ON THE INVARIANCE OF THE SPECTRUM IN LOCALLY m-CONVEX ALGEBRAS

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In this paper we consider two closely related problems concerning a complete locally m-convex (LMC) algebra A with identity. Let a be a fixed element of A, and let $P(a)$ be the smallest closed subalgebra containing a and 1. If B is any subalgebra containing a and 1, we let $\sigma(a; B)$ denote the spectrum of a as an element of B. (I) Describe the set $\sigma(a; P(a))$ in terms of $\sigma(a; A)$. (II) Give necessary and sufficient conditions in order that $\sigma(a; B) = \sigma(a; A)$ for every closed subalgebra B of A which contains a and 1.

If A is a Banach algebra, one can give a simply stated condition for (II) and a nice description of $\sigma(a; P(a))$; cf. (**3**, Theorem 1.6.13). The condition for (II) is that $\sigma(a; A)$ fail to separate the plane. Examples 2.1 and 2.3 below show that this condition is neither necessary nor sufficient for (II) in our setting. In fact, knowing only the set $\sigma(a; A)$, one can neither describe $\sigma(a; P(a))$ nor even decide whether it is larger than $\sigma(a; A)$. The extent of the pathology is shown in Example 2.4, where we have two elements a and b of an algebra A such that $\sigma(a; P(a)) = \sigma(a; A) = \sigma(b; A) \neq \sigma(b; P(b))$.

The key to the solutions of (I) and (II) is the fact that a complete LMC algebra A may be realized as an inverse limit of a family $\{A_n: n \in D\}$ of Banach algebras, where each A_n contains a dense homomorphic image $\pi_n A$ of A. If we identify $a \in A$ and its image $\{\pi_n a\}$ in $\lim \operatorname{inv}\{A_n\}$, we have $\sigma(a; A) = \bigcup_n \sigma(a; A, n)$, where $\sigma(a; A, n) = \sigma(\pi_n a; A_n)$ (by definition). The description of $\sigma(a; P(a))$ and the conditions for the invariance of the spectrum of a are given in terms of the ordered pair $(\sigma(a; A), \{\sigma(a; A, n): n \in D\})$. Specifically,

$$\sigma(a; P(a)) = \bigcup_n \hat{\sigma}(a; A, n)$$

where, for a compact subset K of $\mathfrak{C}$, $\hat{K}$ denotes the polynomially convex hull (defined below). Hence, $\sigma(a; P(a)) = \sigma(a; A)$ if, and only if, for each $n \in D$, $\hat{\sigma}(a; A, n) \subseteq \sigma(a; A)$; i.e., the union of the bounded components of $\mathfrak{C}\backslash\sigma(a; A, n)$ lies in $\sigma(a; A)$. Each of the sets $\sigma(a; A, n)$ may separate the plane as long as the "holes" are in some other set $\sigma(a; A, i)$ or a union thereof (cf. Example 2.1 below).

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1. Preliminaries. A *locally m-convex* (hereafter LMC) *algebra* is a locally convex (Hausdorff) topological algebra A whose topology is determined by a family $\{||\cdot||_n : n \in D\}$ of pseudonorms (submultiplicative convex symmetric functionals), which may be assumed to be directed by $D(=(D, \leqslant))$; $n \leqslant k$ implies that $||x||_n \leqslant ||x||_k$ for each $x \in A$. We shall call such a family a *directed family of pseudonorms for A*. Each $n \in D$ determines a Banach algebra A_n with norm $||\cdot||_n$ and a continuous homomorphism π_n of A onto a dense subalgebra of A_n (in fact, A_n is usually obtained as the completion of

$$A/\{x: ||x||_n = 0\}$$

with the induced norm). If $n \leqslant k$, then π_n and π_k induce a norm-decreasing homomorphism π_n^k of A_k onto a dense subalgebra of A_n. The resulting family of Banach algebras and continuous maps is a dense inverse limit system and A may be identified topologically and algebraically with a dense subalgebra of $\lim\operatorname{inv}\{A_n\}$, which is a closed subalgebra of $\Pi_n A_n$. If A is complete, we identify A and $\lim\operatorname{inv}\{A_n\}$, the correspondence being $x \to \{\pi_n x\}$.

