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On a Question of Igusa: Towards a Theory of Several Variable Asymptotic Expansions: I

BEN LICHTIN
e-mail: lichtin@math.rochester.edu

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Abstract. A question of Igusa (from 1978) inquires about the singular behavior of the singular series, determined by a polynomial mapping $\mathbf{P}: K^n \to K^m$, $m \leqslant n$, where K is a local field of characteristic zero. This paper describes in geometric terms the singularities of the singular series for two classes of polynomial maps $\mathbf{P} = (P_1, P_2): K^n \to K^2$. The main result, which makes possible this description, is a type of uniformization of $\mathbf{P}$ by finitely many monomial maps $\mu(\mathbf{x}) = (\mathbf{x}^{\mathbf{M}_1}, \mathbf{x}^{\mathbf{M}_2})$, such that *rank* $\binom{\mathbf{M}_1}{\mathbf{M}_2} = 2$. This is proved using resolution of singularities. Using this result, nontrivial estimates of oscillatory integrals with phase $\lambda_1 P_1 + \lambda_2 P_2$ are possible. These will be described elsewhere.

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Introduction

Mellin inversion over p-adic fields is the method used by Igusa to determine the singular behavior of the local singular series determined by one polynomial. In particular, for $P \in \mathbb{Q}_p[x_1, \dots, x_n]$, this is the function

$$\begin{aligned} t &= \textit{regular value of } \ P \to F(t) \\ &=_{\mathrm{def}} \lim_{e\to\infty} p^{-e(n-1)} \cdot \#\{\xi \in (\mathbb{Z}/p^e)^n : P(\xi) \equiv t(p^e)\}. \end{aligned}$$

It is a locally constant function in a neighborhood of a regular value of P but has nontrivial singular behavior as t converges to a critical value of P. A standard argument [I, pg. 83ff.] also shows that

$$F(t) = \int_{\{P=t\}\cap\mathbb{Z}_p^n} |\mathrm{d}x/\mathrm{d}P|,$$

where $|\mathrm{d}x/\mathrm{d}P|$ denotes a Borel measure on the fiber that is induced by the Leray residue differential form.

The inverse Mellin transform [T] over $\mathbb{Q}_p$ expresses the local singular series as an infinite trigonometric sum, each summand of which has a factor equal to the residue of a certain zeta function, or of its twist by one of an infinite set of characters on $\mathcal{U}_p =_{\text{def}} \mathbb{Z}_p - p\mathbb{Z}_p$. The key to Igusa's method is a finiteness result [I, pg. 96], proved by applying (embedded) resolution of singularities to the divisor $\{P = 0\}$ in $\mathbb{Q}_p^n$. This shows that only finitely many twists of the zeta function contribute to the inverse Mellin transform. Thus, describing the asymptotic behavior (as t converges to a critical value of P) of the local singular series reduces to a problem that can be solved in terms of a finite set of numerical data, created by the resolution of singularities.

Igusa's principal motivation, however, was to estimate the asymptotic behavior of the Fourier transform of the local singular series. For this, knowledge of the precise singular behavior of the singular series is an essential prerequisite. The value of the Fourier transform at the argument a/p^r, $(a, p) = 1$, equals the generalized Gaussian sum

$$G(a/p^r) = p^{-rn} \sum_{\xi \in (\mathbb{Z}/p^r\mathbb{Z})^n} \mathrm{e}^{\frac{2\pi i a P(\xi)}{p^r}}.$$

The ability to obtain strong decay in p^{-r}, for each prime p, such as $G(a/p^r) = O(p^{-r\alpha})$ for some $\alpha > 2$, when certain geometric properties were satisfied, was an important consequence of his theory of 'asymptotic expansions' in one p-adic variable.

Motivated by this work, Igusa asked in his book [I, pg. 32] how one could extend his method to analyze the singularities of the local singular series, determined by a polynomial map $\mathbf{P} = (P_1, \ldots, P_k)$, $k \geqslant 2$, along the critical values of $\mathbf{P}$. As in the one variable case, when each P_i is defined over $\mathbb{Q}_p$, this is the function

$$\begin{aligned} \mathbf{t} = (t_1, \ldots, t_k) &\to F(\mathbf{t}) \\ &=_{\text{def}} \lim_{e \to \infty} \frac{\#\{\xi \in (\mathbb{Z}_p/p^e)^n : P_i(\xi) \equiv t_i \ (p^e) i = 1, \ldots, k\}}{p^{e(n-k)}}, \end{aligned}$$

where $\mathbf{t}$ is a regular value of $\mathbf{P}$. This function is also a fiber integral [Y]:

$$\mathbf{F}(\mathbf{t}) = \int_{\{\mathbf{P}=\mathbf{t}\} \cap \mathbb{Z}_p^n} |\mathrm{d}x_1 \cdots \mathrm{d}x_n / \mathrm{d}P_1 \wedge \cdots \wedge \mathrm{d}P_k|,$$

where the measure is, again, induced by the Leray residue form on each smooth fiber of dimension $n - k$.

Typically, though not necessarily always, an effective estimate for the Fourier transform of $F(\mathbf{t})$ is of interest. A standard argument [I, pg. 83] shows that the transform determines a 'generalized Gaussian sum',

$$G(a_1/p^r, \ldots, a_k/p^r) = p^{-rn} \sum_{x \in (\mathbb{Z}_p/p^r)^n} \mathrm{e}^{\frac{2\pi i}{p^r} \sum_i a_i P_i(x)},$$

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when evaluated at a vector $(a_1/p^r, \dots, a_k/p^r)$ such that $\gcd(p, a_1, \dots, a_k) = 1$. Estimating the sum for large r is a basic problem in the circle method.

Unfortunately, a multivariable extension of Igusa's geometric method is not at all immediate. Indeed, the proof of his finiteness theorem does not extend to treat k-tuples of polynomials except in a very limited way. In attempting to discover such a generalization, the most important problem seems to occur for pairs of polynomials since a solution for pairs should indicate what one needs to do for $k \geqslant 3$ vectors.

The main result of this paper extends Igusa's method to two classes of pairs of polynomials, defined over $\mathbb{Q}_p$. These are as follows

$\mathcal{C}\ell_{\mathrm{I}} =_{\mathrm{def}}$ all $\mathbf{P}: \mathbb{Q}_p^2 \to \mathbb{Q}_p^2$ satisfying the very mild hypothesis (1.5).

$\mathcal{C}\ell_{\mathrm{II}} =_{\mathrm{def}}$ all $\mathbf{P}: \mathbb{Q}_p^n \to \mathbb{Q}_p^2$, $n \geqslant 2$, such that each P_i is homogeneous with at most one singular point, and so that the singular locus $\mathrm{Sing}_{\mathbf{P}}$ of $\mathbf{P}$ (as a mapping) is a curve with at most one singular point (see Sections 2,3).

For each pair, a finite amount of geometric data is shown to determine the precise singular behavior of the corresponding local singular series. Further, this data is created by a finite number of blowing up morphisms. Each morphism determines an embedded resolution of singularities. Thus, each is a finite composition of monoidal transformations with center a nonsingular variety. The same property also holds if each polynomial is defined over any finite extension of $\mathbb{Q}_p$.

The first three sections of this paper are geometric in nature. Here one derives all the needed ingredients for the local analysis of the singular series in Sections 5–6. Section 4 contains the statement and proof of the main result of the paper, Theorem 4.3. The essential new idea required to establish this is a type of local uniformisation for a map $\mathbf{P}$ by 'good $\mathbf{P}$ wedges' (see (4.2)). That is, one shows that a compact neighborhood of $\mathrm{Sing}_{\mathbf{P}}$ can be decomposed into the union of finitely many good $\mathbf{P}$ wedges. This is not a difficult result for elements of $\mathcal{C}\ell_{\mathrm{I}}$. However, for elements of $\mathcal{C}\ell_{\mathrm{II}}$, the underlying geometric argument is a bit intricate (see sections 2,3). Work in progress intends to show that Theorem 4.3 is true in far greater generality.

Igusa's finiteness theorem extends easily within any good $\mathbf{P}$ wedge. This is shown in Proposition 5.3. The combination of (4.3) and (5.3) proves that the singularities of the local singular series are determined by a finite amount of effectively determined geometric data. This is shown in (5.7), (5.11), and (6.11). Section 6, part ii also works out a complete description of the singular behavior from within the image (by $\mathbf{P}$) of any good $\mathbf{P}$ wedge when P_1 is a linear form.

In the case of one polynomial, this geometric data takes the form of a finite set of $n + 1$ tuples of nonnegative integers, each determined by that polynomial. In the case of pairs $\mathbf{P}$, the data consists of a finite number of explicitly computed $3 \times n$ matrices with nonnegative integer entries. However, unlike the one variable case, a finite number of additional pairs of maps is needed to generate all the matrices. A pair $\mathbf{P}$, by itself, is incapable of generating all the data needed to describe the singularities of the singular series of $\mathbf{P}$. It should also be noted that a convenient geometric encoding of this data produces a finite set of polygons (see (6.12)), in terms of which one can describe certain essential features of the singular behavior of the singular series of **P**. This property is a simple consequence of Theorem 4.3.

Evidently, there needs to be a connection between each ancillary pair (F, G) and **P** itself. The relation between the two pairs is described by a transformation $(F, G) \to \mathbf{P}$ that is called an 'amelioration' (or 'improvement') (see (1.2)). This is a local notion that is defined in the image of the maps. So, it is a type of 'base change' morphism.

The main observation of this paper is that local resolution data (in the domain) can be used to construct all necessary ancillary pairs (F, G), as well as the maps $(F, G) \to \mathbf{P}$ (in the range). Establishing this principle is the key to extending Theorem 4.3 to other classes of polynomial maps. The reader is encouraged to read the introductory remarks of Section 3 where this is discussed more completely.

Theorem 4.3 applied to $\mathbf{P} \in \mathcal{C}\ell_{\mathrm{I}}$ extends a result of Loeser [Lo, Thm. 1.4.3] to a large class of nonfinite maps on $\mathbb{Q}_p^2$. However, since Loeser's method does not extend to treat pairs of polynomials in more than two variables, (4.3), applied to $\mathcal{C}\ell_{\mathrm{II}}$, is the first case known to the author in which an effective (and explicit) description has been given of the singularities of the local singular series of a mapping in which the number of variables exceeds the number of functions.

Since the first version of this paper was completed and circulated in the spring of 1997, generalizations of the principal result have been found by Denef [De] in the p-adic field case, and Lion–Rolin [L–R] in the real case. Abramovich and Karu [Ab–K] also have found a generalization that applies (in the algebraic case) over algebraically closed fields of characteristic zero. These papers imply very general results on the singular behavior of a fiber integral, determined by integrating a compactly supported differential form over the nonsingular fibers of a polynomial (or analytic in [L–R]) map, as a critical value is approached.

What however is, so far, lacking with all three of these papers is the effectiveness of the procedure by means of which this behavior can be described. Moreover, one is limited to fairly general assertions about the form of the singularities, and cannot, as yet, say anything very precise. As a result, it does not yet seem possible to apply these results to estimate, in any useful sense, the Fourier transform of the fiber integral. Since the ability to do just that is one of the principal motivations for studying the singularities of a fiber integral in the first place, it would appear to be very interesting to extend the more constructive methods of this paper. This should make it possible to determine nontrivial estimates for the Fourier transform of many classes of mappings.

Indeed, the article [Li-1] uses the purely geometric results of this paper to derive a nontrivial decay estimate for $|G(a_1/p^r, a_2/p^r)|$ (as $r \to \infty$) for $\mathbf{P} \in \mathcal{C}\ell_{\mathrm{I}} \cup \mathcal{C}\ell_{\mathrm{II}}$, that is uniform in (a_1, a_2). The estimate is most conveniently described in terms of the polygons, defined in (6.12). In particular, when $P_1(x) = \beta \cdot x$, $\beta \in \mathbb{Z}_p^n$, and P_2 has degree $d_2 \geqslant 2$, a simple estimate for the rate of decay can explicitly be given

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and seen to be independent of β. This requires the data obtained in Section 6 part ii.

In this way, for fixed β, a nontrivial decay estimate is found for generalized Kloosterman sums as follows

$$S_u(p^r,\beta) =_{\mathrm{def}} p^{rn} \sum_{\substack{a_1,a_2\in[1,p^r)\\ a_1a_2\equiv 1\,\mathrm{mod}(p^r)}} \mathrm{e}^{\frac{2\pi i u a_1}{p^r}} G(-1/p^r, a_2/p^r)$$
$$= O_\varepsilon(p^{r(n-\frac{n}{d_2}+1+\varepsilon)}), \tag{0.1}$$

for any (sufficiently small) $\varepsilon > 0$. These sums have been intensively studied in the cubic case in [H-B-1]. It has also been suggested in [H-B-2, pg. 152] that an estimate for the averages $\sum_{\{\beta\in\mathbb{Z}^n:|\beta|\leqslant B\}} |S_u(p^r,\beta)|$ is a needed ingredient to detect rational points on $\{P_2 = 0\}$ when P_2 is of higher degree, in particular, of degree 4. Using very different methods than those developed in [Li-1], the article [Li-5] was able to establish estimates on average that are comparable to (0.1). It would be interesting to know if similar estimates could be found that use only the methods from [*op cit*].

The methods of this paper, because they are purely geometric, also apply over $\mathbb{R}$ and $\mathbb{C}$ (see Remark 6.13). Using them, problems in classical analysis can be addressed. Over $\mathbb{R}$, the pairs $\mathcal{C}\ell_{\mathrm{I}}$, $\mathcal{C}\ell_{\mathrm{II}}$ can be defined, and the asymptotic behavior of the oscillatory integral with phase $\lambda_1 P_1 + \lambda_2 P_2$ can be analyzed for large $|\lambda_1|$, $|\lambda_2|$. Although the techniques of integration are completely different than those of [Li-1], the estimates obtained are entirely similar. The results are given in [Li-3]. These too should be true in much greater generality.

In contrast to estimating oscillatory integrals, the works of Phong–Stein (e.g. [P–S-1, 2, 3]) have emphasized the estimate of oscillatory integral *operators* on various L^p spaces. A related problem is the 'stability' of oscillatory integral operators, or of integrals of the type $\int_B |f|^{-s}\,\mathrm{d}x$, where $B \subset \mathbb{R}^n$ is a small box containing a singularity of the polynomial/analytic function f (see [P–S–St]). The proximity to the classical subject of oscillatory integrals suggests that algebro-geometric methods based upon Singularity theory should exist to deal with such problems. So far, this has proved elusive. It seems that one can attack such problems using the existence of a several variable asymptotic expansion for appropriately defined fiber integrals. The article [Li-6] shows that this is possible when the phase function is homogeneous in two variables. To do this, the singular behavior of a three variable fiber integral needs to be understood very precisely.

Over $\mathbb{C}$, there is the problem of obtaining a good definition of residual current (see [P]), using families of paths other than 'admissible' paths. This problem was posed in [Be-Y, Thm. 5.23ff], and solved in the case of complete intersections in [P–T-1,2]. It is not difficult to show, see [Li-4], that this question can also be understood in terms of the singularities of a fiber integral. It follows that the methods of this paper also apply to this problem. As a result, for $(P_1, P_2) \in \mathcal{C}\ell_{\mathrm{I}} \cup \mathcal{C}\ell_{\mathrm{II}}$, one can analyze the limiting behavior of the integrals (where φ denotes a compactly supported $(n, n-2)$ form)

$$\int_{\{|P_1|=t_1\}\cap\{|P_2|=t_2\}} \frac{\varphi}{P_1 P_2} \quad \text{as} \quad t_1, t_2 \to 0.$$

This is particularly of interest for $\mathcal{C}\ell_{\mathrm{I}}$ since such pairs need not define a complete intersection.

To simplify the discussion, this paper has only treated the case of $\mathbb{Q}_p$. The reader, who is assumed to be familiar with Igusa's theory in 1 variable, for which [I] is an excellent reference, will easily be able to make the needed changes for any finite extension K of $\mathbb{Q}_p$. For the reader who is not, it suffices to replace $\mathbb{Z}_p$ resp. the ideal $(p)\mathbb{Z}_p$ by $\mathcal{O}_K$, the ring of elements of norm at most 1 resp. ideal $(\mu)\mathcal{O}_K$, where μ is a generator for the elements of norm strictly smaller than 1. Then, in the expressions below, one replaces the number p by $p^f = \#\mathcal{O}/(\mu)$.

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Section 1

Part i gives the two basic definitions that are used in the rest of the article. Part ii treats the case of $\mathcal{C}\ell_{\mathrm{I}}$.

Part i. *Good-Bad points*

Let $(f, g){:}U(\mathbf{p}) \subset \mathbb{Q}_p^n \to \mathbb{Q}_p^2$ denote any pair of nonzero p-adic analyticfunctions, defined on a compact open neighborhood of a point $\mathbf{p}$, and such that $f(\mathbf{p}) = g(\mathbf{p}) = 0$.

DEFINITION 1.1. $\mathbf{p}$ is a good point of (f, g) if there exist analytic coordinates $x = (x_1, \ldots, x_n)$, defined on $U(\mathbf{p})$ and centered at $\mathbf{p}$, such that

$$f(x) = \prod_{i=1}^{n} x_i^{N_i} \cdot u_1, \qquad g(x) = \prod_{i=1}^{n} x_i^{M_i} \cdot u_2, \qquad u_1, u_2 \text{ analytic units on } U(\mathbf{p}),$$

$$\operatorname{rank} A(\mathbf{p}) =_{\text{def}} \begin{pmatrix} N_1 & \cdots & N_n \\ M_1 & \cdots & M_n \end{pmatrix} = 2.$$

$\mathbf{p}$ is a bad point if it is not good. □

So, if each function has the form of monomial times a unit (in the same coordinate system) but rank $A(\mathbf{p}) = 1$, then $\mathbf{p}$ is bad. As a second example, if at least one function is singular at $\mathbf{p}$ but cannot be expressed as the product of a monomial and unit, then $\mathbf{p}$ is bad.

Note. In this article, when a point and compact open neighborhood of the point are given, one writes 'coordinates on the neighborhood' to mean a system of p-adic analytic coordinates defined on the neighborhood and centered at the point. If no neighborhood is indicated, 'coordinates at the point' means a system of coordinates defined on some unspecified compact open neighborhood of the point. Further, units (on some neighborhood) are always p-adic analytic functions that do not vanish on the neighborhood. Given a point $\mathbf{y}$, the notation $U(\mathbf{y})$ denotes a compact open neighborhood of $\mathbf{y}$. □

DEFINITION 1.2. Let $\mathbf{p} \in \mathbb{Q}_p^n$ Let (f, g), (F, G) be two pairs of analytic functions defined on some $U(\mathbf{p})$. A *permissible modification* is a map $(F, G) \to (f, g)$ such that (up to a permutation of f, g)

$$f = cF^\delta \quad \text{and} \quad g = \psi(F) + G \text{ on } U(\mathbf{p}), \tag{1.2.1}$$

and where $c \neq 0$, $\delta \in \mathbb{N}$, and $\psi(\tau)$ is a p-adic analytic function.

A permissible modification $(F, G) \to (f, g)$ is an *amelioration* at $\mathbf{p}$ if $\mathbf{p}$ is bad for (f, g) but good for (F, G). □

Note. Given some neighborhood U, the expression $(F, G) \to (f, g)$ on U will mean that equations of the form (1.2.1) hold between the two pairs of functions on U. Unless greater precision is required, the neighborhood U will not be written.

Part ii. *The case* $n = 2$

In this part, $\mathbf{p}$ always denotes a bad point of a map $(f, g){:}U(\mathbf{p}) \subset \mathbb{Q}_p^2 \to \mathbb{Q}_p^2$. At first, a simple criterion is given that 'improves' a map at $\mathbf{p}$. Then, the definition of $\mathcal{C}\ell_{\mathrm{I}}$ is presented, and one shows how to apply the criterion to improve any element of $\mathcal{C}\ell_{\mathrm{I}}$ in a neighborhood of a bad point.

Suppose the following two properties hold in a neighborhood $W(\mathbf{p})$ (always up to a permutation of f, g, if needed)

(1.3a) There exist coordinates $z = (z_1, z_2)$ on $W(\mathbf{p})$, nonnegative integers, N_j, $j = 1, 2$, at least one of which is positive, and a unit u_1 on $W(\mathbf{p})$ such that $f(z) = z_1^{N_1} z_2^{N_2} u_1(z)$.

(1.3b) Let $\mathcal{J}_{f,g}$ denote the Jacobian of f, g in the z coordinates. There exist nonnegative integers, A_j, $j = 1, 2$, at least one of which is positive, and a unit u on $W(\mathbf{p})$ such that $\mathcal{J}_{f,g}(z) = z_1^{A_1} z_2^{A_2} u(z)$.

PROPOSITION 1.4. *If* (1.3a), (1.3b) *are satisfied, then there exists an amelioration* $(F, G) \to (f, g)$ *on some (possibly smaller) subneighborhood* $U(\mathbf{p})$ *of* $W(\mathbf{p})$.

Proof. By permuting the coordinates, one may assume that $N_1 > 0$. By factoring out from u_1 the constant term $u_1(\mathbf{p})$, and shrinking $W(\mathbf{p})$ to a subneighborhood

$U(\mathbf{p})$, if needed, one may assume that the binomial series for $u_1(z)^{1/N_1}$ converges for all $z \in U(\mathbf{p})$ and that

$$w_1 = z_1 \cdot (u_1)^{1/N_1}, \quad w_2 = z_2$$

determine coordinates on $U(\mathbf{p})$. One can then write

$$f(w) = u_1(\mathbf{p}) w_1^{N_1} w_2^{N_2}, \qquad \mathcal{J}_{f,g}(w) = w_1^{A_1} w_2^{A_2} u(w), \tag{1.4.1}$$

where $u(w)$ is a unit on $U(\mathbf{p})$.

Let the power series expansion in $U(\mathbf{p})$ for $g(w)$ be given by

$$g(w) = \sum_{J \neq (0,0)} g_J w^J, \quad \text{where} \quad w^J = w_1^{j_1} w_2^{j_2} \text{ if } J = (j_1, j_2).$$

For simplicity in the following, set $|N; J| = \begin{vmatrix} N_1 & j_1 \\ N_2 & j_2 \end{vmatrix}$.

A simple calculation then indicates that

$$\begin{aligned} \mathcal{J}_{f,g}(w) &= u_1(\mathbf{p}) w_1^{N_1-1} w_2^{N_2-1} \sum_{J \neq (0,0)} g_J |N; J| w^J \quad \text{if } N_2 > 0, \\ w_2 \mathcal{J}_{f,g}(w) &= u_1(\mathbf{p}) w_1^{N_1-1} \sum_{J \neq (0,0)} g_J |N; J| w^J \qquad \text{if } N_2 = 0. \end{aligned} \tag{1.4.2}$$

Since $\mathcal{J}_{f,g}(w)$ is given by (1.4.1), it follows that $N_j - 1 \leqslant A_j$, $j = 1, 2$, if each $N_j > 0$, and $N_1 - 1 \leqslant A_1$ if $N_2 = 0$. Set

$$(\varepsilon_1, \varepsilon_2) = (A_1 - N_1 + 1, A_2 - N_2 + 1). \tag{1.4.3}$$

Dividing both sides of (1.4.2) by the monomial factor written on the right side of (1.4.2), one concludes

$$\sum_{J \neq (0,0)} g_J |N; J| w^J = w_1^{\varepsilon_1} w_2^{\varepsilon_2} \cdot u(w) \quad \text{where } u \text{ is a unit on } U(\mathbf{p}). \tag{1.4.4}$$

Next, one writes $g(w) = g_1(w) + g_2(w)$, where

$$g_1(w) = \sum_{|N;J|=0} g_J w^J \quad \text{and} \quad g_2(w) = \sum_{|N;J| \neq 0} g_J w^J.$$

Now set

$$\delta = \gcd(N_1, N_2), \qquad \mathbf{n} =_{\text{def}} (n_1, n_2) = (N_1/\delta, N_2/\delta),$$
$$F = w^{\mathbf{n}}, \qquad G = g_2(w).$$

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Since each J in the index set for g_1 is an integral multiple of $\mathbf{n}$, set $\mathcal{J} = \{j : j\,\mathbf{n} \in \operatorname{supp} g_1\}$. It follows that $g_1 = \psi(F)$ where $\psi(\tau) = \sum_{j \in \mathcal{J}} g_{j\mathbf{n}} \tau^j$.

It is then clear that $(F, G) \to (f, g)$ determines a permissible modification on $U(\mathbf{p})$. To show that it is an amelioration at $\mathbf{p}$, one first notes that $G \neq 0$, since $g = \psi(F)$ implies that $\mathcal{J}_{f,g}$ would be identically zero in $U(\mathbf{p})$. In addition, it is clear that

$$\begin{aligned}\operatorname{supp} g_2 &=_{\text{def}} \{J : |N; J| \neq 0 \quad \text{and} \quad g_J \neq 0\} \\ &= \operatorname{supp}\,(\text{the function defined on the left side of (1.4.4)}).\end{aligned}$$

Thus, (1.4.4) implies $g_2(w) = w_1^{\varepsilon_1} w_2^{\varepsilon_2} \cdot$ (an analytic unit in $U(\mathbf{p})$). Since $(\varepsilon_1, \varepsilon_2)$ cannot be linearly dependent with (N_1, N_2) it follows that $\mathbf{p}$ is a good point for (F, G). $\square$

Now let $\mathbf{P} = (P_1, P_2)$, where P_1, P_2 are polynomial functions on $\mathbb{Q}_p^2$. Recall that $\mathrm{Sing}_{\mathbf{P}}$ denote the singular set of the mapping $\mathbf{P}$. Set $\mathcal{J}_{\mathbf{P}}$ to equal the jacobian of $\mathbf{P}$ with respect to a fixed set of affine coordinates (x_1, x_2) on $\mathbb{Q}_p^2$.

