrow{X(a_0)} & X(w_0) \end{array}$$

Putting the diagram (1) on top of (2) gives the commutative diagram that asserts that $\bar\sigma$ is a morphism. Now to prove that $\bar\tau$ as defined above is a morphism, let $a'\colon v' \to w'$ be an arrow in Q'. For each vertex $w \in Q$ with $f(w) = w'$ let a_w be the unique arrow of Q

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with $f(a_w) = a'$ and $t(a_w) = w$. Then a morphism $\tau\colon f^*(Y) \to X$ gives a commutative diagram

$$\begin{array}{ccc} f^*(Y)\big(i(a_w)\big) & \xrightarrow{f^*(Y)(a_w)} & f^*(Y)(w) \\ \Big\downarrow{\scriptstyle \tau(i(a_w))} & & \Big\downarrow{\scriptstyle \tau(w)} \\ X\big(i(a_w)\big) & \xrightarrow{X(a_w)} & X(w) \end{array}$$

for every vertex $w \in Q$ with $f(w) = w'$. This is just the diagram

$$\begin{array}{ccc} Y(v') & \xrightarrow{Y(a')} & Y(w') \\ \Big\downarrow{\scriptstyle \tau(i(a_w))} & & \Big\downarrow{\scriptstyle \tau(w)} \\ X\big(i(a_w)\big) & \xrightarrow{X(a_w)} & X(w). \end{array}$$

Now $\prod X(a_w) = \prod_{f(a)=a'} X(a)$ has the property that the diagram

$$\begin{array}{ccc} Y(v') & \xrightarrow{Y(a')} & Y(w') \\ \Big\downarrow{\scriptstyle \bar{\tau}(v')} & & \Big\downarrow{\scriptstyle \bar{\tau}(w')} \\ \prod_{f(v)=v'} X(v) & \xrightarrow{\prod X(a_w)} & \prod_{f(w)=w'} X(w) \end{array}$$

also commutes. But that just means that $\bar{\tau}\colon Y \to f_*(X)$ is a morphism of representations. ■

Let $Q \subset Q'$ be a subquiver of Q' satisfying the condition that whenever $a' \in Q'$ is an arrow such that $t(a') \in Q$, then $a' \in Q$. The embedding morphism $e\colon Q \subset Q'$ is then a covering. If X is a representation of Q, then by the construction above $f_*(X)$ is defined on the vertices of Q' by

$$f_*(X)(v') = \begin{cases} X(v') & \text{if } v' \in Q \\ 0 & \text{if } v' \notin Q \end{cases}$$

and for arrows $a' \in Q$ by

$$f_*(X)(a') = \begin{cases} X(a') & \text{if } a' \in Q \\ 0 & \text{if } a' \notin Q. \end{cases}$$

Thus $f_*(X)$ is just X extended to Q' by 0.

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5 Injective Representations of Cyclic Quivers

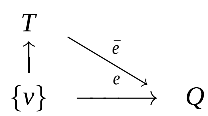
Let R be a ring and $f\colon Q \to Q'$ a morphism of quivers. Then the restriction functor $f^*\colon (Q', R\text{-Mod}) \to (Q, R\text{-Mod})$ is exact. If f_* is the right adjoint of f^* and X is an injective representation of Q, the natural isomorphism

$$\mathrm{Hom}\big(-, f_*(X)\big) \cong \mathrm{Hom}\big(f^*(-), X\big)$$

shows that $f_*(X)$ is an injective representation of Q'. Let $v \in Q$ be a vertex and consider the embedding $e\colon \{v\} \subset Q$. If E is an injective left R-module (so an injective representation of $\{v\}$) then $e_*(E)$ is an injective representation of Q. If X is a representation of Q, then $e^*(X)$ is just the module $X(v)$. We have the natural isomorphism

$$\mathrm{Hom}_R\big(X(v), E\big) = \mathrm{Hom}_R\big(e^*(X), E\big) \cong \mathrm{Hom}\big(X, e_*(E)\big).$$

From this it can be seen that as v ranges over the vertices of Q and we allow E to be any injective left R-module (or some specified cogenerator of R-Mod) then the $e_*(E)$ cogenerate $(Q, R\text{-Mod})$. In the next examples, recall that by Theorem 2.1, $e\colon \{v\} \to Q$ has the factorization



where $\bar{e}$ is a covering and $T \subset P(Q)$ is the subtree of paths terminating at v.

