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# 2-LOCAL ISOMETRIES OF SOME NEST ALGEBRAS

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**Abstract**

Let $H$ be a complex separable Hilbert space with $\dim H \geq 2$. Let $\mathcal{N}$ be a nest on $H$ such that $E_+ \neq E$ for any $E \neq H, E \in \mathcal{N}$. We prove that every 2-local isometry of Alg $\mathcal{N}$ is a surjective linear isometry.

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## 1. Introduction

Let $X$ be a Banach space and $B(X)$ the algebra of all bounded linear operators on $X$. Suppose that $S$ is a subset of $B(X)$. Following [4, 6], a map $\phi : X \to X$ (which is not assumed to be linear) is called a *2-local S-map* if for any $a, b \in X$, there exists $\phi_{a,b} \in S$, depending on $a$ and $b$, such that

$$\phi_{a,b}(a) = \phi(a) \quad \text{and} \quad \phi_{a,b}(b) = \phi(b).$$

Here, $X$ is said to be *2-S-reflexive* if every 2-local $S$-map belongs to $S$.

The concept of a 2-local $S$-map dates back to the paper [13], where Šemrl investigated 2-local automorphisms and 2-local derivations, motivated by Kowalski and Słodkowski [5]. Then in [8], the earliest investigation of 2-local Iso($X$)-maps (also called *2-local isometries* in some papers) was carried out by Molnár, where Iso($X$) denotes the set of all surjective linear isometries of $X$. By an *isometry* of $X$, we mean a function $\varphi : X \to X$ such that $\|\varphi(a) - \varphi(b)\| = \|a - b\|$ for all $a, b \in X$. In [8], Molnár proved that $B(H)$ is 2-Iso($B(H)$)-reflexive, where $H$ is an infinite-dimensional separable Hilbert space. Recently, there has been a growing interest in 2-Iso($X$)-reflexive problems for several operator algebras and function algebras (see, for example, [1, 9, 12]). However, the 2-Iso($X$)-reflexivity in the context of nest algebras has not yet been considered. In this paper, we study 2-Iso($X$)-reflexivity in some nest algebras.

Throughout, $H$ will denote a separable Hilbert space over $\mathbb{C}$ with $\dim H \geq 2$, along with its dual space $H^*$. For a subset $S \subseteq H$, we set $S^\perp := \{f \in H^* : f(S) = 0\}$.

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By a *subspace lattice* on $H$, we mean a collection $\mathcal{L}$ of closed subspaces of $H$ with $(0)$ and $H$ in $\mathcal{L}$ such that, for every family $\{E_r\}$ of elements of $\mathcal{L}$, both $\bigvee\{E_r\}$ and $\bigwedge\{E_r\}$ belong to $\mathcal{L}$, where $\bigvee\{E_r\}$ denotes the closed linear span of $\{E_r\}$ and $\bigwedge\{E_r\}$ denotes the intersection of $\{E_r\}$. We say a subspace lattice is a *nest* if it is totally ordered with respect to inclusion. When there is no confusion, we identify the closed subspace and the orthogonal projection on it.

Let $\mathcal{L}$ be a subspace lattice on $H$ and $E \in \mathcal{L}$. Define

$$E_- = \bigvee\{F \in \mathcal{L} : F \not\supseteq E\} \quad \text{for } E \neq (0); \quad (0)_- = (0),$$
$$E_+ = \bigwedge\{F \in \mathcal{L} : F \not\subseteq E\} \quad \text{for } E \neq H; \quad H_+ = H,$$
$$\mathcal{J}(\mathcal{L}) = \{E \in \mathcal{L} : E \neq (0) \text{ and } E_- \neq H\}.$$

If $\mathcal{N}$ is a nest on $H$, then it is not difficult to verify that

$$H = \bigvee\{E : E \in \mathcal{J}(\mathcal{N})\} \quad \text{and} \quad (0) = \bigwedge\{E_- : E \in \mathcal{J}(\mathcal{N})\}.$$

It follows that the subspaces $\bigcup\{E : E \in \mathcal{J}(\mathcal{N})\}$ and $\bigcup\{E_-^\perp : E \in \mathcal{J}(\mathcal{N})\}$ are both dense in $H$ and $H^*$, respectively, where $E_-^\perp = (E_-)^\perp$.