DEFINITION 1.5. $\mathbf{P} \in \mathcal{C}\ell_{\mathrm{I}}$ iff for any compact open neighborhood U of a point in $\mathrm{Sing}_{\mathbf{P}}$, $\mathbf{P}(U)$ has positive measure in $\mathbb{Q}_p^2$, and $\mathbf{P}(U - \mathcal{C}\ell_{\mathbf{P}})$ is dense in $\mathbf{P}(U)$. $\square$

Remark. This property avoids a certain degeneracy in the image of $\mathbf{P}$ that can arise when one works with a pair of polynomials. One wants the image $\mathbf{P}(U)$ to have positive Haar measure, so that, in particular, it does not lie on an analytic curve, as would be the case if P_1, P_2 satisfied an analytic relation on U. One also wants to insure that any singular fiber of $\mathbf{P}|_U$ can be approached along nonsingular fibers. $\square$

The main observation is the following.

THEOREM 1.6. *Let $\mathbf{P} \in \mathcal{C}\ell_{\mathrm{I}}$ and $q \in \mathrm{Sing}_{\mathbf{P}}$ be a bad point for $\mathbf{P} - \mathbf{P}(q)$.*

(a) *If at least one P_i is nonsingular at q, and $\mathrm{Sing}_{\mathbf{P}}$ is a normal crossing divisor in a compact open neighborhood $U(q)$ of q, then there exists an amelioration $(F, G) \to \mathbf{P} - \mathbf{P}(q)$ at q.*

(b) *If (a) fails to hold at q, then there exist a compact open neighborhood $U(q)$ of q, a nonsingular p-adic manifold X, and a morphism $\eta : X \to U(q)$ such that:*

(i) *η is the composition of finitely many point blowing ups,*

(ii) *the exceptional divisor $\mathcal{E}$ of η blows down to q,*

(iii) *each point $\mathbf{p}$ on $\mathcal{E}$ is either good for $\mathbf{P} \circ \eta - \mathbf{P}(q)$, or there exists an amelioration $(F, G) \to \mathbf{P} \circ \eta - \mathbf{P}(q)$ at $\mathbf{p}$.*

Proof of (a). By a permutation if necessary, one may assume there exist coordinates $y = (y_1, y_2)$ such that $P_1 - P_1(q) = y_1$ and $\mathcal{J}_{\mathbf{P}} = \partial P_2/\partial y_2 = y_1^{A_1} y_2^{A_2} u$, where u is a unit at q. Setting $(f, g) = \mathbf{P} - \mathbf{P}(q)$, it follows that Proposition 1.4 applies immediately to finish the proof.

To prove (b), note first that in the ring of analytic germs at q, there exist distinct, irreducibles $h_1, \ldots, h_r \neq 0$, each vanishing at q, and units u_1, u_2, u at q such that for suitable nonnegative integers a_i, b_i, c_i, one has

$$P_1 - P_1(q) = h_1^{a_1} \ldots h_q^{a_q} \cdot u_1, \qquad P_2 - P_2(q) = h_1^{b_1} \ldots h_q^{b_q} \cdot u_2,$$
$$\mathcal{J}_{\mathbf{P}} = h_1^{c_1} \ldots h_q^{c_q} \cdot u.$$

Notation. For a function germ h at a point q, one writes here and in the rest of the article h_{rd} to denote the 'reduced' function germ, obtained by setting any positive exponent on the right side to 1. A similar notation is used for a representative function defined in a neighborhood of q. □

Let $U(q)$ denote a compact open neighborhood on which all these functions are defined, and so that the above equations hold at each point.

Now observe that the hypothesis in (b) implies either (i) q is a singular point of $(P_1 - P_1(q))_{rd}$ and $(P_2 - P_2(q))_{rd}$, or (ii) both $(P_1 - P_1(q))_{rd}$, $(P_2 - P_2(q))_{rd}$ are nonsingular at q, and $(\mathcal{J}_{\mathbf{P}})_{rd}$ is singular at q, but does not define a normal crossing divisor at q. It follows that the set of points q, satisfying the hypothesis in (b), is a discrete subset since it is defined purely in terms of the underlying reduced functions. Thus, by shrinking $U(q)$, if needed, one may assume that the only singular point in $U(q)$ *not* satisfying (a) is the point q.

In either case, there exist finitely many point blow ups η_i, $i = 1, \ldots, R$, so that the following hold

(1) the first blowing up, η_1, blows up q;
(2) for each $i \geqslant 2$, η_i blows up exactly one point in the exceptional divisor of η_{i-1};
(3) defining $\eta = \eta_1 \circ \cdots \circ \eta_R$, and $X = \eta^{-1} U(q)$, the divisor
$$\mathcal{D} =_{\text{def}} \{(P_1 - P_1(q)) \cdot (P_2 - P_2(q)) \cdot \mathcal{J}_{\mathbf{P}} \circ \eta = 0\} \cap X$$
is a locally normal crossing divisor.

Now set $\mathcal{J}_{\mathbf{P}\circ\eta}$ to denote the jacobian of $\mathbf{P} \circ \eta$. Then, because $n = 2$, one concludes:

$$\mathcal{J}_{\mathbf{P}\circ\eta} = (\mathcal{J}_{\mathbf{P}} \circ \eta) \cdot (\det \mathrm{d}\, \eta). \tag{1.6.1}$$

Since $\mathcal{J}_{\mathbf{P}} \circ \eta$ and $\det \mathrm{d}\eta$ are both locally the product of a monomial and unit at each point of $\mathcal{D}$, it follows from (1.6.1) that the same holds for $\mathcal{J}_{\mathbf{P}\circ\eta}$. Hence, one can now apply Proposition 1.4 to the pair $(f, g) = \mathbf{P} \circ \eta - \mathbf{P}(q)$ at any bad point $\mathbf{p}$ on the exceptional divisor of η.

To conclude the proof of (b), it suffices to observe that if $\mathbf{y} \in X \cap \{\mathcal{J}_{\mathbf{P}} \circ \eta = 0\}$ is a bad point for $\mathbf{P} \circ \eta - \mathbf{P} \circ \eta(\mathbf{y})$, *and* $\eta(\mathbf{y}) \neq q$, then (a) implies the existence of

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an amelioration at $\mathbf{y}$. That is, no additional blowing up of $\mathbf{y}$ is required. $\square$

Remark. The elementary argument that proves (1.6), it turns out, was first observed in the work of Akbulut and King [A–K] on an entirely different problem in topology. The author rediscovered the same idea in the course of work that led to this paper. This should be another indication that a general method of 'improving' the singularities of mappings would find important applications in subjects far removed from Singularity theory.

2. Local Normal Forms at Singular points for $\mathbf{P} \in \mathcal{C}\ell_{\mathrm{II}}$

Convenient local forms are given in a neighborhood of a point $q \in \mathrm{Sing}_{\mathbf{P}}$. There are two cases. The first assumes q is not the unique singular point of each P_i, denoted subsequently as $\bar{\mathbf{0}}$. (2.2) gives a local form that is sufficient for this paper. Here the neighborhood $U(q)$ will lie in the original affine n space on which $\mathbf{P}$ is defined. The rest of the section treats the case of a point p on the exceptional divisor of the blowing up of $\bar{\mathbf{0}}$ in $\mathbb{Q}_p^n$. A system of affine coordinates $x = (x_1, \dots, x_n)$ for $\mathbb{Q}_p^n$ is chosen and fixed throughout the discussion.

The assumption $\mathbf{P} \in \mathcal{C}\ell_{\mathrm{II}}$ easily is seen to imply the following.

LEMMA 2.1. *There exist local coordinates $(y_1, \dots, y_n)$ at $q \neq \bar{\mathbf{0}} \in \mathrm{Sing}_{\mathbf{P}}$, such that*

$$P_1(y) - P_1(q) = y_1, \qquad P_2(y) - P_2(q) = \psi_q(y_1) + p_2(y),$$

where ψ_q is a convergent power series with $\mathrm{ord}_{y_1}\psi_q = 1$, *and so that*

$$\mathrm{d}y_1 \wedge d(\partial p_2/\partial y_2) \wedge \cdots \wedge d(\partial p_2/\partial y_n) \neq 0.$$

The standard proof of Morse's Lemma [C–M, pg. 24] also applies in the ring of germs of p-adic analytic functions at the origin in $\mathbb{Q}_p^n$. Using this and the Preparation Theorem applied to $P_2 - P_2(q)$, more explicit local forms for $P_1 - P_1(q)$, $P_2 - P_2(q)$ can be found.

LEMMA 2.2. *There exist coordinates $(Y_1, \dots, Y_n)$, defined on a compact open neighborhood $U(q)$, such that*

$$P_1(Y) - P_1(q) = Y_1, \tag{2.2.1}$$

$$P_2(Y) - P_2(q) = \psi_q(Y_1) + \sum_{j=2}^{n} c_j Y_j^2 + \sum_{\ell \geqslant 2} Y_1^\ell g_\ell(Y_2, \dots, Y_n),$$

where each $c_j \neq 0$ and $g_\ell(0, \dots, 0) = 0$ for each ℓ.

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Remark. The ability to describe the function $P_2 - P_2(q) - \psi_q(Y_1)$ with such precision is a key ingredient to the arguments of Section 3. □

To analyze what occurs at the origin, one blows it up. The following is used.

Notation. $\pi: X \to \mathbb{Q}_p^n$ denotes the blowing up of $\bar{\mathbf{0}}$, where X is covered by n affine charts X_j, with coordinates $x(j) =_{\text{def}} (x_{1j}, \ldots, x_{nj})$ so that

$$\pi(x(j)) = (x_{1j}x_{jj}, \ldots, x_{jj}, \ldots, x_{nj}x_{jj}) = (x_1, \ldots, x_n).$$

The exceptional divisor $\pi^{-1}(\bar{\mathbf{0}})$ is denoted $\mathcal{D}$ and one sets $\mathcal{D} \cap X_j = \mathcal{D}_j$ for simplicity. For each j and $p = (p_{1j}, \ldots, p_{nj}) \in \mathcal{D}_j$, set $p^{(j)} = (p_{1j}, \ldots, p_{j-1,j}, 1, p_{j+1,j}, \ldots, p_{nj})$, thought of as a point in the x coordinate plane where $\mathbf{P}$ is defined. The strict transform of each P_i in any chart X_j is written $\hat{P}_i$. The context will make clear in which chart the strict transform is being considered. In X_j, it follows that $\hat{P}_i = P_i(x_{1j}, \ldots, x_{j-1,j}, 1, x_{j+1,j}, \ldots, x_{nj})$. □

It is then easy to see that $\mathbf{P} \in \mathcal{C}\ell_{\text{II}}$ implies the following.

LEMMA 2.3.

(i) *For each* i, j, $X_j \cap \{\hat{P}_i = 0\}$ *is a nonsingular hypersurface;*
(ii) *For each* i, j, $X_j \cap \{\hat{P}_i = 0\}$ *is transverse to* $\mathcal{D}_j$.

Set $d_i = \deg P_i$, $i = 1, 2$.

LEMMA 2.4. *Suppose* $p \in \mathcal{D}_j \cap \{\hat{P}_1 = \hat{P}_2 = 0\}$ *is a singular point of the mapping* $\hat{P} =_{\text{def}} (\hat{P}_1, \hat{P}_2): \{x_{jj} = 0\} \subset X_j \to \mathbb{Q}_p^2$. *Then*

(i) $p^{(j)}$ *is a singular point of* $\mathbf{P}$;
(ii) *If* $d_1 = d_2$, *then the singular locus of* $\{\hat{P}_1 = \hat{P}_2 = 0\} \cap \pi^{-1}(\bar{\mathbf{0}})$ *is a smooth curve near* p.

Proof of (i). First, one notes that $P_i(p^{(j)}) = 0$ for each i. Secondly, Euler's identity implies that for each i,

$$\frac{\partial P_i}{\partial x_j}(p^{(j)}) = -\sum_{k \neq j} p_{kj} \frac{\partial P_i}{\partial x_k}(p^{(j)}). \tag{2.4.1}$$

Now, for each $k \neq j$, $\partial P_i / \partial x_k(p^{(j)}) = \partial \hat{P}_i / \partial x_{kj}(p)$. So, if p singular for the map $\hat{P}$, and some $p_{kj} \neq 0$, then (2.3)(i) implies the rank of the Jacobian matrix of this mapping is precisely 1, and the preceding equation implies the same holds for the rank of the Jacobian of $\mathbf{P}$ at $p^{(j)}$. If all $p_{kj} = 0$, then $\partial P_i / \partial x_j(p^{(j)}) = 0$ for each i and (i) is immediate. This shows (i).

To prove (ii), one shows that $n-2$ of the 2×2 minors of the Jacobian matrix of $\hat{P}$ have linearly independent differentials at p. To do so, one first notes that since $\mathbf{p}\in \mathcal{C}\ell_{\mathrm{II}}$, there are $n-1$ of the 2×2 minors of the Jacobian matrix of $\mathbf{P}$ at $p^{(j)}$ whose differentials are linearly independent at $p^{(j)}$. Denote these as $h_e = \Delta_{k_e,\ell_e}(\mathbf{P})$, $e=1,\dots,n-1$, where $k_e<\ell_e$ denote the columns from which the minor is formed. Set d to denote the common degree of each P_i. It follows that each h_e is homogeneous of degree $2(d-1)$, and, by (i) vanishes at $p^{(j)}$. Further, $P_i(p^{(j)})=0$ for each i.

Now, if $j\notin\{k_e,\ell_e\}$, for each e, then it is clear that

$$h_e|_{\{x_j=1\}} = \Delta_{k_e,\ell_e}(\hat{P}) =_{\mathrm{def}} \begin{vmatrix} \partial\hat{P}_1/\partial x_{k_e j} & \partial\hat{P}_1/\partial x_{\ell_e j} \\ \partial\hat{P}_2/\partial x_{k_e j} & \partial\hat{P}_2/\partial x_{\ell_e j} \end{vmatrix}.$$

It then follows that for any $u\neq j$, $\partial h_e/\partial x_u|_{\{x_j=1\}} = \partial\Delta_{k_e,\ell_e}(\hat{P})/\partial x_{uj}$. Forming the matrix $M=\left(\frac{\partial h_e}{\partial x_u}\big|_{\{x_j=1\}}\right)$, with $1\leqslant e\leqslant n-1$, $u\neq j$, it follows that a priori its rank is at least $n-2$, provided one evaluates at a singular point of $\mathbf{P}$. However, by Lemma 2.3, one can always choose local coordinates $(\tilde{x}_2,\dots,\tilde{x}_n)$, centered at p in $\mathcal{D}_j$ such that $\hat{P}=(\tilde{x}_2, c\tilde{x}_2+\tilde{\varphi}(\tilde{x}_2,\dots,\tilde{x}_n))$ for some $c\neq 0$. This implies there exist at most $n-2$ minors of the Jacobian matrix of $\hat{P}$ in these coordinates that can be nonzero. Since the rank of M is independent of local coordinates, its rank can not be more than $n-2$. This shows (ii) in this case.

If, however, $j\in\{k_e,\ell_e\}$ for some e, then one uses the following easily verified formula, obtained by applying Euler's formula. For any ℓ such that $j<\ell$,

$$\Delta_{j,\ell}(\mathbf{P}) = \frac{d}{x_j}\left(P_1\frac{\partial P_2}{\partial x_\ell} - P_2\frac{\partial P_1}{\partial x_\ell}\right) - \sum_{k<\ell}\frac{x_k}{x_j}\Delta_{k,\ell}(\mathbf{P}) + \sum_{k>\ell}\frac{x_k}{x_j}\Delta_{\ell,k}(\mathbf{P}).$$

Since $p^{(j)}$ is a singular point of $\mathbf{P}$ at which each P_i vanishes, it follows that for any $r\neq j$,

$$\begin{aligned}\frac{\partial\Delta_{j,\ell}(\mathbf{P})}{\partial x_r}(p^{(j)}) &= -d\sum_{k\neq j}\varepsilon_k p_{kj}\frac{\partial\Delta_{k,\ell}(\mathbf{P})}{\partial x_r}(p^{(j)}) \\ &= -d\sum_{k\neq j}\varepsilon_k p_{kj}\frac{\partial\Delta_{k,\ell}(\hat{P})}{\partial x_{rj}}(p), \end{aligned} \tag{2.4.2}$$

where $\varepsilon_k=1$ if $k<\ell$ and $\varepsilon_k=-1$ if $k>\ell$.

Assuming $p_{kj}\neq 0$, for some k, (2.4.2) then shows

$$\begin{aligned}\operatorname{rank}\left(\frac{\partial\Delta_{i,\ell}(\mathbf{P})}{\partial x_r}(p^{(j)})\right)_{r\neq j} &= \operatorname{rank}\left(\frac{\partial\Delta_{k,\ell}(\mathbf{P})}{\partial x_r}(p^{(j)})\right)_{\substack{j\notin\{k,\ell\}\\ r\neq j}} \\ &= \operatorname{rank}\left(\frac{\partial\Delta_{k,\ell}(\hat{P})}{\partial x_{rj}}(p)\right)_{\substack{j\notin\{k,\ell\}\\ r\neq j}}.\end{aligned}$$

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The matrix on the left includes all but one column of a matrix of n columns whose rank equals $n-1$, which implies its rank is at least $n-2$. Arguing exactly as above shows that the rank of the matrix on the right is at most $n-2$. This verifies (ii), assuming some $p_{kj} \neq 0$.

On the other hand, if $p_{kj} = 0$ for each $k \neq j$, then homogeneity of each $\Delta_{j,\ell}(\mathbf{P})$, and the fact that each $\Delta_{a,b}(\mathbf{P})(p^{(j)}) = 0$ for all (a, b), implies that $p^{(j)}$ is a singular point of each $\Delta_{j,\ell}(\mathbf{P})$. So, in that case, each row of the matrix on the left in the preceding equation that contains the partials of $\Delta_{j,\ell}(\mathbf{P})$ must be all zero. So then it is clear that the rank of the matrix involving all other minors equals $n-1$, which implies that the rank of the submatrix formed by deleting the jth column must be at least $n-2$. Applying the prior argument then completes the proof of (ii). □

Using the prior results, one then deduces convenient local forms near any point on the exceptional divisor in each chart X_j.

LEMMA 2.5. *Suppose* $p \in \mathcal{D}_j \cap \{\hat{P}_1 = \hat{P}_2 = 0\}$. *Then*

(i) *If p is a nonsingular point of $\hat{P}$, then there exist local coordinates $(y_1, \dots, y_n)$ on some compact open $U(p)$, such that*

$$P_1 \circ \pi = y_1^{d_1} y_2, \qquad P_2 \circ \pi = y_1^{d_2} y_3.$$

(ii) *If $d_1 = d_2 =_{\mathrm{def}} d$, and p is a singular point of $\hat{P}$, then there exist local coordinates $(y_1, \dots, y_n)$ on some compact open $U(p)$ so that*

$$\begin{aligned} P_1 \circ \pi &= y_1^d y_2, \\ P_2 \circ \pi &= y_1^d \left[\psi_p(y_2) + \sum_{i \geqslant 3} c_i y_i^2 + \sum_{\ell \geqslant 2} y_2^\ell g_\ell(y_3, \dots, y_n) \right], \end{aligned}$$

where ψ_p and each g_ℓ is analytic, $\mathrm{ord}_{y_2} \psi_p = 1$, *and each g_ℓ vanishes at* $(y_3, \dots, y_n) = (0, \dots, 0)$.

(iii) *If $d_1 \neq d_2$, and p is a singular point of $\hat{P}$, then there exist local coordinates $(y_1, \dots, y_n)$ on some compact open $U(p)$ so that*

$$P_1 \circ \pi = y_1^{d_1} y_2, \qquad P_2 \circ \pi = y_1^{d_2} [y_2 + \phi_p(y_3, \dots, y_n)][\textit{unit on } U(p)],$$

where the norm of the unit factor is constant and ϕ_p is a p-adic analytic function in $(y_3, \dots, y_n)$ that vanishes at $(0, \dots, 0)$.

Proof. (i) is evident. (ii) follows by applying Lemmas 2.2, 2.4 to $\hat{P}$. (iii) follows from Lemma 2.3 and the Preparation Theorem. □

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One concludes the section by considering points on the exceptional divisor of π that are on fibers of $\hat{P}$ other than $\hat{P}^{-1}(0, 0)$.

LEMMA 2.6. *Suppose* $p \in \pi^{-1}(\bar{\mathbf{0}})$ *is such that* $\hat{P}_1(p) \neq 0$ *and* $\hat{P}_2(p) = 0$. *Then there exist local coordinates* $(y_1, \ldots, y_n)$ *on some compact open* $U(p)$ *such that*

$$P_1 \circ \pi = y_1^{d_1}[\hat{P}_1(p) + p_1(y_2, \ldots, y_n)], \qquad P_2 \circ \pi = y_1^{d_2} y_2,$$

where p_1 *is analytic on* $U(p)$ *and* $|p_1(y)| < |\hat{P}_1(p)|$.

Proof. This is clear from Lemma 2.3. $\square$

LEMMA 2.7. *Suppose* $p \in \pi^{-1}(\bar{\mathbf{0}})$ *is such that* $\hat{P}_1(p) \cdot \hat{P}_2(p) \neq 0$. *Then there exist local coordinates* $(y_1, \ldots, y_n)$ *on some compact open* $U(p)$ *such that*

$$P_1 \circ \pi = \hat{P}_1(p) y_1^{d_1}, \qquad P_2 \circ \pi = y_1^{d_2}[\hat{P}_2(p) + p_2(y_2, \ldots, y_n)],$$

where p_2 *is analytic on* $U(p)$, $p_2(0, \ldots, 0) = 0$, $|p_2(y_2, \ldots, y_n)| < |\hat{P}_2(p)|$, *and* p_2 *is nonsingular outside* $\{p_2 = 0\} \cap U(p) \cap \{y_1 = 0\}$.

Proof. By assumption each P_i has the following form in the chart X_j:

$$P_i \circ \pi = x_{jj}^{d_i} \hat{P}_i, \qquad \hat{P}_i = \hat{P}_i(p) + \hat{p}_i(x_{1j}, \ldots, x_{j-1,j}, x_{j+1,j}, \ldots, x_{nj}),$$

where $\hat{p}_i$ vanishes at p.

One first restricts to a neighborhood $W_1(p)$ on which the binomial series for $(1 + \hat{p}_1/\hat{P}_1(p))^{1/d_1}$ is a p-adic analytic function. Inside $W_1(p)$, there is a possibly smaller open compact $U(p)$ so that $y_1 = x_{jj}(1 + \hat{p}_1/\hat{P}_1(p))^{1/d_1}$ satisfies the condition $|\mathrm{d}y_1/\mathrm{d}x_{jj}| = 1$. Then $(y_1, \ldots, y_n) =_{\mathrm{def}} (y_1, x_{1j}, \ldots, x_{j-1,j}, x_{j+1,j}, \ldots, x_{nj})$ are local coordinates centered at p on $U(p)$ such that

$$\begin{aligned} P_1 \circ \pi &= \hat{P}_1(p) y_1^{d_1}, \\ P_2 \circ \pi &= y_1^{d_2}[\hat{P}_2(p) + \hat{p}_2(y_2, \ldots, y_n)] \left[1 + \frac{\hat{p}_1(y_2, \ldots, y_n)}{\hat{P}_1(p)}\right]^{-d_2/d_1}. \end{aligned}$$

One can now write

$$\left[1 + \frac{\hat{p}_1(y_2, \ldots, y_n)}{\hat{P}_1(p)}\right]^{-d_2/d_1} = 1 + \mu(y_2, \ldots, y_n) \quad \text{with} \quad \mu(0, \ldots, 0) = 0,$$

where $\mu(y_2, \ldots, y_n)$ is p-adic analytic, and set $p_2(y_2, \ldots, y_n) = \hat{P}_2(p)\mu + \hat{p}_2 + \hat{p}_2\mu$. One then achieves the last property in the Lemma's statement by shrinking $U(p) \cap \{y_1 = 0\}$, if needed, so that the only critical value of p_2 can occur at 0. This completes the proof of the Lemma.

3. Finding Good Points for Elements of $\mathcal{C}\ell_{\mathrm{II}}$

By (1.1), it is clear that any point $p \in \pi^{-1}(\bar{\mathbf{0}})$ is a good point if the local form of $\mathbf{P} \circ \pi$ is given by (2.5)(i) or (2.6). Any other point $q \neq \bar{\mathbf{0}} \in \mathrm{Sing}_{\mathbf{P}}$, whose local form is given by (2.2), is bad for the map $\mathbf{P} - \mathbf{P}(q)$. Furthermore, any $p \in \pi^{-1}(\bar{\mathbf{0}})$ with local form given by (2.5)(ii), (iii), or (2.7) is bad for $\mathbf{P} \circ \pi$. Part i of this section treats (2.7), Part ii treats (2.5)(iii), Part iii treats (2.5)(ii), and Part iv treats (2.2). This sequence is determined by increasing difficulty.

The goal in each part is to cover a compact open neighborhood of the point q resp. p with finitely many images of neighborhoods of good points by monoidal transformations (i.e. blowing up morphisms with nonsingular centers). The good points will be good for some pair, obtained from $\mathbf{P} - \mathbf{P}(q)$ resp. $\mathbf{P} \circ \pi$ by combining a permissible modification (see (1.2)) in the range and the composition of finitely many blowing ups in the domain. As a result, it is essential to pay attention to *all* the points on the exceptional divisor of any blowing up.