Examples Let $Q = \tilde{A}_0$ be the quiver with one vertex v and one arrow a. In the factorization above we see that $T = A_\infty^-$ (*i.e.*, T is a line, infinite to the left). The embedding $e\colon \{v\} \subset Q$ decomposes according to Theorem 2.1 as a composition $e = \bar{e}e'$ where $e'\colon \{v\} \to T$ and $\bar{e}\colon T \to \tilde{A}_0$. Let E be an injective left R-module, so a representation of v. By Example 3.1, $(e')_*(E)$ is the constant representation on A_∞^- determined by E. But then by Theorem 4.1, $e_*(E) = (\bar{e})_*\big((e')_*(E)\big)$ is such that $e_*(E)(v) = E \times E \times E \times \cdots$ and $e_*(E)(a)$ is just the shift operator, namely

$$(z_0, z_1, z_2, \dots) \mapsto (z_1, z_2, z_3, \dots).$$

This suggests we use Northcott's notation [6] and denote $E \times E \times E \times \cdots$ by $E[[x^{-1}]]$ (so $(z_0, z_1, z_2, \dots)$ corresponds to $z_0 + z_1x^{-1} + z_2x^{-2} + \cdots$) and then the shift operator above is denoted by x. So

$$x(z_0 + z_1x^{-1} + \cdots) = z_1 + z_2x^{-1} + \cdots .$$

Since $R\tilde{A}_0$ is just the polynomial ring $R[a]$ (or changing notation, $R[x]$), we recover Northcott's observation in [6] that $E[[x^{-1}]]$ is an injective $R[x]$-module when E is an injective left R-module. In fact when R is left noetherian, $E[x^{-1}] \subset E[[x^{-1}]]$ is also an injective left R-module.

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Let us now consider the cyclic quiver $\tilde{A}_1$, represented as $\bullet \rightleftarrows \bullet$ with vertices v_0 and v_1. According to Theorem 2.1, the embedding $e_0\colon \{v_0\} \subset \tilde{A}_1$ decomposes as $\{v_0\} \to A_\infty^- \to \tilde{A}_1$ with the obvious maps. If a left R-module E is considered as a representation of $\{v_0\}$, then $(e_0)_*(E)$ is the representation

$$E \times E \times \cdots \rightleftarrows E \times E \times E \times \cdots$$

where the right arrow denotes the identity map and the left arrow the shift operator. In the notation above, this representation can be written

$$E[[x^{-1}]] \underset{x}{\overset{\mathrm{id}}{\rightleftarrows}} E[[x^{-1}]].$$

If E is an injective left R-module, then each of these is an injective representation of $\tilde{A}_1$ over R. Recall from [2] that the cyclic quiver $\tilde{A}_n$ is the quiver with $n+1$ vertices $v_0, \dots, v_n$ and $n+1$ arrows $a_i\colon v_i \to v_{i+1}$ for $0 \le i < n$ and $a_n\colon v_n \to v_0$. If ${}_{R[x]}M$ is a left $R[x]$-module and a an arrow of $\tilde{A}_n$, denote by $F_a(M)$ the representation of $\tilde{A}_n$ over R that assigns to every vertex the restricted R-module ${}_RM$ and where

$$F_a(M)(a_i) = \begin{cases} x & \text{if } a_i = a \\ 1_M & \text{otherwise.} \end{cases}$$

As above, the representation $F_a(E[[x^{-1}]])$ is injective for every arrow of $\tilde{A}_n$. We want to argue that if R is left noetherian and E is an injective left R-module, then for every arrow a of $\tilde{A}_n$ the representation $F_a(E[x^{-1}])$ is also an injective representation of $\tilde{A}_n$ over R. For this we need the next result which was observed by Matlis [5] when the f below is given by scalar multiplication. The more general version we need was noted in [3, p. 198, Proposition 4.2]. We include a proof here for completeness.