Denote by $B(H)$, $K(H)$ and $F(H)$ the algebra of all bounded linear operators on $H$, the algebra of all compact operators on $H$ and the algebra of all bounded finite rank operators on $H$, respectively.

By a *nest algebra* $\operatorname{Alg}\mathcal{N}$, we mean the set of all operators in $B(H)$ leaving each element in $\mathcal{N}$ invariant, that is, $\operatorname{Alg}\mathcal{N} = \{T \in B(H) : TE \subseteq E \text{ for all } E \in \mathcal{N}\}$. Denote $F(\mathcal{N}) = \operatorname{Alg}\mathcal{N} \cap F(H)$ and $K(\mathcal{N}) = \operatorname{Alg}\mathcal{N} \cap K(H)$.

For $x \in H$ and $f \in H^*$, the rank-one operator $x \otimes f$ is defined as the map $z \mapsto f(z)x$. The following well-known result about rank-one operators will be repeatedly used.

Proposition 1.1 [7]. *If $\mathcal{L}$ is a subspace lattice, then $x \otimes y \in \operatorname{Alg}\mathcal{L}$ if and only if there exists an element $E \in \mathcal{L}$ such that $x \in E$ and $y \in E_-^\perp$.*

## 2. Main result

Our main result is the following theorem.

Theorem 2.1. *Let $\mathcal{N}$ be a nest on $H$ such that $E_+ \neq E$ for any $E \neq H, E \in \mathcal{N}$. If $\phi$ is a 2-local isometry of $\operatorname{Alg}\mathcal{N}$, then $\phi$ is a surjective linear isometry.*

The proof of Theorem 2.1 will be organised in a series of lemmas. In what follows, $\mathcal{N}$ is a nest on $H$ such that $E_+ \neq E$ for any $E \neq H, E \in \mathcal{N}$ and $\phi$ is a 2-local isometry of $\operatorname{Alg}\mathcal{N}$. For $A, B \in \operatorname{Alg}\mathcal{N}$, the symbol $\phi_{A,B}$ stands for a surjective linear isometry from $\operatorname{Alg}\mathcal{N}$ to itself such that $\phi_{A,B}(A) = \phi(A)$ and $\phi_{A,B}(B) = \phi(B)$. For a nest $\mathcal{M}$, we denote by $\mathcal{M}^\perp$ the nest $\{I - E : E \in \mathcal{M}\}$. A conjugation is a conjugate linear map on $H$ such that $J^2 = I$ and $\langle Jx, y\rangle = \langle Jy, x\rangle$ for all $x, y \in H$.

Proposition 2.2 below is cited from the paper by Moore and Trent [11] where they summarise the results in [2, 10] and characterise the surjective linear isometries on nest algebras.

PROPOSITION 2.2. *Let $\mathcal{M}$ be a nest on $H$ and $\rho : \mathrm{Alg}\,\mathcal{M} \to \mathrm{Alg}\,\mathcal{M}$ be a surjective linear isometry. Then there are unitary operators $U$ and $V$ in $B(H)$ such that $U$ and $U^*$ lie in $\mathrm{Alg}\,\mathcal{M}$. Moreover, one of the following cases holds:*

(1) *$\rho(A) = UV^*AV$ for every $A \in \mathrm{Alg}\,\mathcal{M}$ and the map $E \mapsto V^*EV$ is an order isomorphism of $\mathcal{M}$;*
(2) *$\rho(A) = UV^*JA^*JV$ for every $A \in \mathrm{Alg}\,\mathcal{M}$, where $J$ is a conjugation on $H$ such that $JE = EJ$ for each $E \in \mathcal{M}$ and the map $E \mapsto V^*JEJV$ is an order isomorphism from $\mathcal{M}$ onto $\mathcal{M}^\perp$.*

REMARK 2.3. (1) It can be easily verified that the map $T \mapsto JT^*J$ is a *-anti-isomorphism of $B(H)$ and $J$ maps an orthonormal basis onto another orthonormal basis.

(2) For any $a, b \in H$,

$$\begin{aligned}\langle (Jf \otimes Jx)a, b\rangle &= \langle \langle a, Jx\rangle Jf, b\rangle = \langle a, Jx\rangle\langle Jf, b\rangle = \langle x, Ja\rangle\langle Jb, f\rangle \\ &= \langle \langle Jb, f\rangle x, Ja\rangle = \langle (x \otimes f)Jb, Ja\rangle = \langle a, J(x \otimes f)Jb\rangle,\end{aligned}$$

so $(Jf \otimes Jx)^* = J(x \otimes f)J$.