We consider here only complete LMC algebras with identity. Completeness is essential if we are to exploit the existence of the family $\{A_n\}$, and the assumption that A has an identity is no real restriction, since we could always adjoin one, after defining the spectrum in terms of quasi-inverses, and obtain the same results.

Let A be a complex LMC algebra with identity 1 and let $\{||\cdot||_n : n \in D\}$ be a directed family of pseudonorms for A. If $a \in A$, then $\sigma(a; A)$ is the spectrum of a as an element of A, and $\sigma(a; A, n)$ is defined to be $\sigma(\pi_n a; A_n)$. We have that $\sigma(a; A) = \bigcup_n \sigma(a; A, n)$. If B is a closed subalgebra of A, then B with the family of restricted pseudonorms is a complete LMC algebra and for each $n \in D$, $B_n = \overline{\pi_n B} \subseteq A_n$. One special closed subalgebra which we need is $P(a)$, $a \in A$, which is the closure in A of the algebra $\{p(a): p \in P\}$, where P is the family of all polynomials in one variable with complex coefficients.

If A is a commutative complete LMC algebra with identity, the *spectrum of* A is the space M of non-zero continuous complex-valued homomorphisms with the Gelfand topology. If $A = \lim\operatorname{inv}\{A_n\}$, then π_n induces a homeomorphism of $M(A_n)$ onto a compact equicontinuous subset M_n of M. This family $\{M_n\}$ covers every equicontinuous subset of M. Each $a \in A$ defines an element of $C(M)$ by $\hat{a}(m) = m(a)$ for each $m \in M$. Moreover, $\sigma(a; A) = \hat{a}(M)$, and $\sigma(a; A, n) = \hat{a}(M_n)$ for each $n \in D$. For proofs of the facts listed above, the reader is referred to **(2)**.

2. Some examples. We assume throughout this section that A is a complete LMC algebra with identity 1.

Definition 2.1. For $a \in A$ we say that $\sigma(a; A)$ is *invariant* provided that $\sigma(a; B) = \sigma(a; A)$ for every closed subalgebra B of A which contains a and 1.

Definition 2.2. If $a \in A$ and if $\{||\cdot||_n: n \in D\}$ is a directed family of pseudonorms for A, we say that the ordered pair $(\sigma(a; A), \{\sigma(a; A, n) : n \in D\})$ is *invariant* provided that $(\sigma(a; A), \{\sigma(a; A, n)\}) = (\sigma(a; B), \{\sigma(a; B, n)\})$ for every closed subalgebra B of a which contains a and 1. We say that $\sigma(a; A)$ is *strongly invariant* if there exists a directed family $\{||\cdot||_n\}$ of pseudonorms such that $(\sigma(a; A), \{\sigma(a; A, n)\})$ is invariant.

We state, without proof, two theorems. The first is Theorem 1.6.13 of **(3)** and the second is an immediate consequence of the first and the fact that failure to separate the plane is a topological property of compact sets.

THEOREM 2.1. *If A is a Banach algebra and $a \in A$, then $\sigma(a; A)$ is invariant if, and only if, $\sigma(a; A)$ fails to separate the plane.*

THEOREM 2.2. *If A and B are Banach algebras, and if a and b are elements of A and B, respectively, such that $\sigma(a; A)$ and $\sigma(b; B)$ are homeomorphic, then $\sigma(a; A)$ is invariant if, and only if, $\sigma(b; B)$ is.*

We consider the following statements about an element a of A:

(1) $\sigma(a; A)$ is invariant.

(2) $\sigma(a; A)$ fails to separate the plane.

(3) $\sigma(a; A)$ is strongly invariant.

(4) There exists a directed family $\{||\cdot||_n: n \in D\}$ of pseudonorms such that $\sigma(a; A, n)$ fails to separate the plane, for each $n \in D$.

We note that the equivalence of (3) and (4) is an immediate consequence of Theorem 2.1. We also have that (3) implies (1), since

$$\sigma(a; P(a)) = \bigcup_n \sigma(a; P(a), n)$$

for every family of pseudonorms and we can choose one for which we have $\sigma(a; A, n) = \sigma(a; P(a), n)$ for each $n \in D$. We now give examples to show that this is the only implication which holds among (1), (2), and (3).