NOTATION. In the following, morphisms will be noted by η with sub or superscripts appended when needed. A bold faced letter $\mathbf{q}$ denotes a point on the exceptional divisor of some η that is mapped to $q \neq \bar{\mathbf{0}} \in \mathrm{Sing}_{\mathbf{P}}$ at which the local form is (2.2). A bold faced $\mathbf{p}$ denotes a point on the exceptional divisor of some η that maps to a point $p \in \pi^{-1}(\bar{\mathbf{0}})$, at which the local form is given by (2.5) or (2.7). Any η is defined so that it maps onto some neighborhood $U(q)$ or $U(p)$. If the point and its image do not need to be stressed in the discussion, then one uses $\mathbf{x}$ to denote the point on the exceptional divisor of η. $\square$

The main problem that one meets can be illustrated by considering a point $\mathbf{x}$ on the exceptional divisor of π that is bad for $\mathbf{P} \circ \pi$. Let (P_1', P_2') denote the pair obtained by a permissible modification $(P_1', P_2') \to \mathbf{P} \circ \pi$. If $\mathbf{x}$ is good for (P_1', P_2'), then one is finished with the local analysis at $\mathbf{x}$, and proceeds to another point on the exceptional divisor. If, however, $\mathbf{x}$ is bad for (P_1', P_2') as well, then one applies a second blowing up η' of some nonsingular subvariety containing $\mathbf{x}$. One then must analyze the behavior of $(P_1', P_2') \circ \eta'$ at *each* point of the exceptional divisor of η'. Some points will be good, but others might be bad. At each bad point $\mathbf{x}'$, a second permissible transformation would be made $(P_1'', P_2'') \to (P_1', P_2')$ and, if $\mathbf{x}'$ is still bad for (P_1'', P_2''), then another blowing up η'' with smooth center containing $\mathbf{x}'$ would need to be found, leading to the analysis of all the points on the exceptional divisor of η''. Some would be good, others could be bad, and so forth and so on.

In principle, the combination of blowing up and permissible modification could go on indefinitely. A priori, there is no reason that it should ever terminate, that is, *each* point of each exceptional divisor should be good for some pair of functions, obtained by a combination of permissible modification in the range and blowing up (with nonsingular center) in the domain.

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It turns out, however, that the procedure *always* terminates in finitely many steps. In the two simpler cases of parts i, ii, this requires a use of resolution of singularities, applied to the original pair of functions, followed by at most one permissible transformation. It is then easy to see that the latter is an amelioration. Thus, there is no need to understand the singularities of the new pair.

In the more difficult cases studied in parts iii, iv, this is no longer the case. One must keep track of the singularities of a new pair, whenever it is necessary to create one at a bad point. The finiteness of the procedure described above is proved by introducing the indices M_0, M_1, M_2 at each bad point treated in case iii (i.e. with local form (2.5)(ii)). These are defined in (3.5)(4). The analogues for case iv (local form (2.2)) are defined after the statement of (3.10).

These numbers can be thought of as numerical ways of measuring how bad the point is. As indicated above, at each such point, the first step in the procedure defines a blowing up morphism whose center contains the point. The choice of morphism will be (more or less) evident in the discussion. This is primarily due to the simplicity of the local forms derived in Section 2. The assumption that $\mathbf{P} \in \mathcal{C}\ell_{\mathrm{II}}$ is evidently used here.

The main problem is to describe with sufficient precision how the strict transform of the pair behaves under this morphism. The principal difficulty is to do this for the function, denoted by P_2' in (iii), (iv). One expects the generic point of the exceptional divisor of the morphism to be good. For case iii, this is true. For case iv this is true only after an additional permissible transformation, even at the generic point of the exceptional divisor. At any exceptional point that is bad, one then needs to measure how bad it is in terms of the M_i. Thanks to the M_i, it is possible to show that it is not as bad as the original point was for P_2'. This implies that an iterative procedure exists to improve points because a similar analysis can then be carried out at each new bad point. Since there is always some improvement, this implies the termination of the 'infinitesimal improvement' procedure in finitely many steps.

In more general cases, it is tempting to believe that analogues of the M_i should exist at any bad point of a mapping. It also seems quite reasonable to expect that such invariants can be found by extending the methods of Bierstone–Milman [Bi-M] to each point of an exceptional divisor, not just a point on the strict transform of some variety.

Part i. *The local form (2.7)*

LEMMA 3.1. *Assume $p \in \pi^{-1}(\bar{\mathbf{0}})$ is a point at (2.7) occurs. Then there exists a smooth p-adic variety Y and proper birational map $\pi' : Y \to U(p)$ so that the following properties hold*

(i) *π' is an isomorphism outside the singular locus of $\{p_{2,rd} = 0\}$;*

(ii) *If $\pi'(y)$ is not a singular point of $p_2(y_2, \ldots, y_n)$, then local coordinates $v = (v_1, \ldots, v_n)$ exist on a compact open $V(y)$ so that*

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$$P_1 \circ \pi \circ \pi'(v) = \hat{P}_1(p) v_1^{d_1},$$

$$P_2 \circ \pi \circ \pi'(v) = v_1^{d_2}[\hat{P}_2(p) + p_2(\pi'(y)) + v_2];$$

(iii) *If $\pi'(\mathbf{p})$ is a singular point of $\{p_{2,rd} = 0\}$, then local coordinates $v = (v_1, \ldots, v_n)$ exist on a compact open $V(\mathbf{p})$ so that*

$$P_1 \circ \pi \circ \pi'(v) = \hat{P}_1(p) v_1^{d_1},$$

$$P_2 \circ \pi \circ \pi'(v) = v_1^{d_2}[\hat{P}_2(p) + v_2^{m_2} \cdots v_n^{m_n} \cdot u(v_2, \ldots, v_n)],$$

where u is a unit of constant norm on $V(\mathbf{p})$.

Proof. This is just the embedded resolution theorem applied to p_2. □

One concludes the following.

COROLLARY 3.2.

(i) *Define the pair*

$$(F, G) =_{\text{def}} \begin{cases} (v_1, v_1^{d_2} v_2) & \textit{if } (3.1)\ \text{(ii)}\ \textit{holds}, \\ (v_1, v_1^{d_2} v_2^{m_2} \cdots v_n^{m_n} \cdot u_{\mathbf{p}}(v_2, \ldots, v_n)) & \textit{if } (3.1)\ \text{(iii)}\ \textit{holds}. \end{cases}$$

Then the map $(F, G) \to \mathbf{P} \circ \pi \circ \pi'$ is an amelioration at $\mathbf{p}$.

(ii) *There exist finitely many $\mathbf{p}_i$ so that $\cup_i \pi'(V(\mathbf{p}_i)) = U(p)$;*

(iii) *The $V(\mathbf{p}_i)$ can be made pairwise disjoint, and $\pi'(V(\mathbf{p}_i)) \cap \pi'(V(\mathbf{p}_\ell))$ is a subset of the singular locus of $\{p_{2,rd} = 0\}$ inside $\mathcal{D}$ if $\mathbf{p}_i \neq \mathbf{p}_\ell$.*

Proof. (iii) follows from (3.1)(i) and the total disconnectedness of the p-adic topology on Y. (ii) follows from the fact that π' is proper. (1.2) implies (i). □

NOTATION 3.3. For purposes of Section 6 and [Li-1], it is useful to extend the notation introduced in (1.1). Let $\eta: X \to \mathbb{Q}_p^n$ be a morphism obtained by composing finitely many blowing ups along smooth subvarieties. Let $\mathbf{y}$ be a point on the exceptional divisor of η. Let f, g be analytic functions defined in a neighborhood of $\mathbf{y}$ for which there exist local coordinates $w = (w_1, \ldots, w_n)$ such that

$$f = \prod_{i=1}^{n} w_i^{N_i} \cdot u_1, \qquad g = \prod_{i=1}^{n} w_i^{L_i} \cdot u_2, \qquad \det \mathrm{d}\eta = \prod_{i=1}^{n} w_i^{\mu_i - 1} \cdot u,$$

where u_1, u_2, u are units. Then, one defines a $3 \times n$ matrix

$$\mathbf{A}(\mathbf{y}) = \begin{pmatrix} N_1 & N_2 & \cdots & N_n \\ L_1 & L_2 & \cdots & L_n \\ \mu_1 - 1 & \mu_2 - 1 & \cdots & \mu_n - 1 \end{pmatrix}.$$

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If, however, only $R < n$ columns are nonzero, then one chooses the indexing so that the first R columns of $\mathbf{A}(\mathbf{y})$ are the nonzero ones, and for simplicity in presenting the data, the remaining $n - R$ columns are not included as columns of $\mathbf{A}(\mathbf{y})$. This notation will be used when $\mathbf{y} = \mathbf{p}$ or $\mathbf{q}$ (see the above Notation). To coordinate with notation of [Li-1], if $\mathbf{y}$ is bad for $(f, g) = \mathbf{P} \circ \eta - \mathbf{P} \circ \eta(\mathbf{y})$ but an amelioration exists at $\mathbf{y}$, then the matrix is written $\mathbf{A}^{\#}(\mathbf{y})$. *No* '#' is written if $\mathbf{y}$ is good for (f, g).

For example, if (3.1)(ii) resp. (3.1)(iii) holds at $\mathbf{p}$, then, for $\eta = \pi \circ \pi'$, and the pair of maps (F, G), the matrix $\mathbf{A}^{\#}(\mathbf{p})$ is given by

$$\begin{pmatrix} 1 & 0 \\ d_2 & 1 \\ n-1 & 0 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 1 & 0 & \dots & 0 \\ d_2 & m_2 & \dots & m_n \\ n-1 & \mu_2 - 1 & \dots & \mu_n - 1 \end{pmatrix}.$$

Part ii. *The local form (2.5)(iii)*

The difficulty of this case is considerably less because the degrees of P_1, P_2 are not equal. Since the blow ups will be centered inside $y_1 = 0$, the multiplicities along this divisor remain unchanged throughout. This tends to promote the presence of good points after a blowing up of the affine space $y_1 = 0$ along a smooth subvariety.

LEMMA 3.4. *Suppose the local form is* (2.5)(iii), *and* $p \in \mathcal{D}_j \cap \{\hat{P}_1 = \hat{P}_2 = 0\}$ *is a singular point of the mapping* $\hat{P}$. *Then there exist a smooth* p*-adic variety* Z *and proper birational map* $\eta{:}Z \to U(p)$, *obtained by composing finitely many point blow ups, so that each point* $\mathbf{p} \in \eta^{-1}(p)$ *is good for* $\mathbf{P} \circ \pi \circ \eta$.

Proof. One first may assume that $U(p)$ is so small that the only singular value of $\phi_p|_{U(p)\cap\{y_1=y_2=0\}}$ equals 0. Then, let $\eta_0{:}Z_0 \to U(p) \cap \{y_1 = y_2 = 0\}$ be a proper birational map, an isomorphism outside the singular locus of $\{\phi_{p,rd} = 0\} \cap U(p) \cap \{y_1 = y_2 = 0\}$, so that $\phi_p \circ \eta_0$ is locally the product of a monomial and unit in a neighborhood of each point in the preimage of the singular locus by η_0.

Set $\hat{Z}_0 = \{|y_1| \leqslant \varepsilon\} \times \{|y_2| \leqslant \varepsilon\} \times Z_0$ and $\hat{\eta}_0 = \mathrm{id} \times \eta_0{:}\hat{Z}_0 \to U(p)$.

It is clear that if $\mathbf{v}_0 \notin (\phi_p \circ \eta_0)^{-1}(0)$, then $\mathbf{v}_0$ is good for $(P_1 \circ \pi \circ \hat{\eta}_0, P_2 \circ \pi \circ \hat{\eta}_0)$. So, it suffices to assume $\mathbf{v}_0 \in (\phi_p \circ \eta_0)^{-1}(0)$. Thus, one may assume local coordinates $(y_1, y_2, z_3, \dots, z_n)$ are defined on a compact open neighborhood $U(\mathbf{v}_0) \subset \hat{Z}_0$ of $\mathbf{v}_0$ so that for some $k \in [3, n]$ there exist $e_i > 0$, $i = 3, \dots, k$, and a nonzero constant c such that on $U(\mathbf{v}_0)$ one has:

$$P_1 \circ \pi \circ \hat{\eta}_0 = y_1^{d_1} y_2, \qquad P_2 \circ \pi \circ \hat{\eta}_0 = y_1^{d_2}[y_2 + c z_3^{e_3} \cdots z_k^{e_k}] \cdot (\text{unit}).$$

Remark. Since this is obtained by absorbing a local unit into one of the local defining equations for an irreducible component of $\phi_p \circ \eta_0 = 0$ at $\mathbf{v}_0$, it follows, a priori, that one may have to factor out some p-adic number that does not have a root of any order e_i in $\mathbb{Q}_p$. This is the explanation for the factor c. □

One now converts the bracketed expression into the product of a monomial and analytic unit everywhere along $\{y_1 = 0\}$. First blow up along the subvariety $\{y_2 = z_3 = 0\}$ in $U(\mathbf{v}_0)$. Let $\eta_1{:}Z(1) \to U(\mathbf{v}_0)$ denote this morphism, and set $\eta^{(1)} = \pi \circ \hat{\eta}_0 \circ \eta_1$. Then $Z(1) = Z_2(1) \cup Z_3(1)$ such that

$$\begin{aligned} P_1 \circ \eta^{(1)}|_{Z_2(1)} &= y_1^{d_1} z_{22}, \\ P_2 \circ \eta^{(1)}|_{Z_2(1)} &= y_1^{d_2} z_{22}(1 + c z_{22}^{e_3-1} z_{32}^{e_3} \cdots z_k^{e_k}) \cdot (\text{unit}) \end{aligned}$$

$$\begin{aligned} P_1 \circ \eta^{(1)}|_{Z_3(1)} &= y_1^{d_1} z_{23} z_{33}, \\ P_2 \circ \eta^{(1)}|_{Z_3(1)} &= y_1^{d_2} z_{33}(z_{23} + c z_{33}^{e_3-1} z_4^{e_4} \cdots z_k^{e_k}) \cdot (\text{unit}). \end{aligned}$$

Thus, $d_1 \neq d_2$ implies that any point $\mathbf{x}$ on $\{y_1 = z_{22} = 0\}$ is necessarily a good point. Additionally, at $\mathbf{x}$, if $e_3 > 1$, then

$$\mathbf{A}(\mathbf{x}) = \begin{pmatrix} d_1 & 1 & 0\ldots \\ d_2 & 1 & 0\ldots \\ n-1 & \mu_2 - 1 & \mu_3 - 1\ldots \end{pmatrix},$$

where $\mu_2, \mu_3, \ldots$ are certain positive integers with $\mu_2 \geqslant 2$. If $e_3 = 1$, however, then if $\mathbf{x}$ is a zero of $(1 + c z_{32}^{e_3} \cdots z_k^{e_k})$, then by reindexing, if necessary, one replaces the third column by $(0, 1, \mu - 1)$, where $\mu = \mu_\ell$, and ℓ is such that $e_\ell \geqslant 1$. If $\mathbf{x}$ is not a zero, then there is no change in $\mathbf{A}(\mathbf{x})$.

In the other chart it is evident that if $z_{23} \neq 0$, then $(z_{23}, 0)$ is good. Further, if $k = 3$ and $e_3 = 1$, then even $(0, 0)$ is good. So, one is not yet finished only if $k \geqslant 4$ or if $e_3 \geqslant 2$. In either case, one again blows up along the subvariety $\{z_{23} = z_{33} = 0\}$ in a neighborhood of the origin, and repeats the preceding argument $e_3 - 1$ additional times before losing the factor z_{33} in the monomial. Doing so produces the chain of blow ups $Z(i) \to Z(i-1) \to \cdots \to Z(1)$, where each $Z(j) = Z_2(j) \cup Z_3(j)$, and exactly one point, the origin in the chart $Z_3(i)$, $i \leqslant e_3$, is bad. Let $\eta^{(i)}{:}Z(i) \to U(p)$, $i \leqslant e_3$, denote the composition of these blow ups with $\pi \circ \hat{\eta}_0$. Then it is simple to check that

$$\begin{aligned} P_1 \circ \eta^{(e_3)}|_{Z_2(e_3)} &= y_1^{d_1} z_{22}^{e_3} z_{32}^{e_3-1}, \\ P_2 \circ \eta^{(e_3)}|_{Z_2(e_3)} &= y_1^{d_2} z_{22}^{e_3} z_{32}^{e_3-1}(1 + c z_{32} \cdots z_{k2}^{e_k}) \cdot (\text{unit}) \end{aligned}$$

$$\begin{aligned} P_1 \circ \eta^{(e_3)}|_{Z_3(e_3)} &= y_1^{d_1} z_{23} z_{33}^{e_3}, \\ P_2 \circ \eta^{(e_3)}|_{Z_3(e_3)} &= y_1^{d_2} z_{33}^{e_3}(z_{23} + c z_{43}^{e_4} \cdots z_{k3}^{e_k}) \cdot (\text{unit}). \end{aligned}$$

If $k = 3$, then one is finished, since it is clear that each point of the exceptional divisor is a good point for the pair, precisely because $d_1 \neq d_2$. If however $k > 3$,

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then one needs to repeat the above reasoning. But now, the number of distinct factors of the monomial in $z_{43}, \dots, z_{k3}$ has decreased by 1. So, an evident induction on k then completes the proof of the Lemma.

Part iii. *The local form (2.5)(ii)*
Although the local forms (2.2), (2.5)(ii) are similar, the presence of the factor y_1^d in the latter leads to a somewhat different argument. So, these require separate attention.

Using the notation of (2.5)(ii), write $\psi_p(y_2) = \sum_{i \geqslant 1} \gamma_i y_2^i$, and define

$$P_2' = P_2 \circ \pi - \gamma_1 y_1^d y_2 = P_2 \circ \pi - \gamma_1 (P_1 \circ \pi).$$

The map $(P_1 \circ \pi, P_2') \to (P_1 \circ \pi, P_2 \circ \pi)$ is an example of a permissible transformation (see (1.2)) that is not necessarily an amelioration.

The following will be used in addition to (3.3). Throughout, notation from (2.5)(ii) is used.

NOTATION 3.5. (1) One chooses $U(p)$ to be a polycylinder $\{|y_1| \leqslant \varepsilon\} \times \cdots \times \{|y_n| \leqslant \varepsilon\}$, for some $\varepsilon > 0$.

(2) The ith blowing up in the argument is denoted η_i and will always have the form

$$\eta_i = \mathrm{id}_{y_1} \times \eta_i' \colon Z(i) =_{\mathrm{def}} \{|y_1| \leqslant \varepsilon\} \times Z'(i)$$
$$\to Z(i-1) =_{\mathrm{def}} \{|y_1| \leqslant \varepsilon\} \times Z'(i-1),$$

where $Z'(0) = \{|y_2| \leqslant \varepsilon\} \times \cdots \times \{|y_n| \leqslant \varepsilon\}$. Thus, the blowing up will always be done inside a space of dimension $n-1$.

If one blows up a point, then each $Z'(i)$ is the union of $n-1$ open charts written $Z_j'(i)$, $j = 2, \dots, n$ with coordinates $z'(j) = (z_{2j}, \dots, z_{nj})$ so that

$$\eta_i'|_{Z_j'(i)}(z'(j)) = (z_{2j} z_{jj}, \dots, z_{j-1,j} z_{jj}, z_{jj}, z_{j+1,j} z_{jj}, \dots, z_{nj} z_{jj}).$$

One sets $Z_j(i) = \{|y_1| \leqslant \varepsilon\} \times Z_j'(i)$ for each j, i. The precise coordinates in the range of η_i' will be clear from the discussion to follow. Similarly, the index i will not be included in the notation for the components of η_i', since context will make clear which blowing up one is considering. A similar notation is used if one blows up a line.

One writes $\eta^{(i)} = \eta_1 \circ \eta_2 \circ \cdots \circ \eta_i$.

(3) The rightmost column, with entries written as $(0, \varepsilon, 0)$, of any of the matrices $\mathbf{A}_i^{\#}(\mathbf{p})$ (or $\mathbf{A}_i^{\#}(\mathbf{q})$ in part iv), always equals $(0, 1, 0)$ resp. $(0, 0, 0)$ if the point $\mathbf{p}$ belongs resp. does not belong to the strict transform of the hypersurface, defined by the second function in the pair with matrix of multiplicities equal to $\mathbf{A}_i(\mathbf{p})$. The use of # is needed since all points in this part and part iv will be bad for the original pair $\mathbf{P}$.

(4) The form of P_2' is the subtle point and depends upon three integers. First, define

$$M_0 = \begin{cases} \deg_{y_2}(\psi_p - \gamma_1 y_2) & \text{if } \psi_p \neq \gamma_1 y_2 \\ 0 & \text{if } \psi_p = \gamma_1 y_2. \end{cases}$$

Second, write $\sum_{\ell \geqslant 2} y_2^\ell g_\ell = p_1' + p_2'$ where

$$p_1' = \sum_{\ell \geqslant 2} y_2^\ell \sum_{e \geqslant 2} H_{e,\ell}, \qquad p_2' = \sum_{\ell \geqslant 2} y_2^\ell H_{1,\ell},$$

and $H_{e,\ell} = H_{e,\ell}(y_3, \ldots, y_n)$ is homogeneous of degree e for each ℓ. Then set

$$M_1 = \begin{cases} \operatorname{mult}_{(0,\ldots,0)} p_1' & \text{if } p_1' \neq 0 \\ 0 & \text{if } p_1' = 0, \end{cases}$$

$$M_2 = \begin{cases} \operatorname{mult}_{(0,\ldots,0)} p_2' & \text{if } p_2' \neq 0 \\ +\infty & \text{if } p_2' = 0. \end{cases}$$

These numbers are multiplicities *in* $\mathcal{D}_j$. It is also useful to define in the following

$$\mathcal{L} = \{\ell_1 < \ell_2 < \cdots\} = \{\ell \geqslant 2 : H_{1,\ell} \neq 0\}.$$

LEMMA 3.6. *Let p be a point at which* (2.5)(ii) *holds, and let $U(p)$ be the neighborhood of p in the statement of* (2.5)(ii). *Then there exist a smooth p-adic manifold Z, and proper surjective birational map $\eta : Z \to U(p)$, satisfying these properties:*

(i) *η is an isomorphism outside $\eta^{-1}(p)$;*
(ii) *For each point $\mathbf{p} \in \eta^{-1}(p)$ there exist local coordinates $(z_1, \ldots, z_n)$, defined on a compact open $W(\mathbf{p})$, so that*

$$\begin{aligned} P_1 \circ \pi \circ \eta(z) &= z_1^d z_2^{N_2} \cdots z_n^{N_n} \\ P_2' \circ \pi \circ \eta(z) &= z_1^d z_2^{L_2} \cdots z_n^{L_n} \cdot (\text{unit}), \end{aligned}$$

$$\operatorname{rank} \begin{pmatrix} d & N_2 & \cdots & N_n \\ d & L_2 & \cdots & L_n \end{pmatrix} = 2.$$

Proof. The proof depends upon the three integers defined in (3.5.4). There are four cases to consider.

(A) $(M_0, M_2) = (0, +\infty)$.

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Set $\eta_1' : Z'(1) \to U(p) \cap \{y_1 = 0\}$ to be the blowing up of the y_2 axis. That is, one blows up the line $\{y_3 = y_4 = \ldots = y_n = 0\}$ *in* $\mathcal{D}_j$, and restricts to the preimage of $U(p) \cap \{y_1 = 0\}$. Then in each chart $Z_j(1)$, it is clear that

$$P_1 \circ \pi \circ \eta_1 = y_1^d y_2, \qquad P_2' \circ \eta_1 = y_1^d z_{jj}^2 \left[c_j + \sum_{i \neq j} c_i z_{ij}^2 + z_{jj} \cdot (*) \right],$$

where $(*)$ is some analytic function. It follows that at any point $\mathbf{x}$ on $\{y_1 = z_{jj} = 0\}$, the bracketed term is either nonzero or, if it is zero, then it crosses transversally this divisor. Using the notation in (3.3), it is then clear that

$$\mathbf{A}_1^{\#}(\mathbf{x}) = \begin{pmatrix} d & 1 & 0 & 0 \\ d & 0 & 2 & \varepsilon \\ n-1 & 0 & (n-2) & 0 \end{pmatrix}.$$

So, any such $\mathbf{x}$ is good.

(B) $M_0 \geqslant 1$, $M_1 \geqslant 1$, $M_2 = +\infty$.

It follows that $M_0 \geqslant 2$, $M_1 \geqslant 4$, but no term g_ℓ is linear in $y_3, \ldots, y_n$. So, one first blows up the origin in $\mathbb{Q}_p^{n-1}$ rather than the y_2 axis in $\mathbb{Q}_p^{n-1}$. Denote the intersection of the exceptional divisor of η_1 with the chart $Z_j(1)$ by $\mathcal{D}_j$. It follows that in the jth chart one has:

$$\begin{aligned}
&(P_1 \circ \pi \circ \eta_1, P_2' \circ \eta_1) \\
&= \begin{cases} \Bigg(y_1^d z_{22}, y_1^d z_{22}^2 \Bigg[\sum\limits_{k \geqslant M_0} \gamma_k z_{22}^{k-2} \\ \qquad + \sum\limits_{i \geqslant 3} c_i z_{i2}^2 + z_{22}^{M_1 - 2} \cdot (*) \Bigg] \Bigg), & \text{if } j = 2 \\ \Bigg(y_1^d z_{2j} z_{jj}, y_1^d z_{jj}^2 \Bigg[\sum\limits_{k \geqslant M_0} \gamma_k z_{2j}^k z_{jj}^{k-2} + c_j \\ \qquad + \sum\limits_{\substack{i \geqslant 3 \\ i \neq j}} c_i z_{ij}^2 + z_{jj}^{M_1 - 2} \cdot (*) \Bigg] \Bigg), & \text{if } j \neq 2; \end{cases}
\end{aligned} \tag{3.7}$$

One then observes that the factor $(*)$ satisfies the property that the degree of each of its monomials in the variables z_{ij}, $i \neq 2$, is always at least 2 *for each* j.