(3) If $\rho$ is a surjective linear isometry of $\mathrm{Alg}\,\mathcal{M}$, then according to Proposition 2.2, for any rank-one operator $x \otimes f \in \mathrm{Alg}\,\mathcal{M}$, $\rho$ maps $x \otimes f$ to either $UV^*x \otimes V^*f$ or $UV^*Jf \otimes V^*Jx$, both of which are also rank-one operators. Since every finite rank operator in $\mathrm{Alg}\,\mathcal{M}$ can be written as a sum of finitely many rank-one operators in $\mathrm{Alg}\,\mathcal{M}$ and $\rho$ preserves linear independence, it follows that $\rho$ preserves the rank of a finite rank operator. Since $\rho^{-1}$ is also a surjective linear isometry, $\rho$ preserves the rank in both directions.

LEMMA 2.4. *$\phi$ is rank preserving and $\phi|_{F(\mathcal{N})}$ is linear.*

PROOF. It follows from Remark 2.3 that $\phi$ is rank preserving. According to Proposition 2.2, $\phi_{A,B}(X) = U_{A,B}V_{A,B}^*XV_{A,B}$ or $\phi_{A,B}(X) = U_{A,B}V_{A,B}^*JX^*JV_{A,B}$, where $U_{A,B}$ and $V_{A,B}$ are unitary operators in $B(H)$ depending on $A, B$ and $U_{A,B}, U_{A,B}^*$ lie in $\mathrm{Alg}\,\mathcal{N}$.

First, we show that $\phi$ is complex homogeneous. For any $A \in \mathrm{Alg}\,\mathcal{N}$ and $\lambda \in \mathbb{C}$, $\phi(\lambda A) = \phi_{A,\lambda A}(\lambda A) = \lambda\phi_{A,\lambda A}(A) = \lambda\phi(A)$.

Next, we prove that $\phi$ is additive on $F(\mathcal{N})$. For any $A, B \in F(\mathcal{N})$, since $\phi$ is rank preserving, $\phi(A)$ and $\phi(B)$ are in $F(\mathcal{N})$. We claim that $\mathrm{tr}(\phi(A)\phi(B)^*) = \mathrm{tr}(AB^*)$. Indeed, if $\phi_{A,B}(X) = U_{A,B}V_{A,B}^*XV_{A,B}$, then

$$\mathrm{tr}(\phi(A)\phi(B)^*) = \mathrm{tr}(U_{A,B}V_{A,B}^*AV_{A,B}V_{A,B}^*B^*V_{A,B}U_{A,B}^*) = \mathrm{tr}(AB^*).$$

If $\phi_{A,B}(X) = U_{A,B}V_{A,B}^*JX^*JV_{A,B}$, then

$$\begin{aligned}\mathrm{tr}(\phi(A)\phi(B)^*) &= \mathrm{tr}(U_{A,B}V_{A,B}^*JA^*JV_{A,B}V_{A,B}^*(JB^*J)^*V_{A,B}U_{A,B}^*) \\ &= \mathrm{tr}(U_{A,B}V_{A,B}^*JA^*JV_{A,B}V_{A,B}^*JBJV_{A,B}U_{A,B}^*) = \mathrm{tr}(JA^*BJ) = \mathrm{tr}(AB^*).\end{aligned}$$

Thus, for any $A, A' \in F(\mathcal{N})$, by the linearity of tr,

$$\mathrm{tr}((\phi(A + A') - \phi(A) - \phi(A'))\phi(B)^*) = \mathrm{tr}(((A + A') - A - A')B^*) = 0.$$

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By replacing $B$ with $A + A'$, $A$ and $A'$, we obtain

$$\mathrm{tr}((\phi(A + A') - \phi(A) - \phi(A'))(\phi(A + A') - \phi(A) - \phi(A'))^*) = 0.$$

It follows that $\phi(A + A') - \phi(A) - \phi(A') = 0$, which means that $\phi$ is additive on $F(\mathcal{N})$. □

By Lemma 2.4 and [3, Corollary 2.2] where Hou and Cui characterise rank-1 preserving linear maps between nest algebras acting on Banach spaces, we can easily prove Lemma 2.5.