Lemma 5.1 *Let R be a left noetherian ring and $M \subset E(M)$ the injective envelope of a left R-module M. If $\phi\colon E(M) \to E(M)$ is an R-morphism such that $\phi(M) = 0$, then ϕ is locally nilpotent on $E(M)$, i.e., for any $x \in E(M)$, $\phi^n(x) = 0$ for some $n \ge 1$.*

Proof As R is left noetherian, $E(M \oplus M \oplus \cdots) = E(M) \oplus E(M) \oplus \cdots$. The endomorphism of $E(M) \oplus E(M) \oplus \cdots$ defined by

$$(x_1, x_2, \dots) \mapsto \big(x_1, x_2 - \phi(x_1), x_3 - \phi(x_2), \dots\big)$$

is the identity on $M \oplus M \oplus \cdots$ and is therefore surjective. So if $x \in E(M)$ and $(x_1, x_2, x_3, \dots)$ is mapped onto $(x, 0, 0, \dots)$, it is easy to see that $x_1 = x$, $x_2 = \phi(x)$, $x_3 = \phi^2(x)$, $\dots$. Since the envelope is a direct sum, eventually $x_{n+1} = 0$ and we get $\phi^n(x) = 0$. ∎

Any representation X of $\tilde{A}_1$, say $M_0 \underset{\psi}{\overset{\phi}{\rightleftarrows}} M_1$, has an obvious turning endomorphism,

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given by the commutative diagram

$$\begin{array}{ccc} M_0 & \overset{\phi}{\underset{\psi}{\rightleftarrows}} & M_1 \\ {\scriptstyle \psi\circ\phi}\downarrow & & \downarrow{\scriptstyle \phi\circ\psi} \\ M_0 & \overset{\phi}{\underset{\psi}{\rightleftarrows}} & M_1 \end{array}$$

We shall denote this endomorphism by T_X. For a representation X of $\tilde{A}_n$over R, the turning endomorphism $T_X\colon X \to X$ is defined by

$$T_X(v_i) := X(a_{i-1})\cdots X(a_0)X(a_n)\cdots X(a_{i+1})X(a_i).$$

Proposition 5.2 *Let R be a left noetherian ring and E an injective left R-module. For every arrow a of $\tilde{A}_n$, the representation $F_a(E[x^{-1}])$ is injective.*

Proof We shall freely use the equivalence of the category of left $R\tilde{A}_n$-modules and of $(\tilde{A}_n, R\text{-Mod})$. This means that every representation of $\tilde{A}_n$ will be considered as a left module over $R\tilde{A}_n$. Note that $R\tilde{A}_n$ is left noetherian. Consider the subrepresentation $F_a(E)$ of the injective representation $F_a(E[[x^{-1}]])$. The turning endomorphism of $F_a(E[[x^{-1}]])$ is clearly 0 on $F_a(E)$. Now $F_a(E[[x^{-1}]])$ contains an injective envelope of $F_a(E)$ and, by the lemma above, the turning endomorphism is locally nilpotent on this envelope. This means the injective envelope is contained in $F_a(E[x^{-1}])$. But (as a module), $F_a(E[x^{-1}])$ is an essential extension of $F_a(E)$, and hence $F_a(E[x^{-1}])$ is the injective envelope. ■

Given a vertex $v \in \tilde{A}_n$ and an R-module M, denote by $C_v(M)$ the representation defined by

$$C_v(M)(v_i) = \begin{cases} M & \text{if } v_i = v \\ 0 & \text{otherwise.} \end{cases}$$

If $E = E(M)$ is the injective envelope of M and a is the arrow with initial vertex v, then $C_v(M) \subseteq F_a(E) \subseteq F_a(E[x^{-1}])$ is an essential extension of representations and hence the injective envelope of $C_v(M)$. Because the representation $C_v(E)$ is clearly uniform, its injective envelope $F_a(E[x^{-1}])$ is indecomposable. This proves the following.