Example 2.1. Let $T = \{t \in \mathfrak{C}: 1 < |t| < 2\} \cup \{0\}$. T is a locally compact Hausdorff space and $A = (C(T)$, compact-open topology) is a complete LMC algebra with identity; cf. **(2**, Ex. D.3 and Lemma D.5**)**. Also, $M = T$ **(2**, Ex. 7.6**)**. We define $a: T \to \mathfrak{C}$ by

$$a(0) = 0, \qquad a(t) = (|t| - 1)(2 - |t|)^{-1}t \quad \text{for } t \neq 0.$$

Then $a \in A$ and $\sigma(a; A) = \hat{a}(M) = a(T) = \mathfrak{C}$. Thus, $\sigma(a; A)$ is trivially invariant and $\sigma(a; A)$ fails to separate the plane. We shall prove that (4) fails to hold, thereby showing that neither (1) nor (2) implies (4), equivalently (3).

We choose a decreasing sequence $\{r_n: n = 1, 2, \ldots\}$ such that $r_1 < 2$ and $\lim r_n = 1$, an increasing sequence $\{R_n: n = 1, 2, \ldots\}$ such that $r_1 < R_1$ and $\lim R_n = 2$, and let $K_n = \{0\} \cup \{t: r_n \leqslant |t| \leqslant R_n\}$ for each n. We define $||\cdot||_n$ to be the supremum-on-K_n pseudonorm. Then $\{||\cdot||_n: n = 1, 2, \ldots\}$ is a directed family of pseudonorms for A, and for each $n \in N$ we have

$$\sigma(a; A, n) = \hat{a}(M_n) = a(K_n),$$

which is a compact annulus plus the isolated point 0. Let $\{||\cdot||_n' : n \in D\}$ be a directed family of pseudonorms for A, and fix $n \in \mathfrak{N}$. There exists $k \in D$ and $\epsilon > 0$ such that $||x||_n < \epsilon||x||_k'$ for each $x \in A$. Since $\{||\cdot||_n : n \in \mathfrak{N}\}$ is a directed family for A, there exists $p \in \mathfrak{N}$ and $\delta > 0$ such that

$$||x||_n \leqslant \epsilon||x||_k' \leqslant \delta||x||_p.$$

But then $K_n \subseteq M_k' \subseteq K_p$, where M_k' corresponds to $M(A_k')$ under the induced map (A_k' = completion of $A/\{x: ||x||_k' = 0\}$). Hence, $\sigma(a; A, 'k)$ separates the plane and (4) fails to hold.

Example 2.2. Let $T = [0, 2\pi)$ and let $A = (C(T)$, compact-open topology). We define $a \in A$ by $a(t) = \exp(it)$. Then $\sigma(a; A) = a(T) =$ unit circle. Thus, (2) fails to hold. If $\{t_n, n = 1, 2, \ldots\}$ is an increasing sequence in $[0, 2\pi)$ with limit 2π, then $\{||\cdot||_n : n \in \mathfrak{N}\}$ is a directed family of pseudonorms for A, where $||\cdot||_n$ = supremum on $[0, t_n]$. For this family,

$$\sigma(a; A, n) = \{\exp(it) : 0 \leqslant t \leqslant t_n\}$$

for each $n \in \mathfrak{N}$. Thus, (4) holds; hence, (1) holds; and neither implies (2).

Example 2.3. Let U_1 be the open unit disk, U_δ the open δ-disk ($0 < \delta < 1$), $T = U_1 \backslash U_\delta$, and Γ_δ the δ-circle about 0. We let $A = (C(T)$, compact-open topology), $A(\delta) = \{x \in C(\bar{U}_\delta) : x|U_\delta$ is analytic$\}$, and $B = \{a \in A$: there exists $x \in A(\delta)$ such that $x|\Gamma_\delta = a|\Gamma_\delta\}$. We note that if $a \in A$ and if $a(t) = t$ on Γ_δ, then $a \in B$. Moreover, $M(B) = U_1$ and if $b \in B$, then $\sigma(b: B)$ is the range of the unique extension $\hat{b}$ of b to all of U_1. We define $a: T \to \mathfrak{C}$ by $a(t) = (1 - \delta)(1 - |t|)^{-1}$ for $t \in T$. Then $a \in A$ and $a(t) = t$ on Γ_δ. Thus, $a \in B$. Now, $\sigma(a; A) = a(T) = U_\delta$; $\hat{a}(U_1) = \sigma(a; B)$. Hence, $\sigma(a; A)$ is not invariant, while $\sigma(a; A)$ fails to separate the plane. Thus, (1) does not imply (2).