LEMMA 2.5. *One of the following statements holds.*

(1) *There exist injective linear transformations*

$$D : \bigcup\{E : E \in \mathcal{J}(\mathcal{N})\} \to H \quad \text{and} \quad C : \bigcup\{E_-^{\perp} : E \in \mathcal{J}(\mathcal{N})\} \to H^*$$

*such that $\phi(x \otimes f) = Dx \otimes Cf$ for every $x \otimes f \in F(\mathcal{N})$.*

(2) *There exist injective linear transformations*

$$D : \bigcup\{E_-^{\perp} : E \in \mathcal{J}(\mathcal{N})\} \to H \quad \text{and} \quad C : \bigcup\{E : E \in \mathcal{J}(\mathcal{N})\} \to H^*$$

*such that $\phi(x \otimes f) = Df \otimes Cx$ for every $x \otimes f \in F(\mathcal{N})$.*

By categorising and discussing the two cases in Lemma 2.5, we can obtain the following result.

LEMMA 2.6. *One of the following statements holds.*

(1) *There exist unitary operators $C, D \in B(H)$ such that $\phi(A) = DAC^*$ for any $A \in K(\mathcal{N})$.*
(2) *There exist bounded conjugate linear operators $C, D$ such that $CJ, DJ \in B(H)$ are unitary operators and $\phi(A) = (DJ)JA^*J(CJ)^*$ for any $A \in K(\mathcal{N})$.*

PROOF. We consider two cases.

*Case 1.* If Lemma 2.5(1) holds, then based on the assumption on $\mathcal{N}$, there exist injective linear transformations $D : \bigcup\{E : E \in \mathcal{J}(\mathcal{N})\} \to H$ and $C : H^* \to H^*$ such that $\phi(x \otimes f) = Dx \otimes Cf$ for every $x \otimes f \in F(\mathcal{N})$. Thus, for any $x \otimes f \in \mathrm{Alg}\,\mathcal{N}$,

$$\|Dx\|\,\|Cf\| = \|Dx \otimes Cf\| = \|\phi(x \otimes f) - \phi(0)\| = \|x \otimes f - 0\| = \|x\|\,\|f\|.$$

Fix $x_0 \neq 0 \in (0)_+$. Then $x_0 \otimes f$ is in $\mathrm{Alg}\,\mathcal{N}$ for any $f \neq 0, f \in ((0)_+)_-^{\perp} = H^*$. It follows that $\|Dx_0\|\,\|Cf\| = \|x_0\|\,\|f\|$. So $\|Cf\|/\|f\| = \|x_0\|/\|Dx_0\|$ for any $f \neq 0, f \in H^*$, which means that $C \in B(H^*)$ and $\|C\| = \|x_0\|/\|Dx_0\|$.

For any $E \in \mathcal{J}(\mathcal{N})$, fix $f_0 \neq 0, f \in E_-^{\perp}$. Then $x \otimes f_0 \in \mathrm{Alg}\,\mathcal{N}$ for any $x \neq 0, x \in E$. It follows that $\|Dx\|\,\|Cf_0\| = \|x\|\,\|f_0\|$. Therefore, $\|Dx\|/\|x\| = \|f_0\|/\|Cf_0\| = \|Dx_0\|/\|x_0\|$, which means that $\|D|_E\| = \|Dx_0\|/\|x_0\|$. Since $\bigcup\{E : E \in \mathcal{J}(\mathcal{N})\}$ is dense in $H$, we can extend $D$ to an operator in $B(H)$ also denoted by $D$ such that $\|Dx\|/\|x\| = \|Dx_0\|/\|x_0\|$ for any $x \neq 0, x \in H$. So we can assume that $C, D$ are isometries. Since $\phi$ is an isometry, by the linearity of $\phi|_{F(\mathcal{N})}$ and the continuity of $\phi$, we have $\phi(A) = DAC^*$ for all $A \in K(\mathcal{N})$.