Proposition 5.3 *Let R be a left noetherian ring and E an indecomposable injective left R-module. For every arrow a of $\tilde{A}_n$, the injective representation $F_a(E[x^{-1}])$ is indecomposable.*

Suppose that $n \geq 1$ and that $u \xrightarrow{a} v \xrightarrow{b} w$ is a subquiver of $\tilde{A}_n$ where possibly $u = w$. For an R-module M, we have the equation

$$F_b(M[x^{-1}])/C_v(M) = F_a(M[x^{-1}]).$$

If $M = E$ is an injective indecomposable R-module, this shows there is a nontrivial morphism $\eta\colon F_b(E[x^{-1}]) \to F_a(E[x^{-1}])$ though the two indecomposable injective representations are not isomorphic.

6 The Torsion Free Case

We now classify the remaining indecomposable injective representations of $\tilde{A}_n$. We say that the turning endomorphism T_X of a representation X is locally nilpotent on X if when we view X as an $R\tilde{A}_n$-module, the corresponding endomorphism of the module is locally nilpotent. In this case, we say that X is a torsion representation. If a representation Y has no non-trivial torsion subrepresentations, we say that Y is torsion free. Given a representation X of $\tilde{A}_n$ over R define $C(X)$ to be the maximal subrepresentation of X of the form $\bigoplus_{v\in\tilde{A}_n} C_v(M_v)$. Then Y is torsion free iff $C(Y) = 0$ and X is torsion iff it is an essential extension of $C(X)$. Let $\mathcal{X}$ be the class of torsion representations and $\mathcal{Y}$ the torsion free representations of $\tilde{A}_n$ over R. For the terminology in the next result see [9, Chapter VI].

Proposition 6.1 *If R is left noetherian, then $(\mathcal{X}, \mathcal{Y})$ is a hereditary and stable torsion theory on the category of representations of $\tilde{A}_n$ over R.*

Proof It is straight-forward to see that $(\mathcal{X}, \mathcal{Y})$ is a hereditary torsion theory. The theory is called *stable* if the injective envelope of a torsion X is torsion. But if $X \neq 0$ is torsion, then it is an essential extension of a subrepresentation of the form

$$\bigoplus_{v\in\tilde{A}_n} C_v(M_v).$$

The injective envelope of X is then the direct sum of the envelopes of the respective summands $C_v(M_v)$:

$$\bigoplus_{a\in\tilde{A}_n} F_a\big(E(M_{i(a)})[x^{-1}]\big).$$

We have already noted that this representation is torsion. ■

The proof of Proposition 6.1 also shows the following.

Proposition 6.2 *Let R be a left noetherian ring. Every injective torsion representation of $\tilde{A}_n$ over R has the form*

$$\bigoplus_{a\in\tilde{A}_n} F_a(E_a[x^{-1}])$$

where each E_a is an injective left R-module.

Proposition 6.3 *Let R be a ring and Z is an injective representation of $\tilde{A}_n$. For every arrow $a \in \tilde{A}_n$, the R-morphism $Z(a)$ is surjective.*

Proof We noted at the beginning of the previous section how the category of representations is cogenerated by objects of the form $F_a\big(E\big[[x^{-1}]\big]\big)$, each of which has the desired property. Since the property is preserved when taking products or summands, the result follows. ■

Theorem 6.4 *Let R be a left noetherian ring and a an arrow of the cyclic quiver $\tilde{A}_n$. Every torsion-free injective representation of $\tilde{A}_n$ over R has the form $F_a(\bar{E})$ where $\bar{E}$ is an injective $R[x, x^{-1}]$-module. Conversely, if $\bar{E}$ is an injective $R[x, x^{-1}]$-module, then $F_a(\bar{E})$ is an injective representation of $\tilde{A}_n$ over R. The injective representation $F_a(\bar{E})$ is indecomposable if and only if $\bar{E}$ is an indecomposable injective left $R[x, x^{-1}]$-module.*