Since $\sigma(a; A)$ can be unbounded, we offer the following alternative to (2).

(2′) $\sigma(a; A)$ fails to separate the extended plane $\mathfrak{C}^* = \mathfrak{C} \cup \{\infty\}$, the one-point compactification of $\mathfrak{C}$.

We shall show below (Corollary 3.3.2) that (2′) is a sufficient condition for the invariance of $\sigma(a; A)$. It is, moreover, independent of the other sufficient condition (3). We briefly indicate the relation of (2′) to the other statements. (i) Neither of (2′) and (3) implies the other. Example 2.2 shows that (3) does not imply (2′), while Example 2.1 shows that the other possible implication fails to hold. (ii) Neither of (2) and (2′) implies the other. Example 2.3 shows that (2) does not imply (2′). Let $A = (C(\mathscr{R})$, compact-open topology) and define $a(t) = t$ on $\mathscr{R}$. Then $\sigma(a; A) = \mathscr{R}$, which separates $\mathfrak{C}$, but not $\mathfrak{C}^*$. (iii) Example 2.2 shows that (2′) is not implied by (1).

Example 2.4. Let $A = (C[0, 2\pi)$, compact-open topology) and define a, b in A by $a(t) = \exp(it)$, $b(t) = \exp(2it)$ for $t \in [0, 2\pi)$. Then, $\sigma(a; A) =$

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$\sigma(b; A)$ = unit circle. We let $\{t_n: n \in \mathfrak{N}\}$ be an increasing sequence in $[0, 2\pi)$ with $t_1 = \pi$ and $\lim t_n = 2\pi$, and define $||\cdot||_n$ = supremum on $[0, t_n]$ for $n \in \mathfrak{N}$. The family $\{||\cdot||_n: n \in \mathfrak{N}\}$ is a directed family of pseudonorms for A and from our observations preceding Example 2.1 we have that $\sigma(a; A)$ is invariant. However, $\sigma(b; A, 1) = \sigma(\pi_1 b; A_1)$ = unit circle, since $A_1 = C[0, \pi]$. If we let B denote the closure in A of the polynomials in b, then $\sigma(b; B, 1)$ = closed unit disk. Hence, $\sigma(b; A)$ is not invariant.

From these examples, especially the last one, we see that the invariance of $\sigma(a; A)$ depends on the pair $(\sigma(a; A), \{\sigma(a; A, n\})$ rather than on $\sigma(a; A)$ alone. We include one more example to show that $\sigma(a; A)$ may be strongly invariant even though some pairs are not invariant.

Example 2.5. Let T denote the open unit disk and let $A = (C(T)$, compact-open topology). We fix an increasing sequence $\{r_n: n \in \mathfrak{N}\}$ of positive real numbers with limit 1. Let $K_n = \{t: |t| \leqslant r_n\}$, $\Gamma_n = \{t: |t| = r_n\}$, and $F_n = K_n \cup \Gamma_{n+1}$; and let $||\cdot||_n$ be the supremum on K_n, and $||\cdot||_n'$ the supremum on F_n for each $n \in \mathfrak{N}$. Finally, we let a be defined by $a(t) = t$ for $t \in T$. We use primes to denote objects related to the family $\{||\cdot||_n': n \in \mathfrak{N}\}$. For each n $\in \mathfrak{N}$ we have

$$\sigma(a; A, n) = a(K_n) = K_n = \sigma(a; P(a), n),$$

and $\sigma(a; A, 'n) = a(K_n) \cup a(\Gamma_{n+1})$. But $\sigma(a; P(a), 'n) = a(K_{n+1})$. Thus, for each $n \in \mathfrak{N}$, $\sigma(a; A, n) = \sigma(a; P(a), n)$, and the pair $(\sigma(a; A), \{\sigma(a; A, n)\})$ is invariant, while the pair $(\sigma(a; A), \{\sigma(a; A, 'n)\})$ is not.

3. The main results.

Definition 3.1. If K is a compact subset of $\mathfrak{C}$, the *polynomially convex hull* $\hat{K}$ of K is the set

$$\{\zeta \in \mathfrak{C}: |p(\zeta)| \leqslant ||p||_K \text{ for each } p \in P\},$$

where $||\cdot||_K$ denotes the supremum on K. We say K is *polynomially convex* if $K = \hat{K}$.