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By the Riesz–Frechet theorem, $H^*$ can be identified with $H$ through a conjugate linear surjective isometry. For any $E \neq H, E \in \mathcal{N}$, we have $(E_+)_- = E$ by the hypothesis on $\mathcal{N}$. Thus, $x$ is in $(E_+)_-^{\perp}$ for any $x \in E_+ \ominus E$, and so $x \otimes x \in \operatorname{Alg}\mathcal{N}$. Let $\mathcal{N} = \{E_j : j \in \Omega\}$ and $\{e_i^j : i \in \Lambda_j\}$ be an orthonormal basis of $(E_j)_+ \ominus E_j$. Then $K := \sum_{i,j} e_i^j \otimes e_i^j/(i \cdot j)$ is a compact operator in $\operatorname{Alg}\mathcal{N}$. Moreover, $K$ is an injective operator with dense range. We claim that $\phi(K)$ is also an injective operator with dense range.

For the case when $\phi(K) = U_{K,0}V_{K,0}^*KV_{K,0}$, since $U_{K,0}, V_{K,0}$ are unitary operators, $\phi(K)$ is also an injective operator with dense range.

For the case when $\phi(K) = U_{K,0}V_{K,0}^*JK^*JV_{K,0}$, since $\operatorname{Ker} K = (\operatorname{Ran} K^*)^{\perp}$, $K^*$ is an injective operator with dense range. As $J$ is a conjugate linear isometry, it follows that $\phi(K)$ is also an injective operator with dense range.

Therefore, $\phi(K) = \sum_{i,j} De_i^j \otimes Ce_i^j/(i \cdot j)$ is an injective operator with dense range, which implies $D$ and $C$ have dense ranges. Consequently, $D$ and $C$ are surjective isometries (unitary operators).

*Case 2.* If Lemma 2.5(2) holds, then there exist injective linear transformations $D : H^* \to H$ and $C : \bigcup\{E \in \mathcal{N} \mid E_- \neq H\} \to H^*$ such that $\phi(x \otimes f) = Df \otimes Cx$ for every $x \otimes f \in F(\mathcal{N})$.

According to the Riesz–Frechet theorem, we can consider $D$ as an injective conjugate linear transformation from $H$ to $H$, and $C$ as an injective conjugate linear transformation from $\bigcup\{E \in \mathcal{N} \mid E_- \neq H\}$ to $H$. Similarly to Case 1, we can conclude that $DJ$ and $CJ$ are unitary operators. By Remark 2.3,

$$\begin{aligned}\phi(x \otimes f) = Df \otimes Cx &= (DJ)(Jf \otimes Jx)(CJ)^* \\ &= (DJ)(J(x \otimes f)J)^*(CJ)^* = (DJ)(J(x \otimes f)^*J)(CJ)^*\end{aligned}$$

for any $x \otimes f \in \operatorname{Alg}\mathcal{N}$. By the linearity of $\phi|_{F(\mathcal{N})}$ and the continuity of $\phi$, we have $\phi(A) = (DJ)(JA^*J)(CJ)^*$ for any $A \in K(\mathcal{N})$. □

LEMMA 2.7. *$\phi(P)\phi(T)^*\phi(P) = \phi(PT^*P)$ for any $T \in \operatorname{Alg}\mathcal{N}$ and any $P = x \otimes f \in \operatorname{Alg}\mathcal{N}$.*

PROOF. By Lemma 2.2, $\phi_{P,T}(X) = U_{P,T}V_{P,T}^*XV_{P,T}$ or $\phi_{P,T}(X) = U_{P,T}V_{P,T}^*JX^*JV_{P,T}$. To simplify the notation, denote $U_{P,T}, V_{P,T}$ by $U, V$, respectively. For $\phi_{P,T}(X) = UV^*XV$,

$$\begin{aligned}\phi(P)\phi(T)^*\phi(P) &= UV^*PV(UV^*TV)^*UV^*PV = UV^*PT^*PV = UV^*\langle T^*x, f\rangle PV \\ &= \langle T^*x, f\rangle UV^*PV = \langle T^*x, f\rangle\phi(P) = \phi(\langle T^*x, f\rangle P) = \phi(PT^*P).\end{aligned}$$

For $\phi_{P,T}(X) = UV^*JX^*JV$, using Remark 2.3,

$$\begin{aligned}\phi(P)\phi(T)^*\phi(P) &= UV^*JP^*JV(UV^*JT^*JV)^*UV^*JP^*JV = UV^*JP^*TP^*JV \\ &= UV^*J(PT^*P)^*JV = UV^*J(\langle T^*x, f\rangle x \otimes f)^*JV \\ &= \langle T^*x, f\rangle UV^*J(x \otimes f)^*JV \\ &= \langle T^*x, f\rangle\phi(P) = \phi(\langle T^*x, f\rangle P) = \phi(PT^*P).\end{aligned}$$