Assume first that $M_0 = 2$. If $\mathbf{p} \in \mathcal{D}_j$ does not belong to the strict transform of $\{P_2' = 0\}$, defined by the bracketed factor in (3.7), it is clear that $\mathbf{p}$ is good. If

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it does belong to the strict transform, then, for $j = 2$, it belongs to the surface $\{\gamma_2 + \sum_{i\geqslant 3} c_i z_{i2}^2 = 0\}$. So, it is clear that $\mathcal{D}_2$ and the strict transform must cross transversally at $\mathbf{p}$. It is also clear that a similar property holds when $j \neq 2$. Thus, if $\mathbf{p} \in \mathcal{D}_2$ resp. $\mathbf{p} \in \mathcal{D}_j$, $j \geqslant 3$, then

$$\mathbf{A}_1^{\#}(\mathbf{p}) = \begin{pmatrix} d & 1 & 0 \\ d & 2 & \varepsilon \\ n-1 & n-2 & 0 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} d & r & 1 & 0 \\ d & 0 & 2 & \varepsilon \\ n-1 & 0 & n-2 & 0 \end{pmatrix}$$

where $r = 1$ if $z_{2j}(\mathbf{p}) = 0$, $r = 0$ if not. This completes the proof of the Lemma in Case B if $M_0 = 2$.

If $M_0 \geqslant 3$, then the only point at which the strict transform of $\{P_2' = 0\}$ fails to intersect transversally the exceptional divisor is the origin $\zeta'(1) =_{\text{def}} \bar{\mathbf{0}}_{n-1}$ in the chart $Z_2'(1)$ since the geometry is determined by that of the conic $\{\sum_{i\geqslant 3} c_i z_{i2}^2 = 0\}$. This is nonsingular except at $\bar{\mathbf{0}}_{n-1}$. At any other point $\mathbf{p}$ on the exceptional divisor, the matrix $A_1^{\#}(\mathbf{p})$ is the same as in the preceding paragraph, so that $\mathbf{p}$ is a good point for $(P_1 \circ \pi \circ \eta_1, P_2' \circ \eta_1)$.

Thus, one only needs to deal further with $\zeta'(1)$ when $M_0 \geqslant 3$. To do so, one must continue blowing it up. The preceding argument can then be repeated since $M_2 = +\infty$ and the fact the multiplicity at $\zeta'(1)$ of the term denoted by (*) is at least 2. Indeed, writing $M_0 = 2\delta_1 + \delta_2$, $\delta_2 \in \{0, 1\}$, $\delta_1 \geqslant 1$, then one shows that at most $\delta_1 + \delta_2 + 1$ additional blow ups of nonsingular subvarieties in an $n-1$ dimensional affine space are needed to insure that the strict transform of $\{P_2' = 0\}$ intersects the exceptional locus transversally. To complete the proof in case (B), the form of both functions along the exceptional locus needs to be made explicit. The simplest way to do this is by induction on the number of blow ups. The preceding paragraphs have given the first step of the induction.

One assumes that for some $i \in [1, \delta_1 + \delta_2)$, i blow ups

$$Z'(i) \to Z'(i-1) \to \cdots \to Z'(1) \to U(p)$$

have been constructed, so that for each $k \leqslant i-1$, exactly one point $\zeta'(k)$ has been blown up in $Z'(k)$ to give $\eta_{k+1}'{:}Z'(k+1) \to Z'(k)$. This point one assumes to be the origin in the chart $Z_2'(k)$. Further, along the exceptional locus of η_{k+1}', one assumes there is exactly one point of nonnormal crossing with the strict transform of $\{P_2' = 0\}$. This point one assumes to be the origin $\zeta'(k+1) =_{\text{def}} \bar{\mathbf{0}} \in Z_2'(k+1)$. At any point $\mathbf{p}$, *other than* this unique point, the rank of the matrix $A_{k+1}^{\#}(\mathbf{p})$ equals 2. Moreover, if $\mathbf{p} \neq \zeta'(k+1) \in Z_2'(k+1)$ resp. $\mathbf{p} \in Z_j'(k+1)$ then one assumes

$$\mathbf{A}_{k+1}^{\#}(\mathbf{p}) = \begin{pmatrix} d & 1 & 0 \\ d & 2(k+1) & \varepsilon \\ n-1 & (k+1)(n-2) & 0 \end{pmatrix}$$

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$$\text{resp.} \quad \begin{pmatrix} d & r & 1 & 0 \\ d & rk & 2(k+1) & \varepsilon \\ n-1 & k(n-2) & (k+1)(n-2) & 0 \end{pmatrix},$$

where $r = 1$ if $z_{2j}(\mathbf{p}) = 0$, and $r = 0$ if not.

Finally, one assumes that in the coordinates $(y_1, z'(2))$ for $Z_2(i)$:

$$\begin{aligned}(P_1 \circ \pi \circ \eta^{(i)}, P_2' \circ \eta^{(i)}) \\ = \left(y_1^d z_{22}, y_1^d z_{22}^{2i}\left[\sum_{k\geqslant M_0} \gamma_k z_{22}^{k-2i} + \sum_{k\geqslant 3} c_k z_{k2}^2 + z_{22}^{M_1-2i}(*)\right]\right),\end{aligned}$$

where the term $z_{22}^{M_1-2i}(*)$ has multiplicity at least M_1 at $\zeta'(i) = (0, \dots, 0)$.

Assume now that $M_0 = 2\delta_1$ and that $i = \delta_1 - 1$. Then the blowing up of $\zeta'(i)$ suffices to finish the proof of the Lemma in this case. This follows because the strict transform of P_2' in the chart $Z_2(i+1)$ is given by

$$\gamma_{M_0} + \sum_{k>M_0} \gamma_k z_{22}^{k-M_0} + \sum_{k\geqslant 3} c_k z_{k2}^2 + z_{22}^{M_1-2i}(*).$$

So, it is clear that any point of intersection of the strict transform with the exceptional divisor is transverse, and that the strict transform does not vanish at the origin $\zeta'(i+1)$. A simple exercise shows that the same holds in the other charts, and that $\mathbf{A}^{\#}_{\delta_1}(\mathbf{p})$ is obtained by setting $k + 1 = \delta_1$ in the preceding matrix.

If, on the other hand, $M_0 = 2\delta_1 + 1$ and $i = \delta_1$, then the strict transform of P_2' in $Z_2(i)$ is given by

$$\gamma_{M_0} z_{22} + \sum_{k>M_0} \gamma_k z_{22}^{k-2\delta_1} + \sum_{j\geqslant 3} c_j z_{j2}^2 + z_{22}^{M_1-2i}(*).$$

When one now blows up the origin in $Z_2'(i)$, it follows that the strict transform of $\{P_2' = 0\}$ in $Z_2(i+1)$ is disjoint from the component of the exceptional divisor $\{z_{22} = 0\}$ since z_{22} is a factor of each monomial $c_j z_{j2}^2$. So, if $\mathbf{p} \in \{z_{22} = 0\}$, then $\mathbf{p}$ is good and

$$\mathbf{A}^{\#}_{\delta_1+1}(\mathbf{p}) = \begin{pmatrix} d & 1 \\ d & M_0 \\ n-1 & M_0(n-2) \end{pmatrix}.$$

On the other hand, in every other chart $Z_k(\delta_1+1)$, $k \neq 2$, all points in the subvariety $\{z_{2k} = z_{kk} = 0\}$ are not transverse to the strict transform of $\{P_2' = 0\}$ since in this chart

$$P_2' \circ \eta'_{\delta_1+1} = y_1^d z_{2k}^{M_0-1} z_{kk}^{M_0} \Bigg[\gamma_{M_0} z_{2k} + c_k z_{kk} + \sum_{\substack{j \geqslant 3 \\ j \neq k}} c_j z_{kk} z_{jk}^2 + \\ + \sum_{\ell \geqslant M_0+1} \gamma_\ell z_{2k}^{\ell - M_0 + 1} z_{kk}^{\ell - M_0} + z_{kk}^2 (*) \Bigg].$$

Now, blow up the subvariety $\{z_{2k} = z_{kk} = 0\}$ in each $Z_k'(\delta_1 + 1), k \neq 2$, by the map $\eta'_{\delta_1+2}: Z'(\delta_1 + 2) \to Z'(\delta_1 + 1)$. At any point of its exceptional divisor, the total transform of $\{P_2' = 0\}$ is a normally crossing divisor. If $\mathbf{p} \in Z_k(\delta_1 + 2)$, $k \neq 2$, resp. $\mathbf{p} \in Z_2(\delta_1 + 2)$ is any point on the exceptional divisor, then it is easily seen that

$$\mathbf{A}^{\#}_{\delta_1+2}(\mathbf{p}) = \begin{pmatrix} d & 1 & 2 & 0 \\ d & M_0 - 1 & 2M_0 & \varepsilon \\ n-1 & \delta_1(n-2) & M_0(n-2)+1 & 0 \end{pmatrix}$$

$$\text{resp.} \quad \begin{pmatrix} d & 2 & 1 & 0 \\ d & 2M_0 & M_0 & \varepsilon \\ n-1 & M_0(n-2)+1 & (\delta_1+1)(n-2) & 0 \end{pmatrix}.$$

This completes the proof of the Lemma in case B.

(C) $M_0 = 0$, $M_2 < +\infty$.

This case implies $p_2' \neq 0$.

Let $\bar{\mathbf{0}}_{n-1}$ denote the origin in $U(p) \cap \{y_1 = 0\}$. Let η_1' denote the blowing up of $\bar{\mathbf{0}}_{n-1}$ in $U(p) \cap \{y_1 = 0\}$. One first notes that if $p_1' \neq 0$, then $M_1 \geqslant 4$. So, in the chart $Z_j'(1)$ of η_1', the term $z_{jj}^{-2} p_1' \circ \eta_1'$ vanishes everywhere along the exceptional divisor $\mathcal{D}_j'$ to order at least 2. It follows that only the strict transform of p_2' determines whether or not the strict transform of $\{P_2' = 0\}$ is transverse to $\mathcal{D}_j'$.

On the other hand, the multiplicity of the strict transform of p_2' behaves in a more complicated manner. For $j \geqslant 3$, the strict transform of $p_2' \circ \eta_1'$ in $Z_j'(1)$ equals

$$z_{jj}^{-2} \cdot \sum_{i \geqslant 1} z_{2j}^{\ell_i} z_{jj}^{1+\ell_i} H'_{1,\ell_i}(\hat{z}_{jj}),$$

where $\hat{z}_{jj}$ denotes the absence of z_{jj} in the expression. Thus, $\ell_1 \geqslant 2$ implies that the multiplicity of the strict transform of p_1' is at least $2\ell_1 - 1 \geqslant 3$ along the subvariety $\{z_{2j} = z_{jj} = 0\}$ of $\mathcal{D}_j'$.

Now, if $\mathbf{p} \in \{z_{2j} = z_{jj} = 0\}$, either $\mathbf{p}$ belongs to the strict transform of $\{P_2' = 0\}$, or it does not. If it does not belong, then clearly $\mathbf{p}$ is good. If it does belong,

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then necessarily $c_j + \sum_{\substack{i\neq j\\ i\geqslant 3}} c_i z_{ij}(\mathbf{p})^2 = 0$ implies $z_{ij}(\mathbf{p}) \neq 0$ for some $i \neq j$, $i \geqslant 3$. Thus, the strict transform is transverse to the exceptional divisor at $\mathbf{p}$. So, one concludes that $\mathbf{p} \in \{z_{2j} = z_{jj} = 0\}$ is always a good point and

$$\mathbf{A}_1^{\#}(\mathbf{p}) = \begin{pmatrix} d & 1 & 1 & 0 \\ d & 0 & 2 & \varepsilon \\ n-1 & 0 & n-2 & 0 \end{pmatrix}.$$

It therefore suffices to consider the behavior along the open subsets $\{z_{2j} \neq 0\}$ of $\mathcal{D}'_j$, $j \geqslant 3$.

By means of the identifications

$$z_{j2} = 1/z_{2j}, \qquad z_{k2} = z_{kj}/z_{2j},$$

each of these subsets can be viewed as lying in $\mathcal{D}'_2 \subset Z'_2(1)$. So, all the remaining analysis in Case C can be done in $Z'_2(1)$. In this chart, it is clear that z_{22} divides $z_{22}^{-2} p'_2 \circ \eta'_1 = \sum_{i\geqslant 1} z_{22}^{\ell_i - 1} H_{1,\ell_i}(\hat{z}_{22})$ since $\ell_1 \geqslant 2$. One now observes the following.

LEMMA 3.8. *Suppose* $\mathbf{p} \in \{z_{22} = 0\}$, $\mathbf{p} \neq \bar{\mathbf{0}}_{n-1}$. *Then*

$$\mathbf{A}_1^{\#}(\mathbf{p}) = \begin{pmatrix} d & 1 & 0 \\ d & 2 & \varepsilon \\ n-1 & n-2 & 0 \end{pmatrix}.$$

Proof. The quadratic form $\sum_{i\geqslant 3} c_i z_{i2}^2$ has non zero gradient at any such $\mathbf{p}$ since each $c_i \neq 0$. So, the strict transform of $\{P'_2 = 0\}$ must be transverse to $\mathcal{D}'_2$ at any such $\mathbf{p}$. It follows that $A_1^{\#}(\mathbf{p})$ is as claimed. □

Thus, only the point $\mathbf{p} = \bar{\mathbf{0}}_{n-1} \in Z'_2(1)$ remains to be analyzed. At this point, however, $\mathrm{mult}_{\mathbf{p}} z_{22}^{-2} p'_1 \circ \eta'_1 \geqslant M_1$, and this function has an expression

$$\sum_{\ell\geqslant 2} z_{22}^{\ell} G_\ell(z_{32}, \ldots, z_{n2}), \qquad \mathrm{mult}_{(0,\ldots,0)} G_\ell \geqslant 2 \text{ for each } \ell,$$

which is entirely similar to that of p'_1. This implies that if $\ell_1 \geqslant 3$, then one can repeat exactly the above reasoning until the lead exponent of z_{22} in the strict transform of p'_2 equals 1. This requires $\ell_1 - 2$ additional blow ups of the origin in each of the charts $Z'_2(i)$, $i = 1, 2, \ldots, \ell_1 - 2$. Each point $\mathbf{p} \neq \bar{\mathbf{0}}_{n-1} \in Z'_2(i)$ resp. $\mathbf{p} \in \{z_{2j} = z_{jj} = 0\} \subset Z'_j(i)$, $j \geqslant 3$ of the exceptional divisor is a good point and a simple check shows that

$$\mathbf{A}_i^{\#}(\mathbf{p}) = \begin{pmatrix} d & 1 & 0 \\ d & 2i & \varepsilon \\ n-1 & i(n-2) & 0 \end{pmatrix}$$

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$$\text{resp.}\quad \begin{pmatrix} d & 1 & 1 & 0 \\ d & 2i & 2(i+1) & \varepsilon \\ n-1 & (i-1)(n-2) & i(n-2) & 0 \end{pmatrix}.$$

At the point $\mathbf{p} = \bar{\mathbf{0}} \in Z_2'(\ell_1 - 1)$, the local forms of the two functions are

$$P_1 \circ \pi \circ \eta^{(\ell_1-1)} = y_1^d z_{22}$$

$$\begin{aligned} P_2' \circ \eta^{(\ell_1-1)} &= y_1^d z_{22}^{2(\ell_1-1)} \Bigg[\sum_{i\geqslant 3} c_i z_{i2}^2 + z_{22} H_{1,\ell_1} + \\ &\qquad + \sum_{j\geqslant 2} z_{22}^{\ell_j-\ell_1+1} H_{1,\ell_j} + z_{22}^{-2(\ell_1-1)} p_1' \circ \eta^{(\ell_1-1)} \Bigg], \end{aligned}$$

where the multiplicity of $z_{22}^{-2(\ell_1-1)}\ p_1' \circ \eta^{(\ell_1-1)}$ at $\mathbf{p}$ is at least M_1, and $H_{1,\ell_i} = H_{1,\ell_i}(\hat{z}_{22})$ for each $i \geqslant 1$. The bracketed expression defines the strict transform of P_2' at $\mathbf{p}$ under the morphism $\eta^{(\ell_1-1)}$.

Set for each $j \geqslant 1$ and $k \geqslant 3$,

$$H_{1,\ell_j}(\hat{z}_{22}) = \sum_{k\geqslant 3} a_{jk} z_{k2}, \qquad \mathcal{Z}_k = \sum_{j\geqslant 1} a_{jk} z_{22}^{\ell_j-\ell_1+1}.$$

Clearly, $\mathcal{Z}_k$ is the factor of z_{k2} in the sum of the second and third terms in the strict transform of P_2'. The (‘Tchirnhausen’) transformation

$$Z_{22} = z_{22}, \qquad Z_{k2} = z_{k2} + \frac{1}{2c_k}\mathcal{Z}_k, \quad k \geqslant 3, \tag{3.9}$$

determines a coordinate transformation at $\mathbf{p}$, and a simple check now shows that the strict transform of P_2' equals

$$Q(Z_{22}) + \sum_{i\geqslant 3} c_i Z_{i2}^2 + \sum_{\ell\geqslant 2} Z_{22}^{\ell} \tilde{G}_\ell(Z_{32}, \ldots, Z_{n2}),$$

where the multiplicity at $(0, \ldots, 0)$ of each $\tilde{G}_\ell$ is at least 2, and

$$Q(Z_{22}) = \left(\sum_{k\geqslant 3} -a_{1k}^2/4c_k \right) Z_{22}^2 + \sum_{i\geqslant \ell_2} \tilde{c}_i Z_{22}^i, \quad \tilde{c}_i \in \mathbb{Q}_p.$$

So one sees that this transformation now reduces Case C to Case B if $Q(Z_{22}) \neq 0$ and to Case A if $Q \equiv 0$. Following the procedure used in these two cases, it is simple to verify (left to the reader) that (3.6) also holds in Case C.

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(**D**) $M_0 \geqslant 1$, $M_2 < +\infty$.

It is convenient to split the argument into two cases. Writing $M_0 = 2\delta_1 + \delta_2$, $\delta_2 \in \{0, 1\}$, these are

$$\text{(i)} \quad M_2 > \delta_1 + \delta_2, \qquad \text{(ii)} \quad M_2 \leqslant \delta_1 + \delta_2.$$

The argument shows that it is possible to reduce to previous cases if either (i) or (ii) holds.

Subcase (i): It is clear that the resolution procedure will be the same as Case B. After $\delta_1 + \delta_2$ blow ups, first of p, and then of the origin in each chart $Z_2'(i)$, $i = 1, \ldots, \delta_1 + \delta_2 - 1$, the expressions of $P_1 \circ \pi \circ \eta^{(\delta_1+\delta_2)}$ and $P_2' \circ \eta^{(\delta_1+\delta_2)}$ in $Z_2'(\delta_1+\delta_2)$ are as follows

$$\begin{aligned}
P_1 \circ \pi \circ \eta^{(\delta_1+\delta_2)} &= y_1^d z_{22} \\
P_2' \circ \eta^{(\delta_1+\delta_2)} &= y_1^d z_{22}^{M_0} \Bigg[\sum_{i \geqslant 3} c_i z_{i2}^2 + \gamma_{M_0} + \sum_{k \geqslant 1} \gamma_{M_0+k} z_{22}^k + \\
&\qquad + \sum_{i \geqslant 1} z_{22}^{1+\ell_i-\delta_1-\delta_2} H_{1,\ell_i} + z_{22}^{-M_0} p_2' \circ \eta^{(\delta_1+\delta_2)} \Bigg].
\end{aligned}$$

In each of the charts $Z_j'(i)$, $j \neq 2$, the strict transform of $P_2' = 0$ is transverse to the exceptional divisor at each point $\mathbf{p}$ of the divisor, and the matrices $\mathbf{A}_i^{\#}(\mathbf{p})$ are precisely those given in Case (B). So, it suffices to consider the points on the exceptional divisor in $Z_2'(\delta_1 + \delta_2)$.

Since $M_2 > \delta_1 + \delta_2$, it follows that the strict transform intersects the exceptional divisor at any point $\mathbf{p}$ such that $\gamma_{M_0} + \sum_{i \geqslant 3} c_i z_{i2}^2(\mathbf{p}) = 0$. At such a point, the intersection is evidently transverse. On the other hand, the strict transform cannot contain the origin of $Z_2'(\delta_1 + \delta_2)$. Thus, since $M_0 \geqslant 1$ actually implies $M_0 \geqslant 2$, one concludes that $\mathbf{p}$ is good and

$$\mathbf{A}_{\delta_1+\delta_2}^{\#}(\mathbf{p}) = \begin{pmatrix} d & 1 & 0 \\ d & M_0 & \varepsilon \\ n-1 & M_0(n-2) & 0 \end{pmatrix}.$$

This completes the proof in subcase (i).

Subcase (ii): One blows up the origin $\eta_1' : Z'(1) \to U(p)$, and then blows up the origin in the charts $Z_2'(i)$, $i \leqslant M_2 - 2$ times in succession. It then follows that at the point $\bar{\mathbf{0}} \in Z_2'(M_2 - 1)$,

$$P_2' \circ \eta^{(M_2-1)} = y_1^d z_{22}^{2(M_2-1)} \Bigg\{ \sum_{j \geqslant M_0} \gamma_j z_{22}^{j-2(M_2-1)} + \sum_{i \geqslant 3} c_i z_{i2}^2 +$$

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$$+\sum_{i\geqslant 1} z_{22}^{\ell_i-(M_2-1)} H_{1,\ell_i} + z_{22}^{-2(M_2-1)} p_2' \circ \eta^{(M_2-1)}\Big\}.$$

Now, (ii) implies that

$$M_0 - 2(M_2-1) \geqslant \delta_1 + M_2 - 2(M_2-1) = \delta_1 - M_2 + 2.$$

Since $\delta_1 \geqslant M_2 - \delta_2 \geqslant M_2 - 1$, it follows that $\delta_1 - M_2 + 2 \geqslant 1$. Thus, the smallest exponent of $\sum_{j\geqslant M_0} \gamma_j z_{22}^{j-2(M_2-1)}$ is at least 1. This implies that the coordinate change $(z_{j2}) \to (Z_{j2})$, $j \geqslant 2$, defined in (3.9), can be used. Then, in the coordinates (Z_{j2}), it follows that

$$P_2' \circ \eta^{(M_2-1)} = y_1^d Z_{22}^{2(M_2-1)} \left[\sum_{i\geqslant 3} c_i Z_{i2}^2 + \tilde{Q}(Z_{22}) + \frac{p_2' \circ \eta^{(M_2-1)}}{Z_{22}^{2(M_2-1)}} \right],$$

where the quotient with $Z_{22}^{2(M_2-1)}$ has the form $\sum_{k\geqslant 2} Z_{22}^k \tilde{G}_k(\hat{Z}_{22})$, and $\mathrm{mult}_{(0,\dots,0)}$ $\tilde{G}_k \geqslant 2$ for each k. So, once again, one has reduced to Case B if $\tilde{Q} \neq 0$ and to Case A if $\tilde{Q} \equiv 0$. The same verification, left to the reader at the end of case C, now shows that only good points can be obtained at all points of each exceptional divisor, created by any subsequent blowing up. This completes the proof of Lemma 3.6.

Part iv. *The local form (2.2)*

In the notation (2.2.1), define, for fixed q,

$$P_1' = P_1(Y) - P_1(q), \qquad P_2' = P_2(Y) - P_2(q) - \psi_q(Y_1).$$

Although both functions depend upon q, this will not be emphasized in the notation. It is clear that the transformation $(P_1', P_2')|_{U(q)} \to \mathbf{P} - \mathbf{P}(q)|_{U(q)}$ is a permissible modification.

LEMMA 3.10. *Given the point q of* (2.2), *there exists a smooth p-adic manifold Z and proper surjective birational map $\eta_q\colon Z \to U(q)$ satisfying these properties*

(a) *η_q is an isomorphism outside $\eta_q^{-1}(q)$;*

(b) *At each point $\mathbf{q} \in \eta_q^{-1}(q)$, there exist local coordinates $z = (z_1, \dots, z_n)$, defined on a compact open neighborhood $W(\mathbf{q})$, such that either*

$$P_1' \circ \eta_q = z_1^{N_1} \dots z_n^{N_n}, \qquad P_2' \circ \eta_q = z_1^{L_1} \dots z_n^{L_n} \cdot (\textit{unit}),$$

$$\textit{and rank } A(\mathbf{q}) = \begin{pmatrix} N_1 & N_2 \dots \\ L_1 & L_2 \dots \end{pmatrix} = 2,$$

or there exists an amelioration $(F, G) \to (P_1' \circ \eta_q, P_2' \circ \eta_q)|_{W(\mathbf{q})}$ at $\mathbf{q}$.

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Remark. In the latter case, the composition of the transformations $(F, G) \to (P_1' \circ \eta_q, P_2' \circ \eta_q) \to \mathbf{P} \circ \eta_q - \mathbf{P}(q)$ is also an amelioration. $\square$

Proof. Since $P_2' = \sum_{i \geqslant 2} c_i Y_i^2 + \sum_{\ell \geqslant 2} Y_1^\ell g_\ell(Y_2, \ldots, Y_n)$, one essentially proceeds as in the proof of Lemma 3.6. The principal difference is that because there is no common factor for the two functions, one cannot hope to obtain a good point without additional terms being subtracted off from $P_2' \circ \eta_q$. This explains why the second possibility in (b) must be explicitly included in the statement of this Lemma and not Lemma 3.6.