We sketch a proof of the following.

THEOREM 3.1. *If A is a Banach algebra with identity, and $a \in A$, then* $\sigma(a; P(a)) = \hat{\sigma}(a; A)$.

Proof. We reduce this immediately to the commutative case by fixing a maximal commutative subalgebra C which contains a. For this algebra we have $\sigma(a; C) = \sigma(a; A)$. We also have that $P(a) \subseteq C$ and $\sigma(a; C) = \{m(a): m \in M(C)\}$. A simple application of the spectral mapping theorem for polynomials yields $\sigma(a; P(a)) \subseteq \hat{\sigma}(a; C)$. For the other inclusion we fix $\zeta \in \hat{\sigma}(a; C)$, define $m(a) = \zeta$, and extend m to the dense subalgebra $\{p(a): p \in P\}$ of $P(a)$. From the definition of $\hat{\sigma}(a; C)$ it follows that m is continuous, hence; extendible to $P(a)$. Thus, $\zeta \in \sigma(a; P(a))$.

COROLLARY 3.1. *If A is a Banach algebra with identity and $a \in A$, then $\sigma(a; A)$ is invariant if, and only if, $\sigma(a; A)$ is polynomially convex.*

THEOREM 3.2. *Let A be a complete LMC algebra with identity, let $\{||\cdot||_n: n \in D\}$ be a directed family of pseudonorms for A, and let a be an element of A. Then*

$$\sigma(a; P(a)) = \bigcup_n \hat{\sigma}(a; A, n).$$

Proof. From Lemma 7.8 of **(2)** we have that $P(a) = \lim \operatorname{inv}\{\pi_n P(a)^-\}$. But $\pi_n P(a)^- = P(\pi_n a) \subseteq A_n$. Hence,

$$\begin{aligned}\sigma(a; P(a)) &= \bigcup_n \sigma(a; P(a), n) = \bigcup_n \sigma(\pi_n a, P(a)_n) \\ &= \bigcup_n \sigma(\pi_n a, P(\pi_n a)) = \bigcup_n \hat{\sigma}(\pi_n a, A_n)\end{aligned}$$

from Theorem 3.1. Since $\sigma(a; A, n) = \sigma(\pi_n a, A_n)$, we have the desired equality.

THEOREM 3.3. *Let A be a complete LMC algebra with identity and let a be an element of A. Then $\sigma(a; A)$ is invariant if, and only if, there exists a directed family $\{||\cdot||_n: n \in D\}$ of pseudonorms for A such that for each $n \in D$ we have $\hat{\sigma}(a; A, n) \subseteq \sigma(a; A)$.*

Proof. Suppose $\sigma(a; A)$ is invariant. Let $\{||\cdot||_n: n \in D\}$ be any directed family of pseudonorms for A. Then by Theorem 3.2 we have

$$\sigma(a; A) = \sigma(a; P(a)) = \bigcup_n \hat{\sigma}(a; A, n),$$

so that $\hat{\sigma}(a; A, n) \subset \sigma(a; A)$ for each $n \in D$. The converse also follows easily from Theorem 3.2.

We note that we have actually shown that if for one family $\{||\cdot||_n\}$ of pseudonorms we have $\hat{\sigma}(a; A, n) \subseteq \sigma(a; A)$ for each n, then the same is true for every family of pseudonorms. Thus, the notion of invariance of $\sigma(a; A)$ is independent of the realization of A as an inverse limit, as contrasted with the notion of invariance of pairs. Also, in order to check for invariance we can choose the most convenient family of pseudonorms without worrying about the strangely constructed ones. (For example, the second family constructed in Example 2.5 is not a natural one to consider, but we need not worry about it if we only want a description of $\sigma(a; P(a))$, rather than a detailed knowledge of its pieces.)

The theorem also indicates that each of the compact sets $\sigma(a; A, n)$ may separate the plane as long as the bounded components of its complement lie inside $\sigma(a; A)$. An example of this is given in Example 2.1 above.

Definition 3.2. A subset S of $\mathfrak{C}$ is called *polynomially convex* if $\hat{K} \subseteq S$ for each compact subset K of S.