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Furthermore, if $\phi$ is the form in Lemma 2.6(1), then $DPC^*\phi(T)^*DPC^* = DPT^*PC^*$, which implies that

$$P(C^*\phi(T)^*D - T^*)P = 0 \tag{2.1}$$

for any $T \in \operatorname{Alg}\mathcal{N}$ and $P = x \otimes f \in \operatorname{Alg}\mathcal{N}$.

If $\phi$ is the form in Lemma 2.6(2), then it follows that

$$\begin{aligned}(DJ)JP^*J(CJ)^*\phi(T)^*(DJ)JP^*J(CJ)^* &= (DJ)J(PT^*P)^*J(CJ)^* \\ &= (DJ)(JP^*J)(JTJ)(JP^*J)(CJ)^*,\end{aligned}$$

which implies that

$$(JP^*J)((CJ)^*\phi(T)^*(DJ) - (JTJ))(JP^*J) = 0 \tag{2.2}$$

for any $T \in \operatorname{Alg}\mathcal{N}$ and any $P = x \otimes f \in \operatorname{Alg}\mathcal{N}$. □

Under the assumption on $\mathcal{N}$, Lemmas 2.8 and 2.9 follow from Proposition 2.2.

LEMMA 2.8. *Let $\rho : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$ be a surjective linear isometry. If Case (1) in Proposition 2.2 holds for $\rho$, then $V, V^*$ are in $\operatorname{Alg}\mathcal{N}$.*

PROOF. It is sufficient to show that $V^*EV = E$ for all $E \in \mathcal{N}$. We prove it by the principle of transfinite induction.

It is evident that $V^*(0)V = (0)$. Moreover, for any given $F \in \mathcal{N}$, if the equation $V^*GV = G$ holds for all $G \in \mathcal{N}$ such that $G < F$, then because $E \mapsto V^*EV$ is an order isomorphism from $\mathcal{N}$ onto $\mathcal{N}$, it follows that $V^*FV = F$. □

LEMMA 2.9. *Let $\rho : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$ be a surjective linear isometry. If Case (2) in Proposition 2.2 holds for $\rho$, then the following statements hold.*

(1) *$E_- \neq E$ for any $E \neq (0), E \in \mathcal{N}$.*
(2) *$\mathcal{N}$ is finite.*
(3) *We denote $\mathcal{N} = \{E_0, E_1, \ldots, E_n\}$ where $(0) = E_0 < E_1 < \cdots < E_n = H$. Then $V^*$ and $V$ both map $E_i$ onto $I - E_{n-i}$ for $0 \le i \le n$.*

PROOF. (1) In the nest $\mathcal{N}^\perp$, we denote $E_+^{\mathcal{N}^\perp} = \bigwedge\{F \in \mathcal{N}^\perp : F \not\subseteq E\}$ for any $E \neq H$, $E \in \mathcal{N}^\perp$, and $E_-^{\mathcal{N}^\perp} = \bigvee\{F \in \mathcal{N}^\perp : F \not\supseteq E\}$ for any $E \neq (0), E \in \mathcal{N}^\perp$.

Since the map $\pi : E \mapsto V^*EV$ is an order isomorphism from $\mathcal{N}$ onto $\mathcal{N}^\perp$, we have $(I - E)_+^{\mathcal{N}^\perp} \neq (I - E)$ for any $I - E \neq H, I - E \in \mathcal{N}^\perp$. So

$$I - E \neq (I - E)_+^{\mathcal{N}^\perp} = \bigwedge\{I - F \in \mathcal{N}^\perp : I - F > I - E\} = \bigwedge\{I - F \in \mathcal{N}^\perp : F < E\} = I - E_-$$

for any $I - E \neq H, I - E \in \mathcal{N}^\perp$. It follows that $E_- \neq E$ for any $E \neq (0) \in \mathcal{N}$.