Proof Let Z be an indecomposable, injective, torsion-free representation of $\tilde{A}_n$. By Proposition 6.3, $Z(b)$ is surjective for every arrow b of $\tilde{A}_n$. Since Z is torsion-free, all the R-morphisms $Z(b)$ are in fact isomorphisms and there is no loss of generality in assuming that for all vertices $v \in \tilde{A}_n$, $Z(v) = \bar{E}$ for some R-module $\bar{E}$ and that $Z(b)$ is the identity map for all arrows $b \neq a$ of $\tilde{A}_n$. The action of $x := Z(a)$ on $\bar{E}$ gives it the structure of an $R[x]$-module. In fact, as Z is torsion free, $\bar{E}$ has the structure of an $R[x, x^{-1}]$-module and as such, we see that $Z \cong F_a(\bar{E})$. Let us note that $\bar{E}$ is an injective $R[x, x^{-1}]$-module. For if $\bar{E}$ is a submodule of the $R[x, x^{-1}]$-module N, then $F_a(\bar{E}) \subset F_a(N)$ is an extension of representations. As $Z = F_a(\bar{E})$ is injective, it is a summand of $F_a(N)$ and hence $\bar{E}$ is a summand of N. Conversely, let $\bar{E}$ be an injective $R[x, x^{-1}]$-module. The injective envelope of $F_a(\bar{E})$ is torsion free so, without loss of generality, has the form $F_a(\bar{E}) \subset F_a(N)$ where $x := F_a(N)(a) \colon N \to N$ is an isomorphism. In this way N becomes an $R[x, x^{-1}]$-module. As an $R[x, x^{-1}]$-module $\bar{E} \subset N$ is a retract of N. Thus $F_a(\bar{E})$ is a summand of $F_a(N)$ and is therefore injective. If $\bar{E}$ is furthermore an indecomposable $R[x, x^{-1}]$-module, then it is obvious that $F_a(\bar{E})$ is also indecomposable. The converse follows from the observation that $F_a(M \oplus N) = F_a(M) \oplus F_a(N)$. ∎

For example, if $R = \mathbb{C}$, the field of complex numbers, then there are $n + 1$ torsion indecomposable injective representations of $\tilde{A}_n$ over $\mathbb{C}$, arising from the injective indecomposable $\mathbb{C}$. If $a_0 \colon v_0 \to v_1$ (where possibly $1 = 0$) is the "first" arrow of $\tilde{A}_n$, the torsion free indecomposable injectives correspond to $F_{a_0}(\mathbb{C}[(x - c)^{-1}]$ for each nonzero $c \in \mathbb{C}$, and there is a generic one $F_{a_0}\big(\mathbb{C}(x)\big)$. Let us summarize the results of this and the previous section.

Theorem 6.5 *Let R be a left noetherian ring and $\tilde{A}_n$ the cyclic quiver on $n + 1$ vertices $v_0, \dots, v_n$ and $n+1$ arrows $a_0, \dots, a_n$ with $i(a_j) = v_j$. Every injective representation X of $\tilde{A}_n$ over R has a decomposition, unique up to isomorphism, of the form*

$$X \cong \Big(\bigoplus_{a \in \tilde{A}_n} F_a(E_a[x^{-1}])\Big) \oplus F_{a_0}(\bar{E})$$

where each E_a is an injective left R-module and $\bar{E}$ is an injective left $R[x, x^{-1}]$-module.

Proof It remains to prove the uniqueness of the decomposition. The injective R-modules E_a may be recovered (up to isomorphism) from the equation

$$C(X) = \bigoplus_{a \in \tilde{A}_n} C_{i(a)}(E_a)$$

and because the torsion representation of an injective splits, the $R[x, x^{-1}]$-module $\bar{E}$ is also unique up to isomorphism. ∎

Theorem 6.6 *Let R be a left noetherian ring. The following is a complete list, without repetition, of the indecomposable injective representations of $\tilde{A}_n$ over R:*

1. *$F_a(E_{\underline{a}}[x^{-1}])$ where a is an arrow of $\tilde{A}_n$ and E_a is an indecomposable injective left R-module.*
2. *$F_{a_0}(\bar{E})$ where $\bar{E}$ is an indecomposable injective left $R[x, x^{-1}]$-module.*

Theorem 6.6 asserts that the spectrum of

$$(\tilde{A}_n, R\text{-Mod}) \cong R\tilde{A}_n\text{-Mod}$$

consists of $n + 1$ copies of the left spectrum of R (corresponding to the indecomposable injective torsion representations) and one copy of the left spectrum of $R[x, x^{-1}]$.

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