Define p_1, p_2 by setting $\sum_{\ell \geqslant 2} Y_1^\ell g_\ell(Y_2, \ldots, Y_n) = p_1 + p_2$, where

$$p_1 = \sum_{\ell \geqslant 2} Y_1^\ell \sum_{e \geqslant 2} H_{e,\ell}(Y_2, \ldots, Y_n),$$

$$p_2 = \sum_{\ell \geqslant 2} Y_1^\ell H_{1,\ell}(Y_2, \ldots, Y_n),$$

where $H_{e,\ell}$ is homogeneous of degree e for each $e \geqslant 1$, $\ell \geqslant 2$. Then set

$$M_1 = \begin{cases} \mathrm{mult}_{(0,\ldots,0)}\, p_1 & \text{if } p_1 \neq 0 \\ 0 & \text{if } p_1 = 0, \end{cases}$$

$$M_2 = \begin{cases} \mathrm{mult}_{(0,\ldots,0)}\, p_2 & \text{if } p_2 \neq 0 \\ +\infty & \text{if } p_2 = 0. \end{cases}$$

As before, write $\mathcal{L} = \{\ell_1 < \ell_2 < \ldots\}$.

The proof depends upon the possible values for (M_1, M_2).

(A): $(M_1, M_2) = (0, +\infty)$.
Let $\eta_1{:}Z(1) \to U(q)$ denote the blowing up of the Y_1 axis in $\mathbb{Q}_p^n$, restricted to $U(q)$. Then $Z(1) = \cup_{j \geqslant 2} Z_j(1)$. Each chart $Z_j(1)$ has coordinates $(z_{1j}, \ldots, z_{nj})$ so that

$$\eta_1|_{Z_j(1)} = (z_{1j}, z_{2j}z_{jj}, \ldots, z_{jj}, \ldots, z_{nj}z_{jj}).$$

It is then clear that

$$P_1' \circ \eta_1|_{Z_j(1)} = z_{1j}, \qquad P_2' \circ \eta_1|_{Z_j(1)} = z_{jj}^2 \left[c_j + \sum_{i \neq j} c_i z_{ij}^2 \right].$$

So, any $\mathbf{q}$ on the exceptional divisor is a good point, and

$$\mathbf{A}_1^{\#}(\mathbf{q}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & \varepsilon \\ 0 & n-1 & 0 \end{pmatrix}.$$

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This proves the Lemma in Case A.

(B): $M_1 > 0$, $M_2 = +\infty$.
It follows that one can again blow up along the Y_1 axis via η_1. Then for each $j \geqslant 2$,

$$\begin{aligned} P_1' \circ \eta_1|_{Z_j(1)} &= z_{1j} \\ P_2' \circ \eta_1|_{Z_j(1)} &= z_{jj}^2 \left[c_j + \sum_{i \neq j} c_i z_{ij}^2 + z_{jj}^{-2}\, p_2 \circ \eta_1 \right]. \end{aligned}$$

Since $M_1 \geqslant 4$, the argument in Case B of (3.6) applies to show that if $\mathbf{q} \in \{z_{1j} = z_{jj} = 0\}$, then the strict transform of $\{P_2' = 0\}$ is transverse to $\{z_{jj} = 0\}$ at $\mathbf{q}$. Thus, $\mathbf{A}_1^{\#}(\mathbf{q})$ is exactly the same as in case (A). This shows Case B.

(C): $M_1 = 0$, $M_2 < +\infty$.
Let $\eta_1 : Z(1) \to U(q)$ denote the blowing up of q in $U(q)$, with exceptional divisor $\mathcal{D}_1$. In the chart $Z_j(1)$, $j \geqslant 2$, one has:

$$\begin{aligned} P_1' \circ \eta_1|_{Z_j(1)} &= z_{1j} z_{jj} \\ P_2' \circ \eta_1|_{Z_j(1)} &= z_{jj}^2 \left[c_j + \sum_{i \neq j} c_i z_{ij}^2 + \sum_{\ell \in \mathcal{L}} z_{jj}^{\ell - 1} H_{1\ell}' \right], \end{aligned}$$

where $H_{1,\ell}'$ denotes the strict transform of $H_{1,\ell}$. Since $\ell \in \mathcal{L}$ implies $\ell \geqslant 2$, it follows that if $\mathbf{q} \in \{z_{1j} = z_{jj} = 0\}$, and $z_{kj}(\mathbf{q}) \neq 0$, for some $k \geqslant 2$, then

$$\mathbf{A}_1^{\#}(\mathbf{q}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & \varepsilon \\ 0 & n-1 & 0 \end{pmatrix}.$$

Such a point is then good. On the other hand, if $z_{1j}(\mathbf{q}) \neq 0$, then $\mathbf{q}$ is identified to a point in $Z_1(1)$ via the identification

$$z_{j1}(\mathbf{q}) = 1/z_{1j}(\mathbf{q}), \qquad z_{i1}(\mathbf{q}) = z_{ij}(\mathbf{q})/z_{1j}(\mathbf{q}).$$

So any such point can be analyzed within the chart $Z_1(1)$, in which

$$\begin{aligned} P_1' \circ \eta_1|_{Z_1(1)} &= z_{11}, \\ P_2' \circ \eta_1|_{Z_1(1)} &= z_{11}^2 \left[\sum_{i \geqslant 2} c_i z_{i1}^2 + \sum_{\ell \in \mathcal{L}} z_{11}^{\ell - 1} H_{1\ell}(\hat{z}_{11}) \right]. \end{aligned}$$

Now, if $\mathbf{q} \neq \bar{\mathbf{0}}_n \in \mathcal{D}_1$ also lies on the strict transform of $\{P_2' = 0\}$, then it is clear that the strict transform is transverse to $\mathcal{D}_1$ at $\mathbf{q}$, and

$$\mathbf{A}_1^{\#}(\mathbf{q}) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ n-1 & 0 \end{pmatrix}. \tag{3.10.1}$$

If, however, $\mathbf{q} \neq \bar{\mathbf{0}}_n$ and $\sum_{i\geqslant 2} c_i z_{i1}^2(\mathbf{q}) \neq 0$, then an amelioration at $\mathbf{q}$ can be found. Indeed, let $U(\mathbf{q})$ be a compact open neighborhood of $\mathbf{q}$ on which $\sum_{i\geqslant 2} c_i z_{i1}^2 \neq 0$. On $U(\mathbf{q})$, define

$$G = P_2' \circ \eta_1 - z_{11}^2 \left[\sum_{i\geqslant 2} c_i z_{i1}^2(\mathbf{q}) + \sum_{\ell\in\mathcal{L}} z_{11}^{\ell-1} H_{1\ell}(z_{21}(\mathbf{q}), \ldots, z_{n1}(\mathbf{q})) \right].$$

Since there exists $k \geqslant 2$ such that $z_{k1}(\mathbf{q}) \neq 0$, this implies that $\{G = 0\}$ is a normal crossing divisor at $\mathbf{q}$. Defining $F = z_{11}|_{U(\mathbf{q})}$, the permissible transformation on $U(\mathbf{q})$

$$(F, G) \to (P_1' \circ \eta_1, P_2' \circ \eta_1)|_{U(\mathbf{q})}$$

determines an amelioration at $\mathbf{q}$, and the matrix of multiplicities for (F, G) at $\mathbf{q}$ is evidently the same as in (3.10.1).

Thus, only the origin in $Z_1(1)$ poses a difficulty. This remains true for $\ell_1 - 2$ additional blow ups

$$Z(\ell_1 - 1) \xrightarrow{\eta_{\ell_1-1}} Z(\ell_1 - 2) \xrightarrow{\eta_{\ell_1-2}} \cdots \xrightarrow{\eta_2} Z(1) \xrightarrow{\eta_1} U(q).$$

Set $\eta^{(j)} = \eta_1 \circ \cdots \circ \eta_j$, $j \leqslant \ell_1 - 1$. For each $j \in [2, \ell_1 - 2]$, there are two types of points of interest. The first type consists of points $\mathbf{q}$ on the exceptional divisor of $\eta^{(j)}$ in $Z_1(j)$ *except the origin*. At these points, either $A_j(\mathbf{q})$ has rank 2, or there exists, as in the preceding paragraph, a simple amelioration $(F, G) \to (P_1' \circ \eta^{(j)}, P_2' \circ \eta^{(j)})$. The second type consists of points $\mathbf{q} \in Z_k(j)$, $k \neq 1$, such that $\mathbf{q} \in \{z_{kj} = z_{jj} = 0\}$. Then one checks easily that for points of the first resp. second type:

$$\mathbf{A}_j^{\#}(\mathbf{q}) = \begin{pmatrix} 1 & 0 \\ 2j & 1 \\ j(n-1) & 0 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 1 & 1 & 0 \\ 2(j-1) & 2j & \varepsilon \\ (j-1)(n-1) & j(n-1) & 0 \end{pmatrix}.$$

At $\bar{\mathbf{0}}_n \in Z_1(\ell_1 - 1)$, one has

$$\begin{aligned} P_1' \circ \eta^{(\ell_1-1)} &= z_{11}, \\ P_2' \circ \eta^{(\ell_1-1)} &= z_{11}^{2(\ell_1-1)} \left[\sum_{i\geqslant 2} c_i z_{i1}^2 + \sum_{k\geqslant 1} z_{11}^{\ell_k-\ell_1+1} H_{1,\ell_k}(\hat{z}_{11}) \right]. \end{aligned}$$

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Now, define the coordinate transformation $(z_{11}, \ldots, z_{n1}) \to (Z_{11}, \ldots, Z_{n1})$ in the same manner as done in (3.9) (here, $Z_{11} = z_{11}$). One then shows in the same way that

$$P_2' \circ \eta^{(\ell_1-1)} = Z_{11}^{2(\ell_1-1)} \left[\sum_{i \geqslant 2} c_i Z_{i1}^2 + \tilde{Q}(Z_{11}) \right]$$

where $\tilde{Q}$ is a p-adically convergent power series. However, here, unlike the use made of the transformation in the proof of Lemma 3.6, one can define a new permissible modification by setting:

$$F = Z_{11}, \qquad G = P_2' \circ \eta^{(\ell_1-1)} - Z_{11}^{2(\ell_1-1)} \tilde{Q}(Z_{11}).$$

Evidently, $\bar{\mathbf{0}}_n$ is still a singularity for (F, G), but it is easy to deal with. It suffices to blow up the Z_{11} axis in $Z_1(\ell_1 - 1)$ via the map $\eta_{\ell_1}: Z(\ell_1) \to Z(\ell_1 - 1)$. It is then clear that for each $j = 2, \ldots, n$,

$$F \circ \eta_{\ell_1}|_{Z_j(\ell_1)} = Z_{11}, \qquad G \circ \eta_{\ell_1}|_{Z_j(\ell_1)} = Z_{11}^{2(\ell_1-1)} z_{jj}^2 \left(c_j + \sum_{i \neq j} c_i z_{ij}^2 \right).$$

Thus, for any $j \geqslant 2$ and $\mathbf{q} \in \{z_{jj} = 0\}$, the matrix of multiplicities of $(F \circ \eta_{\ell_1}, G \circ \eta_{\ell_1})$ equals:

$$\mathbf{A}_{\ell_1}^{\#}(\mathbf{q}) = \begin{pmatrix} 1 & 0 & 0 \\ 2(\ell_1 - 1) & 2 & \varepsilon \\ (\ell_1 - 1)(n - 1) & n - 2 & 0 \end{pmatrix}.$$

Thus, $(F \circ \eta_{\ell_1}, G \circ \eta_{\ell_1}) \to (P_1' \circ \eta^{(\ell_1)}, P_2' \circ \eta^{(\ell_1)})$ is an amelioration at $\mathbf{q}$. This completes the proof of the Lemma in Case C.

(D): $M_1 \geqslant 2$, $M_2 < +\infty$.
As in the proof of Case C, one blows up a point $\ell_1 - 1$ times by composing the maps $\eta_i: Z(i) \to Z(i-1)$, $i \leqslant \ell_1 - 1$. Each η_i, $i \geqslant 2$, blows up the origin in the chart $Z_2(i-1)$, and η_1 is the blowing up of q. At all points of the exceptional divisor of $\eta^{(\ell_1-1)}$ *except* the origin in $Z_2(\ell_1 - 1)$ the assertion in (3.10) holds. Indeed, any such point $\mathbf{q}$ is either a good point for $(P_1' \circ \eta^{(\ell_1-1)}, P_2' \circ \eta^{(\ell_1-1)})$, or, as in the proof of Case C, there is a simple amelioration $(F, G) \to (P_1' \circ \eta^{(\ell_1-1)}, P_2' \circ \eta^{(\ell_1-1)})$ at $\mathbf{q}$.

At $\bar{\mathbf{0}}_n \in Z_2(\ell_1 - 1)$,

$$P_1' \circ \eta^{(\ell_1-1)} = z_{11}$$

$$P_2' \circ \eta^{(\ell_1-1)}$$
$$= z_{11}^{2(\ell_1-1)} \left(\sum_{i \geqslant 2} c_i z_{i1}^2 + \sum_{\ell \in \mathcal{L}} z_{11}^{\ell-\ell_1+1} H_{1,\ell}(\hat{z}_{11}) + z_{11}^{-2(\ell_1-1)} p_2 \circ \eta^{(\ell_1-1)} \right).$$

As in preceding cases when $M_2 < +\infty$, one notes that $z_{11}^{-2(\ell_1-1)} p_2 \circ \eta^{(\ell_1-1)}$ has the same form as p_2. That is, the factor of each z_{11} only has terms of degree at least 2, and each power of z_{11} is at least 2. Then, defining the change of coordinates as in (C) (the one first introduced in (3.9)), it follows that

$$P_2' \circ \eta^{(\ell_1-1)} = Z_{11}^{2(\ell_1-1)} \left(\sum_{i \geqslant 2} c_i Z_{i1}^2 + \tilde{Q}(Z_{11}) + Z_{11}^{-2(\ell_1-1)} p_2 \circ \eta^{(\ell_1-1)} \right),$$

where $\tilde{Q}$ is a p-adically convergent power series in Z_{11}. Thus, defining $P_1'' = P_1' \circ \eta^{(\ell_1-1)}$ and

$$\begin{aligned} P_2'' &= P_2' \circ \eta^{(\ell_1-1)} - Z_{11}^{2(\ell_1-1)} \tilde{Q}(Z_{11}) \\ &= Z_{11}^{2(\ell_1-1)} \left(\sum_{i \geqslant 2} c_i Z_{i1}^2 + Z_{11}^{-2(\ell_1-1)} p_2 \circ \eta^{(\ell_1-1)} \right), \end{aligned}$$

one sees that the transformation $(P_1'', P_2'') \to (P_1' \circ \eta^{(\ell_1-1)}, P_2' \circ \eta^{(\ell_1-1)})$ is a permissible modification in some compact open neighborhood of $\bar{\mathbf{0}}_n$. Moreover, (P_1'', P_2'') now satisfies the property that $M_2 = +\infty$. So, one has reduced to the situation in Case B. The argument used there now completes the proof of the Lemma, as a simple check, left to the reader, will verify. □

Remark 3.11. An important point for applications is the lack of precision implicit in the entries of the matrices $\mathbf{A}^{\#}(\mathbf{p})$ resp. $\mathbf{A}(\mathbf{p})$ when $\mathbf{p}$ is a point treated by Corollary 3.2 (i) resp. Lemma 3.4. For example, the m_i, μ_i from (3.2)(i) are not easy to make more precise unless one has more information about P_1, P_2. This is a problem that is concentrated on the set of singular points of P_1, P_2 restricted to the n hyperplanes $\{x_i = 1\}$. Except for these points, the entries of all other matrices $\mathbf{A}^{\#}(\mathbf{y})$ have been explicitly calculated in the Lemmas of this Section. There is one case, however, in which everything can be made explicit, that is, when $P_1(x) = \beta \cdot x$, P_2 is arbitrary, and $(P_1, P_2) \in \mathcal{C}\ell_{\mathrm{II}}$, see Section 6 part ii.

4. Statement and Proof of Main Result

To formulate the main result, it is first necessary to define the following notions of ‘good wedge’ and ‘good asymptotic wedge’ for $\mathbf{P} \in \mathcal{C}\ell_I \cup \mathcal{C}\ell_{\mathrm{II}}$.

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DEFINITION 4.1. Let $\theta \colon X \to U \subset \mathbb{Q}_p^n$ denote a proper birational mapping onto a compact open neighborhood U. Let $\mathbf{x}$ denote a point on the exceptional divisor of θ, and $U(\mathbf{x})$ a compact open neighborhood of $\mathbf{x}$. Then the image $\theta U(\mathbf{x})$ is called an *analytic wedge*. □

If $\mathbf{P} \in \mathcal{C}\ell_{\mathrm{I}}$, let $q \in \mathrm{Sing}_{\mathbf{P}}$, and (f, g) denote $\mathbf{P} - \mathbf{P}(q)$. Set $U(q)$ to be a neighborhood of q satisfying the property in (1.6)(a) or (b). If $\mathbf{P} \in \mathcal{C}\ell_{II}$, π is the blowing up of $\bar{\mathbf{0}} \in \mathbb{Q}_p^n$, and $q \in \mathrm{Sing}_{\mathbf{P}}$, set

$$(f, g) = \begin{cases} \mathbf{P} - \mathbf{P}(q), & \text{if } q \neq \bar{\mathbf{0}}, \\ \mathbf{P} \circ \pi, & \text{if } q = \bar{\mathbf{0}}. \end{cases}$$

Further, if $q \neq \bar{\mathbf{0}}$, then $U(q)$ denotes a neighborhood on which (2.2) holds.

One now specifies the morphism θ as follows. If $\mathbf{P} \in \mathcal{C}\ell_{\mathrm{I}}$, and q satisfies (1.6)(b), then θ denotes a morphism constructed in Theorem 1.6. If $\mathbf{P} \in \mathcal{C}\ell_{\mathrm{II}}$, and $q \neq \bar{\mathbf{0}}$, then θ denotes a morphism constructed in Lemma 3.10. If $q = \bar{\mathbf{0}}$, one first chooses a point $p \in \pi^{-1}(\bar{\mathbf{0}})$ and neighborhood $U(p)$ in which one of the Lemmas 2.4–2.7 holds. Then, θ denotes the composition with π of the appropriate morphism constructed in Lemma 3.1, 3.4, or 3.6 (the choice of which depends upon the particular Lemma in Section 2 that applies to p).

For either possibility, given a point $\mathbf{x}$ in the exceptional divisor of θ, it follows that $\mathbf{x}$ is a good point either for $(f \circ \theta, g \circ \theta)$, or for a pair (F, G), obtained by a permissible modification $(F, G) \to (f \circ \theta, g \circ \theta)|_{U(\mathbf{x})}$, where $U(\mathbf{x})$ is a compact open neighborhood of $\mathbf{x}$ in some affine chart isomorphic to $\mathbb{Q}_p^n$.

DEFINITION 4.2. Using the notation from the preceding paragraph, the wedge $\theta U(\mathbf{x})$ is called a *good* $\mathbf{P}$ *wedge*. Setting $x = \theta(\mathbf{x})$, the pair

$$(\theta U(\mathbf{x}), (\mathbf{P} \circ \theta) U(\mathbf{x})) = (\theta U(\mathbf{x}), \mathbf{P}(x) + (f \circ \theta, g \circ \theta) U(\mathbf{x})),$$

is called a good asymptotic wedge for $\mathbf{P}$. □

The principal finiteness result of this paper now follows easily from the work in the preceding three sections.

THEOREM 4.3. *Let U be a compact neighborhood of a subset of* $\mathrm{Sing}_{\mathbf{P}}$ *for* $\mathbf{P} \in \mathcal{C}\ell_{\mathrm{I}} \cup \mathcal{C}\ell_{\mathrm{II}}$. *Then there exist finitely many good asymptotic wedges $\mathcal{W}_i = (W_{1,i}, W_{2,i})$ for* $\mathbf{P}$ *satisfying the two properties:*

(i) $\cup_i W_{1,i} = U$;
(ii) *For $i \neq i'$, the Haar measure of $W_{1,i} \cap W_{1,i'}$ equals zero.*

Proof. It suffices to give the proof for $\mathbf{P} \in \mathcal{C}\ell_{\mathrm{II}}$. The proof for $\mathcal{C}\ell_{\mathrm{I}}$ is similar, and the simple modifications are left to the reader.

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Corollary 3.2 and Lemmas 3.4, 3.6, 3.10 have shown that for any $y \in U$, there exists an open compact neighborhood $U(y)$ and a finite set of good asymptotic wedges $\mathcal{W}(y) = (W_{1i}(y), W_{2i}(y))$ satisfying the properties:

$$U(y) = \cup_i W_{1i}(y),$$

$$\text{the Haar measure of } W_{1i}(y) \cap W_{1i'}(y) = 0 \quad \text{if } i \neq i'.$$

To see why this is so, consider first the case analyzed by Lemma 3.10. Given the data in the statement of (3.10), the total disconnectedness of the p-adic topology implies that one can cover the compact set $\eta_q^{-1}(q)$ by finitely many pairwise disjoint compact open neighborhoods $W(\mathbf{q}_i)$, so that for each i, $\eta_q(\mathbf{q}_i) = q$, and part (b) of (3.10) holds in each $W(\mathbf{q}_i)$. Setting $\theta = \eta_q$ in (4.2), each $\eta_q W(\mathbf{q}_i)$ is a good $\mathbf{P}$ wedge and the pair $(\eta_q W(\mathbf{q}_i), \mathbf{P}(q) + (\mathbf{P}' \circ \eta_q) W(\mathbf{q}_i))$ is a good asymptotic wedge for $\mathbf{P}$. Moreover, since η_q is an isomorphism outside $\mathrm{Sing}_{\mathbf{P}}$, it follows that $\eta_q W(\mathbf{q}_i) \cap \eta_q W(\mathbf{q}_{i'}) \subset \mathrm{Sing}_{\mathbf{P}}$, from which one concludes that the Haar measure of the intersection is zero.

Now define the set $A_q = \eta_q^{-1} U(q) - \cup_i W(\mathbf{q}_i)$. It follows that A_q is a closed and compact set disjoint from $\eta_q^{-1}(q)$. Thus, $q \notin \eta_q(A_q)$. Since $\eta_q(A_q)$ is closed and compact, there exists an open compact neighborhood $\mathcal{U}(q) \subset U(q)$ of q such that $\mathcal{U}(q) \cap \eta_q(A_q) = \emptyset$. Thus, $\eta_q^{-1}\mathcal{U}(q) \subset \cup_i W(\mathbf{q}_i)$. Now set $W_q(\mathbf{q}_i) = \eta_q W(\mathbf{q}_i) \cap \mathcal{U}(q)$. The good asymptotic wedges in question are $\mathcal{W}_i(q) =_{\mathrm{def}} (W_q(\mathbf{q}_i), \mathbf{P}(q) + (\mathbf{P}' \circ \eta_q) W_q(\mathbf{q}_i))$. In this way, one has 'uniformized' $\mathcal{U}(q)$ by the good $\mathbf{P}$ wedges $W_q(\mathbf{q}_i)$.

An entirely similar argument applies for $\bar{\mathbf{0}}$. To each point p on $\pi^{-1}(\bar{\mathbf{0}})$, there is a neighborhood $U(p)$ that satisfies the properties stated in (3.2), (3.4), or (3.6). This uniformizes some neighborhood $\mathcal{U}(p) \subset U(p)$ by good wedges, using the same argument as above. Then, by composing with π and using compactness of $\pi^{-1}(\bar{\mathbf{0}})$, one argues, again as above, that some neighborhood of $\bar{\mathbf{0}}$ is a finite union of good $\mathbf{P}$ wedges.

Since U is compact, one can extract a finite subcover of open neighborhoods $\mathcal{U}(q_j)$ of the covering $\{\mathcal{U}(q)\}_{q \in U}$, each of which satisfies the properties of being uniformized by good $\mathbf{P}$ wedges. Moreover, total disconnectedness of the p-adic topology insures that one can always arrange the finite cover so that $\mathcal{U}(q_j) \cap \mathcal{U}(q_{j'}) = \emptyset$ if $j \neq j'$. It then follows that the collection of good asymptotic wedges $\{\mathcal{W}_i(q_j)\}_{j,i}$ is a finite set and satisfies properties (i), (ii), completing the proof.

5. Analysis Restricted to a Good Asymptotic Wedge

The same data and notations from the beginning of Section 4 are used in the following.

Since Theorem 4.3 decomposes a compact neigborhood of $\mathrm{Sing}_{\mathbf{P}}$ into a union of finitely many good $\mathbf{P}$ wedges, the next step is to extend Igusa's finiteness theorem to such wedges. This is not difficult since the wedge comes equipped with a

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convenient parametrization (via the map θ in (4.2)). Thus, one can always work in a compact open neighborhood of a fixed good point $\mathbf{x}$ for a pair (F, G), in which it is straightforward to iterate Igusa's one variable theory. The first part of Section 5 does this by verifying two needed analytical properties of a zeta function associated to a good $\mathbf{P}$ wedge. These properties are the natural two variable analogues of results proved by Igusa in [I, ch. 3] over $\mathbb{Q}_p$. Part ii introduces the fiber integral associated to a good asymptotic wedge for $\mathbf{P}$. A general expression is then given for this fiber integral, using an iterated form of the inverse Mellin transform formula. Both these notions apply only to the pair (F, G), *not necessarily* to $\mathbf{P}\circ\theta - \mathbf{P}\circ\theta(\mathbf{x})$. Part iii relates the results to the fiber integral of $\mathbf{P}$.