(2) Suppose that $\mathcal{N}$ is infinite, then there is a sequence $\{E_i : i \in \mathbb{N}\} \subseteq \mathcal{N}$ such that $E_i \neq (0)$ or $H$ for any $i \in \mathbb{N}$ and $E_i < E_j$ when $i < j$. Let $G = \bigvee\{E_i : i \in \mathbb{N}\}$. Then $G_- = \bigvee\{F \in \mathcal{N} : F < G\} \supseteq \bigvee\{E_i : i \in \mathbb{N}\} = G$ which contradicts $G_- \neq G$. This implies that $\mathcal{N}$ is finite.

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(3) Since $E \mapsto V^*JEJV$ is an order isomorphism from $\mathcal{N}$ onto $\mathcal{N}^\perp$ and $EJ = JE$ for any $E \in \mathcal{N}$, we obtain $E_i \mapsto V^*E_iV = I - E_{n-i}$ for $0 \le i \le n$. Since $V$ is a unitary operator, it follows that $V^*$ and $V$ both map $E_i$ onto $I - E_{n-i}$ for $0 \le i \le n$. □

Using the characterisation of the $\phi_{A,B}$ provided by Proposition 2.2, we divide the proof of Theorem 2.1 into two lemmas based on whether $\mathcal{N}$ is isomorphic to $\mathcal{N}^\perp$.

LEMMA 2.10. *If $\mathcal{N}$ is not order isomorphic to $\mathcal{N}^\perp$, then $\phi$ is a surjective linear isometry.*

PROOF. Since $\mathcal{N}$ is not order isomorphic to $\mathcal{N}^\perp$, every surjective linear isometry of $\operatorname{Alg}\mathcal{N}$ is of the form in Proposition 2.2(1). We distinguish two cases according to Lemma 2.6.

*Case 1.* Suppose that Lemma 2.6(1) holds, that is, $\phi(A) = DAC^*$ for every $A \in K(\mathcal{N})$ where $C, D$ are unitary operators. We claim that $C$ and $D$ are both in $\operatorname{Alg}\mathcal{N} \cap \operatorname{Alg}\mathcal{N}^\perp$.

For any fixed $E \in \mathcal{N}$, if $x \neq 0, x \in E$ and $f \neq 0, f \in E_-^\perp$, then it follows from $\phi(x \otimes f) = Dx \otimes Cf = U_{T,x\otimes f}V^*_{T,x\otimes f}(x \otimes f)V_{T,x\otimes f}$ that

$$Dx = \lambda_{T,x\otimes f}U_{T,x\otimes f}V^*_{T,x\otimes f}x \quad \text{and} \quad Cf = \frac{1}{\overline{\lambda}_{T,x\otimes f}}V^*_{T,x\otimes f}f,$$

where $\lambda_{T,x\otimes f} \in \mathbb{C}$ is on the unit circle.

By Proposition 2.2 and Lemma 2.8, $U_{T,x\otimes f}, V_{T,x\otimes f}$ are both in $\operatorname{Alg}\mathcal{N} \cap \operatorname{Alg}\mathcal{N}^\perp$. Fix $x_0 \neq 0, x_0 \in (0)_+$. Then $x_0 \otimes f$ is in $\operatorname{Alg}\mathcal{N}$ for any $f \neq 0, f \in H$. Thus, for any $E \neq (0), E \in \mathcal{N}$, we have $Cf = V^*_{T,x_0\otimes f}f/\overline{\lambda}_{T,x_0\otimes f} \in E$ for any $f \neq 0, f \in E$. Also, for any $E \neq H, E \in \mathcal{N}$, we have $Cf = V^*_{T,x_0\otimes f}f/\overline{\lambda}_{T,x_0\otimes f} \in E^\perp$ for any $f \neq 0, f \in E^\perp$. This shows that $C$ is in $\operatorname{Alg}\mathcal{N} \cap \operatorname{Alg}\mathcal{N}^\perp$.

For any fixed $E \in \mathcal{J}(\mathcal{N})$, there exists an $f_0 \neq 0, f_0 \in E_-^\perp$. It follows that $Dx = \lambda_{T,x\otimes f_0}U_{T,x\otimes f_0}V^*_{T,x\otimes f_0}x \in E$ for any $x \neq 0, x \in E$, which means that $D \in \operatorname{Alg}\mathcal{N}$.