COROLLARY 3.3.1. *If $\sigma(a; A)$ is polynomially convex, then $\sigma(a; A)$ is invariant. This condition is not necessary.*

Proof. We fix a directed family $\{||\cdot||_n\}$ of pseudonorms. For each $n \in D$, $\sigma(a; A, n)$ is a compact subset of $\sigma(a; A)$; hence, $\hat{\sigma}(a; A, n) \subseteq \sigma(a; A)$.

Example 2.2 above shows that an invariant $\sigma(a; A)$ need not be polynomially convex.

COROLLARY 3.3.2. *If $\sigma(a; A)$ fails to separate $\mathfrak{C}^*$, then $\sigma(a; A)$ is invariant.*

Proof. We shall show that if S is any subset of $\mathfrak{C}$ which is not polynomially convex, then S separates $\mathfrak{C}^*$. The assertion will then follow from Corollary 3.3.1. We shall use the following facts about a compact subset K of $\mathfrak{C}$. We let U_K be the unbounded component of $\mathfrak{C}\backslash K$ and let W_K be the union of the bounded components of $\mathfrak{C}\backslash K$ (may be empty). The sets U_K and W_K are open in the open set $\mathfrak{C}\backslash K$; hence, are open in $\mathfrak{C}$ and are disjoint. For a compact set F in $\mathfrak{C}$ we have that $\mathfrak{C}\backslash F$ is connected if, and only if, $F = \hat{F}$. Now, $K \cup W_K$ is compact, contains K, and is contained in $\hat{K}$. Then, $\hat{K} = (K \cup W_K)^\wedge$. But $\mathfrak{C}\backslash(K \cup W_K) = U_K$ is connected. Hence, $\hat{K} = K \cup W_K$.

If S is not polynomially convex, then there exists a compact subset K of S such that $\hat{K} \not\subseteq S$. Thus, $\mathfrak{C}\backslash K = W_K \cup U_K$, where W_K is bounded and non-empty and $\mathfrak{C}\backslash U_K$ is compact. The set $U_K^* = U_K \cup \{\infty\}$ is, therefore, an open set in $\mathfrak{C}^*$ containing ∞ and disjoint from W_K, and $\mathfrak{C}^*\backslash K = W_K \cup U_K^*$. We let $O_1 = W_K \cap (\mathfrak{C}^*\backslash S)$ and $O_2 = U_K^* \cap (\mathfrak{C}^*\backslash S)$. Then O_1 and O_2 are open, non-empty disjoint subsets of $\mathfrak{C}^*\backslash S$ and $\mathfrak{C}^*\backslash S = O_1 \cup O_2$.

We close with some remarks on a special case and the extension to the joint spectrum of a finite family in the commutative case. If A is a Banach algebra with norm $||\cdot||$, then $\{||\cdot||\}$ is a directed family for A and for $a \in A$ the ordered pair which determines invariance is just $(\sigma(a; A), \{\sigma(a; A)\})$. Hence, Theorems 3.2 and 3.3 reduce to Theorems 3.1 and 2.1 in case A is normed and complete.

If A is commutative and $\{a_1, \ldots, a_k\}$ is a finite family in A, the joint spectrum $\sigma(a_1, \ldots, a_k)$ is $\{(m(a_1), \ldots, m(a_k)) : m \in M\}$, a subset of $\mathfrak{C}^k$. This concept was studied for special LMC algebras (F-algebras) by Arens (**1**). If K is a compact subset of $\mathfrak{C}^k$, then the polynomially convex hull $\hat{K}$ of K is the set $\{\zeta \in \mathfrak{C}^k : |p(\zeta)| \leqslant ||p||_K$ for each $p \in P^k\}$, where P^k is all polynomials in k variables with complex coefficients. The elements $a_1, \ldots, a_k$ generate a smallest closed subalgebra $P(a_1, \ldots, a_k)$ of A and we have the equality

$$\sigma(a_1, \ldots, a_k; P(a_1, \ldots, a_k)) = \bigcup_n \hat{\sigma}(a_1, \ldots, a_k; A, n),$$

where n runs over a set D directing a family $\{||\cdot||_n\}$ of pseudonorms for A. From the equality above we easily obtain a condition for the invariance of $\sigma(a_1, \ldots, a_k; A)$ in terms of pairs $(\sigma(a_1, \ldots, a_k; A), \{\sigma(a_1, \ldots, a_k; A, n)\})$.

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