Part i. *Properties of a zeta function associated to a good* $\mathbf{P}$ *wedge* $\theta U(\mathbf{x})$

One assumes that local coordinates $z = (z_1, \ldots, z_n)$ are defined on a compact open $U(\mathbf{x})$ such that

$$F = \prod_{i=1}^{n} z_i^{N_i} \cdot u_1(z), \qquad G = \prod_{i=1}^{n} z_i^{M_i} \cdot u_2(z), \qquad \det d\theta = \prod_{i=1}^{n} z_i^{\mu_i - 1} \cdot u(z).$$

Moreover, the rank of the matrix $A(\mathbf{x})$ (see (1.2)) equals 2. By permuting coordinates, define the integer R by the condition that $i \leqslant R$ iff $(N_i, M_i) \neq (0, 0)$. Since the exceptional divisor for θ is a subset of $\{F = G = 0\}$, it follows that $\mu_i = 1$ for $i > R$.

Let $\boldsymbol{\chi} = (\chi_1, \chi_2)\colon \mathcal{U}_p^2 \to (S^1)^2$ denote a pair of characters on the units and $\mathbf{s} = (s_1, s_2) \in \mathbb{C}^2$. Let φ denote a compactly supported locally constant function on $U(\mathbf{x})$ (i.e. 'test function'). Define

$$\mathbf{Z}_{\mathbf{x}}(\boldsymbol{\chi}, \mathbf{s}, \varphi) = \int_{U(\mathbf{x}) - \{F\cdot G = 0\}} \varphi \chi_1(acF)\chi_2(acG) \cdot |F|^{s_1}|G|^{s_2}|\theta^* \,\mathrm{d}\mu|,$$

where $acy = y/|y|$ is the 'angular component' of y, and the measure in the integral is the pullback by θ of normalized Haar measure $|d\mu|$ on $\mathbb{Q}_p^n$. This function of $\mathbf{s}$ is called a local zeta function on the wedge $\theta U(\mathbf{x})$. When $\boldsymbol{\chi} = (\chi_0, \chi_0)$, where $\chi_0 \equiv 1$, the zeta function is called the principal zeta function on the wedge. Otherwise it is a 'twist' of this function.

The first of two needed properties is the following.

PROPOSITION 5.2. *Each* $\mathbf{Z}_{\mathbf{x}}(\boldsymbol{\chi}, \mathbf{s}, \varphi)$ *is analytic if each* $\operatorname{Re} s_i > 0$, *and admits an analytic continuation to* $\mathbb{C}^2$ *as a meromorphic function. Defining* $w_i = p^{-s_i}$, $i = 1, 2$, *it is a rational function, which modulo* $\mathbb{C}[w_1, w_2, w_1^{-1}, w_2^{-1}]$, *has the form*

$$\mathbf{Z}_{\mathbf{x}}(\boldsymbol{\chi}, \mathbf{s}, \varphi) = \sum_{I \subset \{1,\ldots,R\}} \frac{u_I(w, \boldsymbol{\chi})}{v_I(w)}, \tag{5.2.1}$$

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where

$$v_I(w) = \prod_{i\in I}(1 - p^{-\mu_i} w_1^{N_i} w_2^{M_i}).$$

Proof. Define

$$\mathcal{I}_0(\mathbf{x}) = \left\{\{i < j\} \subset \{1, \ldots, R\}\text{:rank}\begin{pmatrix} N_i & N_j \\ M_i & M_j \end{pmatrix} = 2\right\}. \tag{5.2.2}$$

Given $\iota = \{i < j\} \in \mathcal{I}_0(\mathbf{x})$, set

$$\Delta_\iota(\mathbf{x}) = \begin{vmatrix} N_i & N_j \\ M_i & M_j \end{vmatrix}, \qquad m_\iota(\mathbf{x}) = \text{ord}\, \Delta_\iota(\mathbf{x}), \qquad m_0(\mathbf{x}) = \max_\iota m_\iota(\mathbf{x}).$$

By shrinking $U(\mathbf{x})$, if needed, one may assume that the coordinates $(z_1, \ldots, z_n)$ identify $U(\mathbf{x})$ with a subset of $(p^{m_0(\mathbf{x})+1})^n$. One then chooses an integer $e_0 \geqslant m_0(\mathbf{x}) + 1$ so that $U(\mathbf{x}) = \bigsqcup[b + (p^{e_0})^n]$ is a disjoint union of cosets, on each of which the following hold:

$$\text{For } i = 1, 2,\ |u_i(z)|_{[b+(p^{e_0})^n]} = |u_i(b)|, \text{ and} |u(z)|_{[b+(p^{e_0})^n]} = |u(b)|. \tag{5.2.3}$$

$$u_i(z)|_{[b+(p^{e_0})^n]} = u_i(b)\cdot\left(1 + \sum_{I:|I|\geqslant 1} a_I(i) z^I\right), \quad \text{and each } a_I(i) \in \mathbb{Z}_p, \tag{5.2.4}$$

$$\varphi|_{[b+(p^{e_0})^n]} = \varphi(b). \tag{5.2.5}$$

Given a character χ on $\mathcal{U}_p$, define its conductor $e_\chi = \inf\{e{:}\chi|_{1+(p)^e} = 1\}$. With e_0 chosen as above, and for a fixed pair of characters $\boldsymbol{\chi}$, define $e = \max\{e_0, e_{\chi_1}, e_{\chi_2}\}$. Now decompose $U(\mathbf{x})$ modulo p^e. Thus, $U(\mathbf{x}) = \bigsqcup[b + b' + (p^e)^n]$. For a fixed choice of b, b', set $c = b + b' = (c_1, c_2, \ldots, c_n)$. It follows that for Re $s_i > 0, i = 1, 2$, the integral defining $\mathbf{Z}_\mathbf{x}$ converges absolutely, and by working mod p^e, the value of $\chi_i(u_i)$ is constant (see [I, pg. 95]). Thus,

$$\begin{aligned}\mathbf{Z}_\mathbf{x}(\boldsymbol{\chi}, \mathbf{s}, \varphi) \;=\; & \sum_c \varphi(c)\left(\prod_{i=1}^{2}|u_i(c)|^{s_i}\cdot|u(c)|\right)\times \\ & \times\prod_{i=1}^{2}\chi_i\left(ac(1+\sum_I a_I(i)c^I)\right)\times \\ & \times\prod_{i=1}^{R}\int_{c_i+(p^e)}\chi_1^{N_i}\chi_2^{M_i}(acz_i)\cdot|z_i|^{N_i s_1+M_i s_2+\mu_i-1}|dz_i|.\end{aligned} \tag{5.2.6}$$

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One evaluates (5.2.6) by iterating the calculation in [I, p. 89]. It follows that the integral over $c+(p^e)^n$ equals 0 if for some i:

$$\begin{aligned}&\chi_1^{N_i}\chi_2^{M_i} \neq 1 \text{ and } c_i \in (p^e),\\ &\text{or}\quad \chi_1^{N_i}\chi_2^{M_i}|_{1+c_i^{-1}(p^e)} \neq 1 \quad\text{and}\quad c_i \notin (p^e).\end{aligned} \tag{5.2.7}$$

On the other hand, if (5.2.7) does not hold, then one evaluates the integral over the coset $[c+(p^e)^n]$ in (5.2.6) as follows. Define $\mathcal{M}_0(c)=\{i\in[1,R]{:}c_i\notin(p^e)\}$, and $\mathcal{M}_1(c)=\{1,\ldots,R\}-\mathcal{M}_0(c)$. Set $M_i(c)=\#\mathcal{M}_i(c)$, $w_i=p^{-s_i}$, $i=1,2$. Then the third line in (5.2.6) equals:

$$\delta(c)\left[\prod_{i\in\mathcal{M}_0(c)} p^{-(\mu_i-1)\operatorname{ord} c_i}\chi_1^{N_i}\chi_2^{M_i}(ac c_i)w_1^{N_i\operatorname{ord} c_i}w_2^{M_i\operatorname{ord} c_i}\right]$$

$$\left[\prod_{i\in\mathcal{M}_1(c)}\frac{p^{-e\mu_i}w_1^{eN_i}w_2^{eM_i}}{1-p^{-\mu_i}w_1^{N_i}w_2^{M_i}}\right],$$

where $\delta(c)=p^{-e(M_0(c)+n-R)}(1-p^{-1})^{M_1(c)}$.

It is then clear that this term has the form of a summand, indexed by the subset $I=\mathcal{M}_1(c)$ of $\{1,\ldots,R\}$, as asserted by the Proposition, completing the proof of (5.2).

The formulae in the proof of (5.2) are now used to prove the following extension of Igusa's finiteness theorem (see Introduction).

PROPOSITION 5.3. *For all but a finite number of* $\boldsymbol{\chi}$, $\mathbf{Z}_{\mathbf{x}}(\boldsymbol{\chi},\mathbf{s},\varphi)=0$ *for all* φ.

Proof. Using the notations from (5.2), it follows that the integral over the coset $[c+(p^e)^n]$ is non zero iff

$$\begin{aligned}&\chi_1^{N_i}\chi_2^{M_i}|_{1+c_i^{-1}(p^e)}=1 \quad \text{if } i\in\mathcal{M}_0(c), \quad\text{and}\\ &\chi_1^{N_i}\chi_2^{M_i}=1 \quad \text{if } i\in\mathcal{M}_1(c).\end{aligned} \tag{5.3.1}$$

For each $\iota=\{i<j\}\in\mathcal{I}_0(\mathbf{x})$, there exist pairs of integers $(\alpha_1(\iota),\beta_1(\iota))$, $(\alpha_2(\iota),\beta_2(\iota))$ such that

$$\begin{aligned}&\chi_1^{N_i\alpha_1(\iota)+N_j\beta_1(\iota)}\chi_2^{M_i\alpha_1(\iota)+M_j\beta_1(\iota)}=\chi_1^{\Delta_\iota(\mathbf{x})},\\ &\chi_1^{N_i\alpha_2(\iota)+N_j\beta_2(\iota)}\chi_2^{M_i\alpha_2(\iota)+M_j\beta_2(\iota)}=\chi_2^{\Delta_\iota(\mathbf{x})}.\end{aligned}$$

The elements i,j of ι are in $\mathcal{M}_0(c)$ or $\mathcal{M}_1(c)$, independently of one another. Suppose first that $i\in\mathcal{M}_0(c)$ and $j\in\mathcal{M}_1(c)$. Then (5.3.1) implies $\chi_1^{\Delta_\iota(\mathbf{x})}=\chi_2^{\Delta_\iota(\mathbf{x})}=1$ on $[1+c_i^{-1}(p^e)]$. If $i\in\mathcal{M}_1(c)$, $j\in\mathcal{M}_0(c)$, then the same equation holds on $[1+$

$c_j^{-1}(p^e)$]. If $i, j \in \mathcal{M}_0(c)$, then the equation holds on $[1+c_i^{-1}(p^e)] \cup [1+c_j^{-1}(p^e)]$. If $i, j \in \mathcal{M}_1(c)$, then $\chi_1^{\Delta_\iota(\mathbf{x})} = \chi_2^{\Delta_\iota(\mathbf{x})} = 1$ on $\mathcal{U}_p$.

Since for each $k \in [1, R]$, $c_k \in p^{m_0(\mathbf{x})+1}$, this implies $(p^{e-m_0(\mathbf{x})-1}) \subset c_k^{-1}(p^e)$ for each k. Hence, $1 + (p^{e-m_0(\mathbf{x})-1}) \subset 1 + c_k^{-1}(p^e)$, for $k \in \mathcal{M}_0(c)$. By (5.3.1), it follows that for each $\ell = 1, 2$,

$$\chi_\ell^{\Delta_\iota(\mathbf{x})} = 1 \quad \text{on the coset} \quad [1 + (p^{e-m_0(\mathbf{x})-1})].$$

Using [I, Lemma 2.5, ch. 3], this implies that if $e - m_0(\mathbf{x}) - 1 > m_0(\mathbf{x})$, then for each ℓ, $\chi_\ell = 1 \quad \text{on} \;\; 1 + \Delta_\iota(\mathbf{x})(p^{e-m_0(\mathbf{x})-1})$.

Suppose now that there exists ℓ such that $e_{\chi_\ell} > e_0 + m_0(\mathbf{x})$. To fix notation, assume $\ell = 1$. Further, one may assume $e_{\chi_1} \geqslant e_{\chi_2}$, so that $e = e_{\chi_1}$. This implies

$$e - m_0(\mathbf{x}) - 1 = e_{\chi_1} - m_0(\mathbf{x}) - 1 \geqslant e_0 \geqslant m_0(\mathbf{x}) + 1,$$

by the choice of e_0 in the proof of (5.2). Hence, $e - m_0(\mathbf{x}) - 1 > m_0(\mathbf{x})$.

By the definition of $m_0(\mathbf{x})$, there exists $r \geqslant 1$ such that $\Delta_\iota(\mathbf{x})(p^{e-m_0(\mathbf{x})-1}) = (p^{e-r})$. One concludes that $\chi_1 = 1 \quad \text{on}\; 1 + (p^{e_{\chi_1}-r})$. This, however, violates the definition of e_{χ_1}. Thus, if $\boldsymbol{\chi}$ is some pair for which the integral, over some coset, in the second line of (5.2.6) is non zero, then necessarily $e_{\chi_1}, e_{\chi_2} \leqslant e_0 + m_0(\mathbf{x}) < 2e_0$. This implies the set of $\boldsymbol{\chi}$ for which $\mathbf{Z}_{\mathbf{x}}(\boldsymbol{\chi}, \cdot) \neq 0$ can only be a finite set, and completes the proof.

Part ii. *The fiber integral for a good asymptotic wedge*
Given the good asymptotic wedge $\mathcal{W}(\mathbf{x}) = (\theta U(\mathbf{x}), (\mathbf{P} \circ \theta)\; U(\mathbf{x}))$, the pair (F, G), satisfying (5.1), and the test function φ, one starts with the following.

DEFINITION 5.4. The fiber integral for $\mathcal{W}(\mathbf{x})$ is the fiber integral along the nonsingular fibers of (F, G) with respect to the measure $|\theta^*(\mathrm{d}x_1 \cdots \mathrm{d}x_n)|$ on $U(\mathbf{x})$.

In other words, letting $\boldsymbol{\tau} = (\tau_1, \tau_2)$ denote a regular value for (F, G), the fiber integral is the function:

$$\boldsymbol{\tau} \to \mathcal{F}_{\mathbf{x}}(\varphi, \boldsymbol{\tau}) =_{\mathrm{def}} \int_{\{(F,G)=\boldsymbol{\tau}\}} \varphi |\det \mathrm{d}\theta| |\omega_{(F,G)}|, \tag{5.4.1}$$

where $\omega_{(F,G)}|_{(F,G)=\boldsymbol{\tau}} = \mathrm{d}z_1\, \mathrm{d}z_2 \cdots \mathrm{d}z_n / \mathrm{d}F \wedge \mathrm{d}G|_{(F,G)=\boldsymbol{\tau}}$ is a globally defined $n-2$ differential form on the indicated fiber whose corresponding measure is denoted by $|\omega_{(F,G)}|$. □

The following Proposition shows that the Mellin transform of $\mathcal{F}_{\mathbf{x}}$ yields the $\mathbf{Z}_{\mathbf{x}}$. Its proof is an iteration of that in [I, ch. 3], and left to the reader.

PROPOSITION 5.5. *If* $\mathrm{Re}(s_i) > 0$ *for each* i, *then*

$$\mathbf{Z}_{\mathbf{x}}(\boldsymbol{\chi}, \mathbf{s}, \varphi) = (1 - p^{-1})^2 \int (|\tau_1 \tau_2| \mathcal{F}_{\mathbf{x}}(\varphi, \boldsymbol{\tau})) \prod_{i=1}^{2} \chi_i(ac\tau_i)|\tau_i|^{s_i} \frac{|\mathrm{d}\,\tau_1\, \mathrm{d}\tau_2|}{|\tau_1 \tau_2|},$$

where the integral is taken over the set of regular values of (F, G) *in* $\{\tau_1 \cdot \tau_2 \neq 0\}$.

Since $\mathbf{x}$ is a good point for (F, G), this formula can be inverted to express $|\tau_1\tau_2|\mathcal{F}_{\mathbf{x}}$ in terms of the $\mathbf{Z}_{\mathbf{x}}$. This is possible, if $\tau_1, \tau_2 \neq 0$. Propositions 5.1, 5.3 imply by a straightforward iteration of the argument in [I, ch. 1], left to the reader, the following.

PROPOSITION 5.6. *For any regular value* $\boldsymbol{\tau}$ *of* (F, G) *such that* $\tau_1\tau_2 \neq 0$, *one has that*

$$\begin{aligned} |\tau_1\, \tau_2|\mathcal{F}_{\mathbf{x}}(\varphi, \boldsymbol{\tau}) \;=\; & \sum_{\boldsymbol{\chi}} (\mathrm{Res}_{w_1=0}\, \mathrm{Res}_{w_2=0} \mathbf{Z}_{\mathbf{x}}(\boldsymbol{\chi}, \mathbf{s}, \varphi) w_1^{-\operatorname{ord}\tau_1 - 1} w_2^{-\operatorname{ord}\tau_2 - 1}) \times \\ & \times \chi_1^{-1}(ac\tau_1)\chi_2^{-1}(ac\tau_2). \end{aligned} \tag{5.6.1}$$

Extensions across $\{\tau_1 = 0\} \cup \{\tau_2 = 0\}$ are sometimes possible. This can be inferred by means of the formula in (6.11). However, to estimate the Fourier transform of the fiber integral, precise information about the behavior of the fiber integral in the open set $\{\tau_1\tau_2 \neq 0\}$ suffices.

Part iii. *Local to global*

Evidently, one needs to connect the $\mathcal{F}_{\mathbf{x}}$ to the 'global' fiber integral determined by $\mathbf{P}$. Given a locally constant and compactly supported function Φ on $\mathbb{Q}_p^n$, recall that the fiber integral for $\mathbf{P}$ is defined at a nonsingular value $\mathbf{t}$ of $\mathbf{P}$ to equal $\mathbf{F}(\Phi, \mathbf{t}) = \int_{\{\mathbf{P}=\mathbf{t}\}} \Phi\, |\omega_{\mathbf{P}}|$, where $\omega_{\mathbf{P}}|_{\mathbf{P}=\mathbf{t}} = \mathrm{d}x_1 \cdots \mathrm{d}x_n / \mathrm{d}P_1 \wedge \mathrm{d}P_2|_{\mathbf{P}=\mathbf{t}}$. This globally defined $n-2$ differential form determines the measure, denoted by $|\omega_{\mathbf{P}}|_{\mathbf{P}=\mathbf{t}}|$. In particular, when Φ is the characteristic function of $\mathbb{Z}_p^n$, the fiber integral equals the function described in the Introduction.

Suppose that supp $\Phi \subset U$, where U is compact and open. Applying (4.3) to U, there exist finitely many good asymptotic wedges $\mathcal{W}_i = (W_{1,i}, W_{2,i})$, satisfying (4.3)(i), (ii). This gives an expression for $\mathbf{F}(\varphi, \mathbf{t})$ as a sum of finitely many 'local' contributions to the fiber integral. Given one such $\mathcal{W}_i$, for which $W_{1,i} = \theta_i U(\mathbf{x}_i)$, the local contribution equals

$$\begin{aligned} \mathbf{F}_{\mathbf{x}_i}(\Phi, \mathbf{t}) &=_{\mathrm{def}} \int_{\{\mathbf{P}=\mathbf{t}\}\cap\theta_i\, U(\mathbf{x}_i)} \Phi|\omega_{\mathbf{P}}| \\ &= \int_{\{\mathbf{P}\circ\theta_i=\mathbf{t}\}\cap U(\mathbf{x}_i)} (\Phi\circ\theta_i)|\det \mathrm{d}\theta_i||\omega_{\mathbf{P}\circ\theta_i}|. \end{aligned} \tag{5.7}$$

The second equation follows since $\mathbf{t}$ is a regular value of $\mathbf{P}\circ\theta_i|_{U(\mathbf{x})}$, and the fact that as measures, $|\det \mathrm{d}\theta_i| \cdot |\omega_{\mathbf{P}\circ\theta_i}|_{\mathbf{P}\circ\theta_i=\mathbf{t}}| = |(\theta_i)^*\omega_{\mathbf{P}}|_{\mathbf{P}=\mathbf{t}}|$. Thus, (4.3)(i), (ii) imply $\mathbf{F}(\varphi, \mathbf{t}) = \sum_i \mathbf{F}_{\mathbf{x}_i}(\Phi, \mathbf{t})$.

There are now two possibilities to consider. Either $\mathbf{x}_i$ is a good point for $\mathbf{P}\circ\theta_i - \mathbf{P}\circ\theta_i(\mathbf{x}_i)$, or it is a bad point. In either case, first define $x_i = \theta_i(\mathbf{x}_i)$. If $\mathbf{x}_i$ is good, then one sets in the preceding discussion

$$\boldsymbol{\tau} = \mathbf{t} - \mathbf{P}(x_i),\, \varphi = (\Phi\circ\theta_i)|_{U(\mathbf{x}_i)}, \qquad (F, G) = \mathbf{P}\circ\theta_i - \mathbf{P}(x_i),$$

$$\mathbf{F}_{\mathbf{x}_i}(\Phi, \boldsymbol{\tau} + \mathbf{P}(x_i)) = \mathcal{F}_{\mathbf{x}_i}(\varphi, \boldsymbol{\tau}).$$

If $\mathbf{x}_i$ is a bad point, one uses a transformation in the range of the two mappings. According to (1.2), this is the mapping (up to a permutation of the coordinates in the image) of the form:

$$\zeta_{\mathbf{x}_i}:\boldsymbol{\tau} \to \mathbf{t} = \mathbf{P}(x_i) + (u_1(\mathbf{x}_i)\tau_1^{\delta}, \tau_2 + \psi(\tau_1)). \tag{5.8}$$

It is clear that the following holds.

LEMMA 5.9. *Let* $\zeta_{\mathbf{x}_i}(\boldsymbol{\tau}) = \mathbf{t}$. *Then*

(1) $\zeta_{\mathbf{x}_i}^{-1}(\mathbf{t})$ *is a finite set of at most* δ *points*;
(2) $w \in U(\mathbf{x}_i)$ *satisfies* $(F(w), G(w)) = \boldsymbol{\tau}$ *iff* $\mathbf{P} \circ \theta_i(w) = \mathbf{t} + \mathbf{P}(x_i)$.

One next connects the differential form $d(P_1 \circ \theta_i) \wedge d(P_2 \circ \theta_i)$ to $\mathrm{d}F \wedge \mathrm{d}G$. Simple verifications show:

LEMMA 5.10.

(A) $d(P_1 \circ \theta_i) \wedge d(P_2 \circ \theta_i) = \delta \cdot u_1(\mathbf{x}_i) \cdot F^{\delta-1} \cdot \mathrm{d}F \wedge \mathrm{d}G$.
(B) *Assume that* $\mathbf{t}$ *is a regular value for* $\mathbf{P} \circ \theta_i|_{U(\mathbf{x}_i)}$. *Then, any* $\boldsymbol{\tau} \in \zeta_{\mathbf{x}_i}^{-1}(\mathbf{t})$ *is a regular value for* (F, G).
(C) *If* $\phi dz = \phi\, \mathrm{d}z_1 \cdots \mathrm{d}z_n$ *is any analytic* n*-form defined on* $U(\mathbf{x}_i)$, *then, as measures on the fibers, one has the following relation:*

$$\begin{aligned}&|\phi\, \mathrm{d}z/d(P_1 \circ \theta_i) \wedge d(P_2 \circ \theta_i)|_{U(\mathbf{x}_i)\cap\{\mathbf{P}\circ\theta_i=\mathbf{P}(x_i)+\mathbf{t}\}}| \\ &\quad= \frac{1}{\delta|u_1(\mathbf{x}_i)||\tau_1|^{\delta-1}}|\phi\, \mathrm{d}z/\mathrm{d}F \wedge \mathrm{d}G|_{U(\mathbf{x}_i)\cap\{(F,G)=\boldsymbol{\tau}\}}|.\end{aligned}$$

Putting together these results, one now can relate the local contribution to the fiber integral in $\theta_i U(\mathbf{x}_i)$, and the fiber integral in the asymptotic wedge $\mathcal{W}_i$. Given the function Φ as above, set $\varphi = \Phi \circ \theta_i|_{U(\mathbf{x}_i)}$.

PROPOSITION 5.11. *If* $\mathbf{t}$ *is a regular value of* $\mathbf{P} \circ \theta_i - \mathbf{P}(x_i)$, *then*

$$\mathbf{F}_{\mathbf{x}_i}(\Phi, \mathbf{t} + \mathbf{P}(x_i)) = \frac{1}{\delta|u_1(\mathbf{x}_i)|} \sum_{\{\boldsymbol{\tau}:\zeta_{\mathbf{x}_i}(\boldsymbol{\tau})=\mathbf{t}\}} \frac{\mathcal{F}_{\mathbf{x}_i}(\varphi, \boldsymbol{\tau})}{|\tau_1|^{\delta-1}}. \tag{5.11.1}$$

6. Some Useful Refinements

The expression (5.6.1) is understood geometrically in part i of this section, using a method of partial fraction decomposition appropriate for the denominators v_I of (5.2.1). The main result is given in Theorem 6.11. The analogues over $\mathbb{R}, \mathbb{C}$ are

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sketched in (6.13). Part ii specializes the discussion to the case when $P_1(x) = \mathbf{b} \cdot x$, where $\mathbf{b}$ is a fixed vector of $\mathbb{Z}_p^n$. Here everything becomes very explicit.

Part i. *A simple geometric interpretation of* (5.6.1)
Substituting the expression (5.2.1) for $\mathbf{Z_x}$ into (5.6.1), and then summing over the index sets I, an explicit determination of the iterated residue of the summand indexed by I is not yet possible whenever three or more factors appear in the factorization of v_I. By the form of v_I, this can only occur if $|I| \geqslant 3$. To get around this difficulty, one needs a constructive method of partial fraction decomposition of rational functions on K^n, K any local field. A useful reference is [Lei]. The discussion is adapted to the particular form of the factors of any v_I, which turns out to be useful in interpreting the procedure in geometric terms. In the following, one $I \neq \emptyset$ with $|I| \geqslant 3$, is fixed. One then shows how to reduce the calculation of the iterated residue to fractions with $|I| = 2$.

Using (5.1) and (5.2.1), define

$$\mathcal{I}_1(\mathbf{x}) = \{\iota = \{i < j\} \subset \{1, \ldots, R\}\text{:rank} \begin{pmatrix} N_i & N_j \\ M_i & M_j \end{pmatrix} = 1\}.$$

For $\iota \in \mathcal{I}_1(\mathbf{x})$, there exist relatively prime positive integers n_i, n_j such that $n_i(N_i, M_i) = n_j(N_j, M_j)$. Now set

$$\mathcal{I}_1^+ = \{\iota \in \mathcal{I}_1(\mathbf{x})\text{:}n_i\mu_i = n_j\mu_j, n_i \neq n_j\}$$
$$\text{and} \quad \mathcal{I}_1^- = \mathcal{I}_1(\mathbf{x}) - \mathcal{I}_1^+ - \{\iota \in \mathcal{I}_1(\mathbf{x})\text{:}\mu_i = \mu_j\}.$$

Define $S_i = 1 - p^{-\mu_i} w_1^{N_i} w_2^{M_i}$, $i = 1, \ldots, R$. For each i define the multiplicity ν_i by the equation

$$v_I(w) = \prod_i S_i^{\nu_i}.$$

The proofs of the following three lemmas are all simple and left to the reader.

LEMMA 6.1.

(1) $\{i < j\} \in \mathcal{I}_1^-$ iff $\{S_i = 0\} \cap \{S_j = 0\} = \emptyset$.
(2) *If* $\{i < j\} \in \mathcal{I}_1^+$, *then* $\{S_i = 0\} = \{S_j = 0\}$.
(3) $\{i < j\} \in \mathcal{I}_0(\mathbf{x})$ *(see (5.2.2)) iff* $\{S_i = 0\} \cap \{S_j = 0\}$ *is a finite set of points.*

LEMMA 6.2. *Suppose* $\{i < j\} \in \mathcal{I}_1^-$. *Then there exist polynomials* $f_i(w), f_j(w)$, *such that* $f_i S_i + f_j S_j = 1$.

LEMMA 6.3. *Suppose* $\{i < j\} \in \mathcal{I}_1^+$. *Then there exists* $F \neq 0 \in \mathbb{Z}[v_1, v_2]$ *such that* $F(S_i, S_j) \equiv 0$.

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Remark. Indeed, using n_i, n_j from the definition of $\mathcal{I}_1^+$, one can use $F = (1-v_1)^{n_i} - (1-v_2)^{n_j}$. $\square$

The last preliminary needed is the following:

LEMMA 6.4. *Let* $\{i < j\} \in \mathcal{I}_0(\mathbf{x})$ *and* $k \notin \{i, j\}$. *Suppose that* $M_i/N_i \leqslant M_k/N_k \leqslant M_j/N_j$ *(at most one equality can then hold). Then, either:*

(1) *there exist polynomials* $f_i(w)$ *such that* $f_i S_i + f_j S_j + f_k S_k \equiv 1$, *or*
(2) *there exist polynomials* $f_i(w)$, $f_j(w)$, *and finitely many* $\alpha_\ell \in \mathbb{Q}$, $\ell = 2, \ldots, N$, *such that* $S_k = f_i S_i + f_j S_j + \sum_{\ell=2}^{N} \alpha_\ell S_k^\ell$.

Proof. There exist $n_i, n_j, n_k \in \mathbb{Z}_+$ such that gcd $(n_i, n_j, n_k) = 1$ and

$$n_i(N_i, M_i) + n_j(N_j, M_j) = n_k(N_k, M_k).$$

Thus, $(1-S_i)^{n_i} \cdot (1-S_j)^{n_j} = c(i, j, k)\,(1-S_k)^{n_k}$, where $c(i, j, k) = p^{n_k\mu_k - n_i\mu_i - n_j\mu_j}$. This implies $F(S_i, S_j, S_k) \equiv 0$ when $F = (1-v_1)^{n_i} \cdot (1-v_2)^{n_j} - c(i, j, k)\,(1-v_3)^{n_k}$. If $c(i, j, k) \neq 1$, then one obtains a low order term in the algebraic relation between the three S_a that is a nonzero constant. Dividing out by the constant one obtains the identity in (1). On the other hand, if $c(i, j, k) = 1$, then v_3 appears with a nonzero coefficient in F. Dividing out by this coefficient and rearranging terms, one obtains the identity in (2) with $N = n_k$. $\square$

An immediate consequence follows.

COROLLARY 6.5.

(1) *Suppose possibility* (1) *occurs in* (6.4). *Then there exist polynomials* g_i, g_j, g_k *such that*

$$\frac{1}{S_i S_j S_k} = \frac{g_i}{S_j S_k} + \frac{g_j}{S_i S_k} + \frac{g_k}{S_i S_j}.$$

(2) *Suppose possibility* (2) *occurs in* (6.4). *Then there exist polynomials* g_i, g_j *such that*

$$\frac{1}{S_i S_j S_k} = \frac{S_k}{S_i S_j S_k^2} = \frac{g_i}{S_j S_k^2} + \frac{g_j}{S_i S_k^2} + \sum_{\ell=2}^{n_k} \frac{\alpha_\ell S_k^\ell}{S_i S_j}.$$

Using Corollary 6.5, a simple induction argument shows the following.

COROLLARY 6.6. *For each summand in* (5.2.1),

$$\frac{u_I}{v_I} = \sum_{\substack{\{i,j\} \in \mathcal{I}_0(\mathbf{x}) \\ \{i,j\} \subset I}} \frac{h_{i,j}}{S_i^{k_i} S_j^{k_j}} + \sum_{\substack{i,\ell \\ i \in I}} \frac{h_{i,\ell}}{S_i^\ell}, \tag{6.6.1}$$

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where each numerator in (6.6.1) is a polynomial, and ℓ runs over a bounded set of positive integers.

Remark 6.7. To use (6.6) in order to estimate the Gaussian sums defined in the Introduction, it will be necessary to have some additional information about the exponents k_i, k_j. In particular, this will be needed to eliminate the contribution of the strict transform of either P_i, whose effect is to give an exponent of decrease equal to -1. One hopes to have exponents of decrease considerably smaller than -1, if favorable geometric conditions hold (see [I, pg. 69, pg. 155ff]). Now, the expression of an S_i, corresponding to the strict transform of P_1 or P_2, is $1 - p^{-1}w$, with $w = w_1$ or w_2. One then would need to insure that such a polynomial only appears to order at most 1 in any denominator of the fractions in (6.6.1). However, (6.6.1) indicates that the order of some S_i can actually be larger than their multiplicity ν_i in φ_I. So, it is important to find a simple condition that insures that the exponents of certain S_i never increase beyond ν_i. This is the point of the following discussion. □

Set $r = \#\{M_i/N_i : i \in [1, R]\}$, and denote the distinct ratios in this set by

$$\alpha(\mathbf{x}) = \rho_1 < \rho_2 < \cdots < \beta(\mathbf{x}) = \rho_r.$$

If some $N_i = 0$, then $\beta(\mathbf{x}) = +\infty$. For each k, set $\mathcal{J}(k) = \{i : M_i/N_i = \rho_k\}$.

For any $a \in \mathcal{J}(1), b \in \mathcal{J}(r)$, define the lattice (i.e. a set closed under addition and scalar multiplication by nonnegative integers)

$$\mathcal{C}(\mathbf{x}) = < (N_a, M_a), (N_b, M_b) >_{\mathbb{Z}_+}.$$

One says that a vector (N_i, M_i) is *extremal* resp. *interior* if it belongs to the boundary resp. interior of $\mathcal{C}(\mathbf{x})$. The following is an easy consequence of (6.5) and suffices for the purposes discussed in (6.7).

PROPOSITION 6.8. *The following two properties hold for $\ell \in \{1, r\}$.*

(A) *Suppose that $\mathcal{J}(\ell) = \{i\}$. Then the exponent of S_i in the denominator of any term appearing in* (6.6.1) *is at most* 1.

(B) *Suppose $\mathcal{J}(\ell) = \{i_1, \ldots, i_k\}, k \geqslant 2$, is such that any doubleton $\{c, d\} \subset \mathcal{J}(\ell)$ satisfies the property that either $S_c = S_d$ or $\{c, d\} \in \mathcal{I}_1^-$. Then for each $j \in \{1, \ldots, k\}$, the exponent of S_{i_j}, appearing in the denominator on the right side of (6.6.1), is at most the multiplicity ν_{i_j} of S_{i_j} in the factorization of v_I.*

Remark 6.9. Since the vector $(1, 0)$ (or $(0, 1)$) must be an extremal vector if it belongs to the set $\{(N_i, M_i)\}_{i=1}^R$, one will apply (6.8) in particular to any point $\mathbf{x}$ contained in the strict transform of some $P_i - P_i(x)$. The factor corresponding to the strict transform, $S = 1 - p^{-1}w_1$, has multiplicity 1. So, (6.8) implies that if no other vector $(N_i, M_i) \neq (1, 0) \in \mathcal{C}(\mathbf{x})$ lies in the same direction as $(1, 0)$, then S can only appear to order at most 1 in any summand of (6.6.1). On the other hand, in looking at the matrices computed in Section 3, one notes that the hypothesis in (A) need not always hold. However, it is then simple to verify that the hypothesis in (B) must be satisfied. This leads to rates of decay for the $|G(a_1/p^r, a_2/p^r)|$ that are considerably better than $O_\varepsilon(p^{r(-1+\varepsilon)})$, see [Li-1]. □

Given $\boldsymbol{\tau} = (\tau_1, \tau_2)$, set $\operatorname{ord} \boldsymbol{\tau} = (\operatorname{ord} \tau_1, \operatorname{ord} \tau_2)$, $ac\boldsymbol{\tau} = (ac\tau_1, ac\tau_2)$.

Each nonzero summand of (6.6.1) can contribute to the asymptotic of $\mathcal{F}_{\mathbf{x}}$ (5.4.1) when $\operatorname{ord} \boldsymbol{\tau}$ is confined to a certain affine translate of a lattice, in the sense of the above definition. Different summands will determine, in general, different translated lattices. The contribution of exactly one summand is simple to make explicit since at most two terms appear in each denominator. To state the following Proposition, one first introduces for each $i \in \{1, \ldots, R\}$, the line

$$\mathcal{L}_i(\mathbf{x}) = \{\mathbf{s} \in \mathbb{C}^2 : N_i s_1 + M_i s_2 = -\mu_i\}.$$

This is a component of the polar divisor of $\mathbf{Z}_{\mathbf{x}}(\boldsymbol{\chi}, \mathbf{s}, \varphi)$. Further, for each $\iota = \{i, j\} \in \mathcal{I}_0(\mathbf{x})$ set

$$\{v_\iota(\mathbf{x}) =_{\text{def}} (v_{1,\iota}(\mathbf{x}), v_{2,\iota}(\mathbf{x}))\} = \mathcal{L}_i(\mathbf{x}) \cap \mathcal{L}_j(\mathbf{x}),$$

$$s_\ell(\mathcal{L}_i(\mathbf{x})) = s_\ell \text{ axis intercept of } \mathcal{L}_i(\mathbf{x}), \ell = 1, 2$$

$$\mathcal{C}_\iota(\mathbf{x}) = \langle (N_i, M_i), (N_j, M_j) \rangle_{\mathbb{Z}_+}.$$

PROPOSITION 6.10. *For any $\iota = \{i, j\} \in \mathcal{I}_0(\mathbf{x})$, positive integers k_i, k_j, and monomial $w_1^{m_1} w_2^{m_2}$, there exists a polynomial $H_\iota(u_1, u_2)$ of degree $k_i - 1$ in u_1 and $k_j - 1$ in u_2 such that for any* $\operatorname{ord} \boldsymbol{\tau} \in (m_1, m_2) + \mathcal{C}_\iota(\mathbf{x})$,

$$\begin{aligned}\operatorname{Res}_{w_1=0} \operatorname{Res}_{w_2=0} &\left(\frac{w_1^{m_1} w_2^{m_2}}{S_i^{k_i} S_j^{k_j}} w_1^{-\operatorname{ord}\tau_1 - 1} w_2^{-\operatorname{ord}\tau_2 - 1} \right) \\ &= H_\iota(\log|\tau_1|, \log|\tau_2|) |\tau_1|^{-v_{1,\iota}(\mathbf{x})} |\tau_2|^{-v_{2,\iota}(\mathbf{x})}.\end{aligned}$$

Further, the iterated residue equals 0 if $\operatorname{ord} \boldsymbol{\tau} \notin (m_1, m_2) + \mathcal{C}_\iota(\mathbf{x})$.

Remark. Exactly the same conclusion holds if the denominator equals S_i^k. In this case, one needs to restrict $\operatorname{ord} \boldsymbol{\tau}$ to $(m_1, m_2) + \langle (N_i, M_i) \rangle_{\mathbb{N}_+}$. It is clear that the exponent in $|\tau_1|, |\tau_2|$ then equals $|\tau_1|^{-s_1(\mathcal{L}_i(\mathbf{x}))} = |\tau_2|^{-s_2(\mathcal{L}_i(\mathbf{x}))}$, provided the axis intercepts exist. □

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Proof. Write

$$\frac{w_1^{m_1} w_2^{m_2}}{S_i^{k_i} S_j^{k_j}} = \frac{1}{(k_i-1)!(k_j-1)!} \times$$
$$\times \sum_{e_1, e_2 \geqslant 0} \sum_{\substack{k,\ell \\ kN_i+\ell N_j=e_1 \\ kM_i+\ell M_j=e_2}} (\mathcal{H}(k,\ell) \cdot p^{-\mu_i k-\mu_j \ell}) w_1^{e_1+m_1} w_2^{e_2+m_2},$$

where $\mathcal{H}(k,\ell) = \prod_{q_1=1}^{k_i-1} \prod_{q_2=1}^{k_j-1} (q_1+k)(q_2+\ell)$. Then, the coefficient of $w_1^{\mathrm{ord}\,\tau_1} w_2^{\mathrm{ord}\,\tau_2}$ can only be nonzero if $(m_1, m_2)+\mathrm{ord}\,\boldsymbol{\tau} \in \mathcal{C}_\iota(\mathbf{x})$. If this occurs, then it is determined by (k, ℓ) so that

$$kN_i + \ell N_j = \mathrm{ord}\,\tau_1 - m_1, \qquad kM_i + \ell M_j = \mathrm{ord}\,\tau_2 - m_2,$$

which, by assumption, consists of exactly one pair (k', ℓ'). One then expresses (k', ℓ') in terms of $v_\iota(\mathbf{x})$. A straightforward calculation then shows

$$\begin{aligned} p^{-\mu_i k' - \mu_j \ell'} &= p^{v_{1,\iota}(\mathbf{x})(\mathrm{ord}\,\tau_1 - m_1) + v_{2,\iota}(\mathbf{x})(\mathrm{ord}\,\tau_2 - m_2)} \\ &= C|\tau_1|^{-v_{1,\iota}(\mathbf{x})} |\tau_2|^{-v_{2,\iota}(\mathbf{x})}, \end{aligned}$$

where $C = C(m_1, m_2, i, j)$ is a constant independent of ord $\boldsymbol{\tau}$.

Moreover, since $(k', \ell') = (\alpha_{11}\,\mathrm{ord}\,\tau_1 + \alpha_{12}\,\mathrm{ord}\,\tau_2, \alpha_{21}\,\mathrm{ord}\,\tau_1 + \alpha_{22}\,\mathrm{ord}\,\tau_2)$, for certain constants $\alpha_{a,b}$, it is clear that $\mathcal{H}(k,\ell)$ is a polynomial H_ι of degree $k_i - 1$ in ord τ_1, and of degree $k_j - 1$ in ord τ_2. Since ord $\tau = -\log|\tau|$, this completes the proof. □

Using the notations introduced above, one summarizes the preceding discussion as follows. In the statement below, the notation $\mathcal{C}_\iota^*(\mathbf{x})$ will be used to denote either the lattice $\mathcal{C}_\iota(\mathbf{x})$ or any of the two sublattices $\langle N_k, M_k \rangle_{\mathbb{Z}_+}$, $k = i, j$. The following also suffices for purposes of [Li-1].

THEOREM 6.11. *There exist*

(i) *finitely many monomials* $\{\mathbf{M}_k\}$,
(ii) *finitely many polyomials* $\{H_\iota(u_1, u_2)\}_{\iota \in \mathcal{I}_0(\mathbf{x})}$, *and finitely many locally constant functions* $\{A_{\boldsymbol{\chi}}\}$ *on* $\mathcal{U}_p^2$,

such that for all sufficiently small and nonzero τ_1, τ_2,

$$|\tau_1||\tau_2|\mathcal{F}_{\mathbf{x}}(\varphi, \boldsymbol{\tau}) = \sum_{\iota \in \mathcal{I}_0(\mathbf{x})} \left(\sum_{k, \boldsymbol{\chi}} \xi_{\mathbf{M}_k + \mathcal{C}_\iota^*(\mathbf{x})}(\mathrm{ord}\,\boldsymbol{\tau}) A_{\boldsymbol{\chi}}(ac\boldsymbol{\tau}) \right) \times$$
$$\times H_\iota(\log|\tau_1|, \log|\tau_2|) |\tau_1|^{-v_{1,\iota}(\mathbf{x})} |\tau_2|^{-v_{2,\iota}(\mathbf{x})},$$

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where $\xi_{\mathbf{M}_k+\mathcal{C}_\iota^*(\mathbf{x})}$ *is the characteristic function of* $\mathbf{M}_k + \mathcal{C}_{i,j}^*(\mathbf{x})$.

The polygons, referred to in the Introduction, are defined as follows.

DEFINITION 6.12. Given $\mathbf{x}$, at which Equations (5.1) hold, and $\iota = \{i, j\} \in \mathcal{I}_0(\mathbf{x})$, denote by $\Gamma_\iota(\mathbf{x})$ the polygon whose sides are contained in the lines $\mathcal{L}_i(\mathbf{x}) \cap \mathbb{R}^2$ and $\mathcal{L}_j(\mathbf{x}) \cap \mathbb{R}^2$. For $\ell = 1, 2$, $s_{\ell,\iota}(\mathbf{x})$ denotes the s_ℓ axis intercept of $\Gamma_\iota(\mathbf{x})$. □

It is then clear from (4.3), and (6.11) that finitely many polygons suffice to describe two main features of the singularities of the local singular series of any $\mathbf{P} \in \mathcal{C}\ell_{\mathrm{I}} \cup \mathcal{C}\ell_{\mathrm{II}}$. Each polygon evidently encodes both the 'dominant monomial' $|\tau_1|^{-1-v_{1,\iota}(\mathbf{x})}|\tau_2|^{-1-v_{1,\iota}(\mathbf{x})}$ for $\mathcal{F}_{\mathbf{x}}$, and the slopes of the boundary of the cone $\mathcal{C}_\iota(\mathbf{x})$. The significance of the latter is that for ord τ_1, ord τ_2 both large, the fiber integral $\mathcal{F}_{\mathbf{x}}$ is defined at ord $\boldsymbol{\tau}$ *if* ord $\tau_2/$ord τ_1 is at least the smaller slope and at most the larger slope of this cone. It follows that the $\Gamma_\iota(\mathbf{x})$ determine both the dominant monomial and the region in which a given monomial is dominant for $\mathcal{F}_{\mathbf{x}}$.

Remark. When $\mathbf{x}$ is a bad point for $\mathbf{P}\circ\theta - \mathbf{P}(x)$, Theorem 6.11 gives an explicit development for the right side of (5.11.1). Thus, the local contribution of the fiber integral for $\mathbf{P}$ at $\mathbf{t} + \mathbf{P}(x)$ is a sum of monomials in $|\tau_1|, |\tau_2|$ and $\log|\tau_1| \log|\tau_2|$, evaluated at the finitely many distinct points in the fiber $\{\zeta_{\mathbf{x}} = \mathbf{t}\}$. As a result, cancellation can occur and a precise 'dominant term' is not obviously identifiable from the sum over the points in the fiber. The question of obtaining a more explicit description in $\mathbf{t}$ for the local contribution $\mathbf{F}_{\mathbf{x}}$ is a problem that can be analyzed using some other ideas from plane curve singularities, see [Li-2]. □

Remark 6.13. It is useful to describe the analogues of (5.2), (5.3), (6.11) over $\mathbb{R}, \mathbb{C}$. A sketch will suffice over $\mathbb{R}$ since the details for both fields involve iterations of standard one variable arguments. It is clear that the notions of permissible transformation, amelioration, and good $\mathbf{P}$ wedge apply equally well to polynomial or analytic maps over $\mathbb{R}, \mathbb{C}$. The notation convention, introduced in the beginning of Section 4, is also used here.

Let $\mathbf{P}$: $\mathbb{R}^n \to \mathbb{R}^2$ satisfy the properties for membership in $\mathcal{C}\ell_{\mathrm{I}}$ (if $n = 2$) or $\mathcal{C}\ell_{\mathrm{II}}$. Let $\theta U(\mathbf{x})$ be a good $\mathbf{P}$ wedge, where Equations (5.1) may be assumed to hold on $U(\mathbf{x}) = \times_{i=1}^n \{|z_i| \leqslant 1\}$. The local zeta function $\mathbf{Z}_{\mathbf{x}}$ can be defined for any pair $\boldsymbol{\chi}$ of characters on $\{\pm 1\}$, and any smooth, compactly supported function φ on $U(\mathbf{x})$. Write $\mathbf{s} = \boldsymbol{\sigma} + i\mathbf{w}$, $\boldsymbol{\sigma}, \mathbf{w} \in \mathbb{R}^2$.

For $\sigma_1, \sigma_2 \gg 1$, $\mathbf{Z}_{\mathbf{x}}$ is analytic and absolutely convergent. A standard argument, using the iteration of the regularization procedure in one variable of [G–S], shows the existence of an analytic continuation of $\mathbf{Z}_{\mathbf{x}}$ to $\mathbb{C}^2$ as a meromorphic function. For each integer $e \geqslant 0$, set

$$\mathcal{L}_i(e) = \{\mathbf{s} \in \mathbb{C}^2 : L_i(e, \mathbf{s}) =_{\mathrm{def}} N_i s_1 + M_i s_2 + \mu_i + e = 0\},$$

$$\mathcal{L}_i^+(e) = \{\boldsymbol{\sigma} \in \mathbb{R}^2 \colon L_i(e, \boldsymbol{\sigma}) \geqslant 0\}, \qquad \Gamma_{\mathbf{x}}(e) = \partial(\cap_{i=1}^R \mathcal{L}_i^+(e)).$$

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Each $\Gamma_{\mathbf{x}}(e)$ is a polygon that separates $\mathbb{R}^2$ into the parts above, below, or on it.

The analogue of (5.2) is:

$$\mathrm{Pol}_{\mathbf{x}} =_{\mathrm{def}} \text{ polar divisor of } \mathbf{Z}_{\mathbf{x}} \subset \bigcup_{i=1}^{R} \bigcup_{e=0}^{\infty} \mathcal{L}_i(e). \tag{6.13.1}$$

The analogue of (5.3) is a strong decay condition in $i\mathbb{R}^2$ as $|w_1|, |w_2| \to \infty$. In the following discussion, K denotes a compact subset of $\mathbb{R}^2$ such that $K \cap \mathrm{Pol}_{\mathbf{x}} = \emptyset$.

CLAIM 6.13.2. *For any monomial $M = M(\mathbf{s})$ there exists $C = C(M)$ such that for any K,*

$$|M(\mathbf{s})\mathbf{Z}_{\mathbf{x}}(\mathbf{s}, \boldsymbol{\chi}, \varphi)| < C \quad \text{for any } \mathbf{s} \text{ such that } \boldsymbol{\sigma} \in K.$$

Sketch of Proof. By reindexing, one may assume that for each $i \geqslant 3$, $(N_i, M_i) \in <(N_1, M_1), (N_2, M_2) >_{\mathbb{R}_+}$ and $M_1/N_1 < M_2/N_2 \leqslant +\infty$. It is convenient to set $S_i = N_i s_1 + M_i s_2$, for $i = 1, 2$. Combining a repeated use of integration by parts with the regularization procedure in each coordinate, the asserted estimate is straightforward to verify in the S_1, S_2 variables, and so, in the s_1, s_2 variables. In particular, this uses the fact that φ and all its derivatives vanish on $\partial U(\mathbf{x})$. The estimate one shows is that for any $k_1, k_2 \in \mathbb{N}$, there exists $C = C(k_1, k_2)$, such that for any K, one has:

$$\left| \prod_{i=0}^{k_1} \prod_{j=0}^{k_2} (S_1 + \mu_1 + i) \cdot (S_2 + \mu_2 + j) \mathbf{Z}_{\mathbf{x}}(S_1, S_2, \boldsymbol{\chi}, \varphi) \right| < C \quad \text{for any } \mathrm{Re}\, S_1, \mathrm{Re}\, S_2 \in K. \qquad \square$$

The analogue of (6.11) is obtained by first applying a partial fraction decomposition to $\mathbf{Z}_{\mathbf{x}}$ inside any tube $K + i\mathbb{R}^2$. Using (6.13.1), for any such K, there exists a smallest vector $\mathbf{e} = (e, \ldots, e) \in \mathbb{N}^R$, such that

$$\mathcal{N}_{\mathbf{x},e}(\mathbf{s}, \boldsymbol{\chi}, \varphi)) =_{\mathrm{def}} \prod_{i=1}^{R} \prod_{j=0}^{e} L_i(j, \mathbf{s}) \cdot \mathbf{Z}_{\mathbf{x}}(\mathbf{s}, \boldsymbol{\chi}, \varphi)$$

is analytic in $K + i\mathbb{R}^2$. By (6.13.2), it follows that $\mathcal{N}_{\mathbf{x},e}$ is also bounded in the tube over K.

Noting that in this discussion there is no need to introduce the index set I that appears in (5.2.1), one adapts the expression (6.6.1) to the following fraction, using the same partial fraction method from (6.1)ff:

$$\frac{1}{\prod_{i=1}^{R}\prod_{j=0}^{e} L_i(j,\mathbf{s})} = \sum_{\substack{\{i,j\}\in\mathcal{I}_0(\mathbf{x})\\ i,j\in[1,R]}} \sum_{m_1=0}^{e}\sum_{m_2=0}^{e} \frac{h_{i,j,m_1,m_2}}{L_i(m_1,\mathbf{s})^{k_i} L_j(m_2,\mathbf{s})^{k_j}} + \sum_{\substack{i,\ell\\ 1\leqslant i\leqslant R}} \sum_{j=0}^{e} \frac{h_{i,j,\ell}(\mathbf{s})}{L_i(j,\mathbf{s})^{\ell}}.$$

Here, the k_i, ℓ are certain positive integers, finite in number, that need not be specified further. Also, the numerator of each fraction is a polynomial. Thus, in $K + i\mathbb{R}^2$,

$$\mathbf{Z_x} = \sum_{\substack{\{i,j\}\in\mathcal{I}_0(\mathbf{x})\\ 1\leqslant i,j\leqslant R}} \sum_{m_1=0}^{e}\sum_{m_2=0}^{e} \frac{F_{i,j,m_1,m_2}(\mathbf{s})}{L_i(m_1,\mathbf{s})^{k_i} L_j(m_2,\mathbf{s})^{k_j}} + \sum_{\substack{i,\ell\\ 1\leqslant i\leqslant R}} \sum_{j=0}^{e} \frac{G_{i,j,\ell}(\mathbf{s})}{L_i(j,\mathbf{s})^{\ell}}, \tag{6.13.3}$$

where each numerator is analytic and satisfies the decay condition (6.13.2) in $K + i\mathbb{R}^2$.

The definition (5.5.1) for the fiber integral $\mathcal{F}_{\mathbf{x}}(\varphi, \boldsymbol{\tau})$ extends to the real case without difficulty. Iterating Mellin inversion on $\mathbb{R}^2$ (see [I, pg. 21] for the one variable case) now shows that for $c \gg 1$,

$$\begin{aligned}\tau_1\tau_2\mathcal{F}_{\mathbf{x}}(\varphi,\boldsymbol{\tau}) &= \frac{1}{(2\pi i)^2}\sum_{\boldsymbol{\chi}} \chi_1(ac\tau_1)\chi_2(ac\tau_2)\times \\ &\quad \times \int_{\sigma_1=c}\int_{\sigma_2=c} \mathbf{Z_x}(\mathbf{s},\boldsymbol{\chi},\varphi)|\tau_1|^{-s_1}|\tau_2|^{-s_2}ds_1ds_2. \end{aligned}\tag{6.13.4}$$

One then applies the discussion in [Li-7]. Starting at (c, c) and choosing $e \geqslant 0$, one forms a rectangle $\mathcal{R}(e)$ with upper right corner at (c, c) such that the three other corners lie below $\Gamma_{\mathbf{x}}(e)$ and above $\Gamma_{\mathbf{x}}(e+1)$. So, each corner is disjoint from $\mathrm{Pol}_{\mathbf{x}}$. By making small modifications along its boundary, one can also assume that $\partial\mathcal{R}(e)$ contains no point of intersection of any two components of $\mathrm{Pol}_{\mathbf{x}}$. In the interior of $\mathcal{R}(e)$, one fixes a simple path $\xi{:}u \to \xi(u) = \boldsymbol{\sigma}$, intersecting $\mathrm{Pol}_{\mathbf{x}}$ transversally at simple points only, along which σ_1, σ_2 are both monotonically decreasing, and ending at $(c', c') =$ lower left corner of $\mathcal{R}(e)$. Deleting from $\mathcal{R}(e)$ the union of small open discs centered at each of the finitely many points in $\mathrm{Pol}_{\mathbf{x}} \cap \mathrm{Im}\,\xi$, one obtains a compact set K such that $\mathcal{N}_{\mathbf{x},e} \cdot Z_{\mathbf{x}}$ is bounded and analytic in $K + i\mathbb{R}^2$. Replacing $\mathbf{Z_x}$ in (6.13.4) by the expression (6.13.3), one then transports the chain of integration from (c, c) to (c', c') along ξ, and uses the Leray residue formula as in [ibid] to rewrite the right side of (6.13.4) as a sum of iterated residues plus an error term.

Each iterated residue has the following form:

$$H(\log|\tau_1|, \log|\tau_2|)|\tau_1|^{-v_1}|\tau_2|^{-v_2}, \quad \text{for some } \; H \in \mathbb{R}[u_1, u_2],$$

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where (v_1, v_2) is a point of intersection of a unique pair of (transversal) components of $\mathrm{Pol}_{\mathbf{x}}$ that lies in the interior of $\mathcal{R}(e)$.

(6.13.2) now allows e to grow without bound. When this occurs, two properties can be shown to hold. The first is that the error term goes to 0. The second is that the coefficient of the iterated residue at the point (v_1, v_2) is multiplied by a characteristic function for a 'wedge' in $[0, \infty)^2$ that can be described as follows. There are exactly two components of the wedge's boundary that contain (0,0). Each component is a monomial curve. Moreover, each curve is determined by the direction vector of one of the two lines in $\mathrm{Pol}_{\mathbf{x}}$ that intersect at (v_1, v_2).

Dividing (6.13.4) by $|\tau_1 \cdot \tau_2|$, the resulting infinite sum, obtained by letting $e \to +\infty$, is the analogue to (6.11) over $\mathbb{R}$. An analogue over $\mathbb{C}$ is obtained by iterating the discussion in [I, pgs. 24–32]. For additional discussion (and all details), see [Li-3, 4].

Part ii. *The case when P_1 is linear*

For fixed $\boldsymbol{\beta} \in \mathbb{Z}_p^n$, set $P_1(x) = \boldsymbol{\beta} \cdot x$ and P_2 a homogeneous polynomial of degree $d_2 \geqslant 2$, so that $\mathbf{P} = (P_1, P_2) \in \mathcal{C}\ell_{\mathrm{II}}$. This part calculates the matrices $\mathbf{A}^{\#}(\mathbf{p})$ resp. $\mathbf{A}(\mathbf{p})$ when $\mathbf{p}$ is a point satisfying the properties in (3.2) resp. (3.4). Notations introduced in Sections 2 and 3 are used here. The first ingredient is the following.

LEMMA 6.14. *Let $p \in \pi^{-1}(\bar{\mathbf{0}})$ at which $\hat{P}_1 \cdot \hat{P}_2(p) \neq 0$. Then $\hat{P}_2 - \hat{P}_2(p)$ has at most a nondegenerate singularity at p.*

Proof. A simple argument uses a preliminary linear coordinate change in $\mathbb{Q}_p^n$ prior to any blowing up. By reindexing one may assume $\beta_1 \neq 0$. Then define the coordinates

$$y_1 = \boldsymbol{\beta} \cdot x,\ y_i = x_i, \quad \text{for any } i \geqslant 2. \tag{6.14.1}$$

In the (y) coordinates, it is clear that $\mathrm{Sing}_{\mathbf{P}}$ is defined by the $n-1$ equations $\partial P_2/\partial y_i = 0$, $i \geqslant 2$. Now apply the blowing up π of $\bar{\mathbf{0}}$. Clearly in the chart $Z_1(1)$, $P_1 \circ \pi(x_{11}, \ldots, x_{n1}) = x_{11}$, and $P_2 \circ \pi(x_{11}, \ldots, x_{n1}) = x_{11}^{d_2} \hat{P}_2$. Since $\hat{P}_1 \equiv 1$, it suffices to show that $\hat{P}_2 - \hat{P}_2(p)$ has at most a nondegenerate critical point at any $p \in \mathcal{D}_1$ for which $\hat{P}_2(p) \neq 0$.

To show this, one first notes that for each $i \geqslant 2$, $\partial \hat{P}_2/\partial x_{i1}(p) = \partial P_2/\partial y_i(p^{(1)})$. Thus, if p is a singular point of $\hat{P}_2$, then $p^{(1)} \in \mathrm{Sing}_{\mathbf{P}}$. Given that $p^{(1)} \in \mathrm{Sing}_{\mathbf{P}}$, the homogeneity of each $\partial P_2/\partial y_i$ and the fact that $y_1(p^{(1)}) = 1$, one concludes by Euler's relation that

$$\frac{\partial^2 P_2}{\partial y_1 \partial y_i}(p^{(1)}) = -\sum_{j \geqslant 2} y_j(p^{(1)}) \frac{\partial^2 P_2}{\partial y_j \partial y_i}(p^{(1)}).$$

So, the first column of the matrix $(\partial^2 P_2/\partial y_i \partial y_j)_{i \geqslant 1, j \geqslant 2}$ is a linear combination of the remaining $n-1$ columns. Since $\mathbf{P} \in \mathcal{C}\ell_{\mathrm{II}}$, it follows that this matrix has rank

$n-1$. This implies the Hessian of $\hat{P}_2$ (with respect to the coordinates $x_{21}, \ldots, x_{n1}$) has rank $n-1$ at p.

In any chart $Z_k(1), k \neq 1$, $P_1 \circ \pi(x_{1k}, \ldots, x_{nk}) = x_{1k}x_{kk}$. Thus, if $p \in \mathcal{D}_k$ satisfies the hypothesis of the Lemma, it follows that $x_{1k}(p) \neq 0$. Then, via the identification $x_{k1} = x_{1k}^{-1}$, and $x_{i1} = x_{ik}/x_{1k}$, $i \neq 1, k$, one can identify p with a point in $\mathcal{D}_1$, and the preceding discussion applies, completing the proof of the Lemma. □

One now evaluates the $m_i, \mu_i - 1$ in (3.2).

COROLLARY 6.15. *Assume $p = (0, p') \in \mathcal{D}_1$ is a singular point of $\hat{P}_2$. Set $\pi' = \mathrm{id} \times \pi_1 : Y \to U(p)$ to be the blowing up of p' in $Z_1(1) \cap \mathcal{D}_1$. Then for any* $\mathbf{p}$ *in the exceptional divisor of π',*

$$\mathbf{A}^{\#}(\mathbf{p}) = \begin{pmatrix} 1 & 0 & 0 \\ d_2 & 2 & \varepsilon \\ n-1 & n-2 & 0 \end{pmatrix}.$$

Thus, $(m_2, \ldots, m_n) = (2, \varepsilon, 0, \ldots, 0)$ *and* $(\mu_2 - 1, \ldots, \mu_n - 1) = (n-2, 0, \ldots, 0)$.

Next, consider a point $p \in \{\hat{P}_1 = \hat{P}_2 = 0\} \cap \mathcal{D}$ that is a singular point of the map $(\hat{P}_1, \hat{P}_2)$ in $n-1$ variables (that is, local form (2.5)(iii)). Using the coordinates from (6.14.1), it is clear that $p \in Z_k(1)$ for some $k \geqslant 2$. Indeed, the hypothesis implies

$$x_{1k}(p) = \partial \hat{P}_2/\partial x_{ik}(p) = 0 \quad \text{for each } i \neq 1, k. \tag{6.16}$$

LEMMA 6.17.

(i) *If $p \in \mathcal{D}_k$ satisfies (6.16), then $p^{(k)} \in \mathrm{Sing}_{\mathbf{P}}$.*
(ii) *If* (i) *holds, then there exist local coordinates $z = (z_1, \ldots, z_n)$, defined in a neighborhood $U(p) = \times_i \{|z_i| \leqslant \varepsilon\} \subset Z_k(1)$, satisfying $U(p) \cap \mathcal{D}_k = \{z_1 = 0\}$, such that*

$$P_1 \circ \pi(z) = z_1 z_2 \cdot (\mathit{unit}),$$

$$P_2 \circ \pi(z) = z_1^{d_2}[z_2 + Q(z_3, \ldots, z_n)], \tag{6.17.1}$$

where $Q = \sum_{i \geqslant 3} c_i z_i^2$, and $c_i \neq 0$ for each $i \geqslant 3$.

Proof of (i). Combining the hypothesis with Euler's relation applied to P_2, and the fact that $P_2(p^{(k)}) = 0$, one sees immediately that for each $i \neq 1, k$, $\partial P_2/\partial y_i(p^{(k)}) = \partial P_2/\partial\, y_k(p^{(k)}) = 0$. Thus, $p^{(k)} \in \mathrm{Sing}_{\mathbf{P}}$.

Proof of (ii). By (i), the rank of the matrix $(\partial^2 P_2/\partial y_i \partial y_j(p^{(k)}))_{i \geqslant 1, j \geqslant 2}$ equals $n-1$. By (6.16), it follows that the kth column is a linear combination of the $n-1$

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other columns. Thus, the rank of the submatrix, formed by deleting the kth column *and* the first column, must equal $n-2$. This however is equivalent to saying that

$$\operatorname{rank}\left(\frac{\partial^2 \hat{P}_2}{\partial x_{ik}\partial x_{jk}}(p)\right)_{\substack{i,j\geqslant 2\\ i,j\neq k}} = n-2.$$

Applying now Lemmas 2.2, 2.3 to $\hat{P}_2(x_{1k},\dots,x_{k-1,k},x_{k+1,k},\dots,x_{nk})$, (ii) implies the existence of local coordinates $(w_1,\dots,w_n)$ (centered at p) such that

$$\begin{aligned} P_1\circ\pi &= w_1w_2,\\ P_2\circ\pi &= w_1^{d_2}[\phi(w_2)+Q(w_3,\dots,w_n)+\sum_{\ell\geqslant 2} w_2^{\ell}H_\ell(w_3,\dots,w_n)], \end{aligned}$$

where $\deg\phi = 1$, $Q=\sum_{i\geqslant 3}c_iw_i^2$, and each H_ℓ vanishes at p. It follows that local coordinates exist (6.17.1) holds, finishing the proof of (ii). □

Define p' to be the point in $\mathcal{D}_k$ whose coordinates equal those of p except for $x_{kk}(p)=0$. Let $\eta_0'{:}Z'(2)\to U(p)\cap\mathcal{D}_k$ denote the blowing up of p' in the ambient space $\mathcal{D}_k$, using the coordinates in (6.17.1). Set $Z(2) =_{\mathrm{def}} \{|z_1|\leqslant\varepsilon\}\times Z'(2) = \cup_{i=2}^n Z_i(2)$, and $\eta_0=\mathrm{id}\times\eta_0'$. From (6.17), the following is now a simple exercise, left to the reader.

COROLLARY 6.18. *The matrices* $\mathbf{A}(\mathbf{p})$ *in the proof of* (3.4) *are obtained as follows* (*see* (3.3) *for the notation convention*).

(i) *Every point* $\mathbf{p}\in Z_i(2)$ *is a good point for* $\mathbf{P}\circ\pi\circ\eta_0$ *except for the origin* $\bar{\mathbf{0}}_i$ *in each chart* $Z_i(2)$, $i\geqslant 3$. *Further,*

$$\mathbf{A}_1(\mathbf{p}) = \begin{pmatrix} 1 & 1\\ d_2 & 1\\ n-1 & n-2\end{pmatrix}.$$

For each $i\geqslant 3$, *set* $Z_i'(2)=Z_i(2)\cap\{z_1=0\}$, *and* $U'(\bar{\mathbf{0}}_i)=U(\bar{\mathbf{0}}_i)\cap\{z_1=0\}$, *where* $U(\bar{\mathbf{0}}_i)$ *denotes a compact open neighborhood of* $\bar{\mathbf{0}}_i$ *in* $Z_i(2)$.

(ii) *There exists a smooth subvariety* Y_i *of codimension 2 in* $U'(\bar{\mathbf{0}}_i)$, *so that the following holds. Let* $\eta'(i){:}Z_i'(3)\to U'(\bar{\mathbf{0}}_i)$ *denote the blowing up of* $U'(\bar{\mathbf{0}}_i)$ *along* Y_i, $Z_i(3)=\{|z_1|\leqslant\varepsilon\}\times Z_i'(3)$, *and* $\eta_1(i)=\mathrm{id}\times\eta'(i)$. *Then each point* $\mathbf{p}$ *of the exceptional divisor of* $\eta_1(i)$ *is a good point for* $\mathbf{P}\circ\pi\circ\eta_0\circ\eta_1(i)$. *Further, for each such* $\mathbf{p}$,

$$\mathbf{A}_2(\mathbf{p}) = \begin{pmatrix} 1 & 2 & r & 0 \\ d_2 & 2 & r & \varepsilon \\ n-1 & n-1 & r(n-2) & 0 \end{pmatrix}$$

$$\textit{or} \quad \begin{pmatrix} 1 & 2 & r & 0 \\ d_2 & 2 & 0 & \varepsilon \\ n-1 & n-1 & 0 & 0 \end{pmatrix},$$

where $r \in \{0, 1\}$.

Using (3.10) for the singular points other than $\bar{\mathbf{0}}$, and the preceding results for any point on $\pi^{-1}(\bar{\mathbf{0}})$, one arrives at the following estimate of the position of any polygon $\Gamma_\iota(\mathbf{x})$ (see (6.12)), as a function of n, d_2, provided P_1 is a linear function.

THEOREM 6.19.

(i) *For any good* $\mathbf{P}$ *wedge of the form* $\theta U(\mathbf{x})$, *and any* $\iota \in \mathcal{I}_0(\mathbf{x})$, *either* $s_{\ell,\iota}(\mathbf{x}) = -1$ *or* $s_{\ell,\iota}(\mathbf{x}) \leqslant -n/d_2$ *for* $\ell = 1, 2$.

(ii) *The multiplicity of* $\{s_\ell + 1 = 0\}$, *as a component of the polar divisor of* $\mathbf{Z}_{\mathbf{x}}(\boldsymbol{\chi}, \mathbf{s}, \varphi)$, *is at most equal to 1, for any* $\boldsymbol{\chi}, \varphi$.

Thus, the only possible obstruction to any axis intercept of a polygon $\Gamma_\iota(\mathbf{x})$ being at most $-n/d_2$ is that at least one of its sides lies on a line $s_\ell = -1$. □

7. An Extension of Theorem 4.3 to Some Pairs not in $\mathcal{C}\ell_{\mathrm{I}} \cup \mathcal{C}\ell_{\mathrm{II}}$

This section is needed to estimate $|\sum_{\boldsymbol{\beta} \in (\mathbb{Z}/p^r)^n} S_u(p^r, \boldsymbol{\beta})|$ for large r (see [Li-1]). Using $\boldsymbol{\beta}, x$ to denote distinct variables on $\mathbb{Q}_p^n$, define the class

$$\mathcal{C}\ell^* = \{\mathbf{P}: \mathbb{Q}_p^{2n} \to \mathbb{Q}_p^2 : P_1(\boldsymbol{\beta}, x) = \boldsymbol{\beta} \cdot x,$$
$$P_2 = P_2(x) \text{ is homogeneous of degree } d \geqslant 2 \text{ and nonsingular outside } \bar{\mathbf{0}}\}.$$

An elementary verification, left to the reader shows

LEMMA 7.1. *If* $\mathbf{P} \in \mathcal{C}\ell^*$, *then* $\mathrm{Sing}_{\mathbf{P}} = \{(\boldsymbol{\beta}, 0) : \boldsymbol{\beta} \in \mathbb{Q}_p^n\}$.

Next, set $\pi: \mathcal{Y} \to \mathbb{Q}_p^{2n}$ the blowing up of $\{(\boldsymbol{\beta}, 0)\}$. Thus, $\mathcal{Y} = \cup_{j=1}^n \mathcal{Y}_j$ and $\pi|_{\mathcal{Y}_j}(\boldsymbol{\beta}, w) = (\boldsymbol{\beta}, w_1 w_j, \ldots, w_{j-1} w_j, w_j, w_{j+1} w_j, \ldots, w_n w_j)$. Moreover,

$$P_1 \circ \pi|_{\mathcal{Y}_j} = w_j[\boldsymbol{\beta}_j + \sum_{i \neq j} \boldsymbol{\beta}_i w_i] =_{\mathrm{def}} w_j \hat{P}_1, \qquad P_2 \circ \pi|_{\mathcal{Y}_j} = w_j^d \hat{P}_2.$$

One now observes the following.

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LEMMA 7.2. *If* $\mathbf{p} = (\boldsymbol{\beta}, \mathbf{w})$ *satisfies* $\hat{P}_1(\mathbf{p}) = 0$, $\hat{P}_2(\mathbf{p}) \neq 0$ resp. $\hat{P}_1(\mathbf{p}) \neq 0$, $\hat{P}_2(\mathbf{p}) = 0$ resp. $\hat{P}_1(\mathbf{p}) = \hat{P}_2(\mathbf{p}) = 0$, *then*

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} 1 & 1 \\ d & 0 \\ n-1 & 0 \end{pmatrix} \text{ resp.}$$

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} 1 & 0 \\ d & 1 \\ n-1 & 0 \end{pmatrix} \text{ resp. } \mathbf{A}(\mathbf{p}) = \begin{pmatrix} 1 & 1 & 0 \\ d & 0 & 1 \\ n-1 & 0 & 0 \end{pmatrix}. \tag{7.2.1}$$

Proof. Assume $\hat{P}_1(\mathbf{p}) = 0$, $\hat{P}_2(\mathbf{p}) \neq 0$. Since $\partial \hat{P}_1/\partial \boldsymbol{\beta}_j(\mathbf{p}) = 1$, it follows, by setting $B_j = \hat{P}_1, B_i = \boldsymbol{\beta}_i, i \neq j$, that $(B_1, \ldots, B_n, w)$ form a system of coordinates at $\mathbf{p}$ so that $P_1 \circ \pi = w_1 B_j$ and $P_2 \circ \pi = w_j^d \cdot$ (unit). So, $\mathbf{A}(\mathbf{p})$ equals the first matrix. If $\hat{P}_1(\mathbf{p}) \neq 0$, $\hat{P}_2(\mathbf{p}) = 0$, then Lemma 2.3 implies $\mathbf{A}(\mathbf{p})$ equals the second matrix. Finally, if $\hat{P}_1(\mathbf{p}) = \hat{P}_2(\mathbf{p}) = 0$, then it is clear, by combining the two previous cases, that w_j and both $\hat{P}_1$, $\hat{P}_2$ can be used as part of local coordinates at $\mathbf{p}$. Using these coordinates, it follows immediately that $\mathbf{A}(\mathbf{p})$ equals the third matrix in (7.2.1). □

The next Lemma treats the remaining possibility.

LEMMA 7.3. *Assume* $\hat{P}_1(\mathbf{p}), \hat{P}_2(\mathbf{p}) \neq 0$. *Then there exist local coordinates* (B, W) *centered at* $\mathbf{p}$ *so that*

$$P_1 \circ \pi = \hat{P}_1(\mathbf{p}) W_j, \qquad P_2 \circ \pi = W_j^d \hat{P}_2(B, W)$$

where $\partial \hat{P}_2/\partial B_j(\mathbf{p}) \neq 0$. *Thus, there exists an amelioration* $(F, G) \to \mathbf{P} \circ \pi$, *and the matrix of multiplicities for* (F, G) *is given by:*

$$\mathbf{A}^{\#}(\mathbf{p}) = \begin{pmatrix} 1 & 0 \\ d & 1 \\ n-1 & 0 \end{pmatrix}. \tag{7.3.1}$$

Proof. For $i = 1, 2$, set $\tilde{P}_i = (\hat{P}_i - \hat{P}_i(\mathbf{p}))/\hat{P}_i(\mathbf{p})$, so that $P_i \circ \pi = \hat{P}_i(\mathbf{p}) w_j [1 + \tilde{P}_i]$. Since $\partial \hat{P}_1/\partial \boldsymbol{\beta}_j(\mathbf{p}) = 1$, there exists a neighborhood $U(\mathbf{p})$ so that $B_j = \tilde{P}_1$, $B_i = \boldsymbol{\beta}_i, i \neq j$, and $W_j = w_j[1 + \tilde{P}_1]$, $W_i = w_i - w_i(\mathbf{p})$, $i \neq j$, has nonzero jacobian on $U(\mathbf{p})$. Thus,

$$P_2 \circ \pi(W) = \hat{P}_2(\mathbf{p}) W_j^d [1 + \tilde{P}_1]^{-d} [1 + \tilde{P}_2] = \hat{P}_2(\mathbf{p}) W_j^d [1 + \tilde{P}_1 \mu + \tilde{P}_2].$$

By shrinking $U(\mathbf{p})$ if needed, one can insure that $\mu|_{U(\mathbf{p})} \neq 0$, and $\partial(\tilde{P}_1\mu)/\partial \boldsymbol{\beta}_j(B, W) \neq 0$ for all $(B, W) \in U(\mathbf{p})$. Since $\partial \tilde{P}_2/\partial \boldsymbol{\beta}_j \equiv 0$, one concludes that $\mathbf{p}$ is a good point for the pair $(F, G) =_{\text{def}} (W_j, \hat{P}_2(\mathbf{p})W_j^d[\tilde{P}_1\mu + \tilde{P}_2])$. It is then immediate that the matrix $\mathbf{A}^{\#}(\mathbf{p})$, computed for (F, G), is the matrix in (7.3.1). $\square$

Combining (7.2), (7.3), it follows that Theorem 4.3 applies to the pairs in $\mathcal{C}\ell^*$.

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