Fix $E \in \mathcal{J}(\mathcal{N})$. Then, for any $y \in E$ and any $x \in E^\perp \cap (\bigcup\{F : F \in \mathcal{J}(\mathcal{N})\})$,

$$\begin{aligned}\langle x, D^*y\rangle &= \langle Dx, y\rangle = \langle \lambda_{T,x\otimes f}U_{T,x\otimes f}V^*_{T,x\otimes f}x, y\rangle \\ &= \langle x, \lambda^*_{T,x\otimes f}V_{T,x\otimes f}U^*_{T,x\otimes f}y\rangle \in \langle x, E\rangle = \{0\}.\end{aligned}$$

So $D^*E \perp (E^\perp \cap (\bigcup\{F : F \in \mathcal{J}(\mathcal{N})\}))$. Since $E^\perp \cap (\bigcup\{F : F \in \mathcal{J}(\mathcal{N})\})$ is dense in $E^\perp$, it follows that $D^* \in \operatorname{Alg}\mathcal{N}$. This completes the claim.

For any $T \in \operatorname{Alg}\mathcal{N}$, denote $G := C^*\phi(T)^*D - T^*$. By (2.1), $f(Gx)x \otimes f = 0$ for any $P = x \otimes f \in \operatorname{Alg}\mathcal{N}$. Thus, $G$ maps $E_+$ into $E$ for any $E \neq H, E \in \mathcal{N}$. It is clear that $G$ is in $\operatorname{Alg}\mathcal{N}^\perp$, and hence $G$ maps every $E^\perp \in \mathcal{N}^\perp$ into $E^\perp$. It follows that $G$ maps $E_+ \ominus E = E_+ \cap E^\perp$ into $E \cap E^\perp$ for any $E \neq H, E \in \mathcal{N}$ which yields $G = 0$ and $\phi(T) = DTC^*$.

*Case 2.* Suppose that Lemma 2.6(2) holds, that is, $\phi(x \otimes f) = Df \otimes Cx$ for every $x \otimes f \in \operatorname{Alg}\mathcal{N}$ where $C, D$ are conjugate linear operators such that $CJ, DJ \in B(H)$ are unitary operators.

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Then for $x_0 \neq 0, x_0 \in (0)_+$ and linear independent $f_1, f_2 \in H$,

$$\phi(x_0 \otimes f_1) = Df_1 \otimes Cx_0 = U_{x_0\otimes f_1, x_0\otimes f_2} V^*_{x_0\otimes f_1, x_0\otimes f_2} x_0 \otimes V^*_{x_0\otimes f_1, x_0\otimes f_2} f_1$$

and

$$\phi(x_0 \otimes f_2) = Df_2 \otimes Cx_0 = U_{x_0\otimes f_1, x_0\otimes f_2} V^*_{x_0\otimes f_1, x_0\otimes f_2} x_0 \otimes V^*_{x_0\otimes f_1, x_0\otimes f_2} f_2.$$

It follows that $Df_1$ and $Df_2$ are linearly dependent which leads to a contradiction.

In conclusion, $\phi(T) = DTC^*$ for any $T \in \operatorname{Alg}\mathcal{N}$ and it is clear that $\phi$ is a surjective linear isometry of $\operatorname{Alg}\mathcal{N}$. □

LEMMA 2.11. *If $\mathcal{N}$ is order isomorphic to $\mathcal{N}^\perp$, then $\phi$ is a surjective linear isometry.*

PROOF. According to Lemma 2.9, $\mathcal{N}$ is finite; denote $\mathcal{N} = \{E_0, E_1, \ldots, E_n\}$ where $(0) = E_0 < E_1 < \cdots < E_n = H$. We distinguish two cases according to Lemma 2.6.

*Case 1.* Suppose that Lemma 2.6(1) holds, that is, $\phi(A) = DAC^*$ for every $A \in K(\mathcal{N})$ where $C, D$ are unitary operators. In this case, for any $E \in \mathcal{J}(\mathcal{N})$ satisfying $\dim E_-^\perp > 1$, fix $x_0 \neq 0, x_0 \in E$. For any linearly independent $f_1, f_2 \in E_-^\perp$, we have $x_0 \otimes f_1, x_0 \otimes f_2 \in \operatorname{Alg}\mathcal{N}$.

We claim that $\phi_{x_0\otimes f_1, x_0\otimes f_2}$ is not of the form in Proposition 2.2(2). Otherwise,

$$\phi(x_0 \otimes f_1) = U_{x_0\otimes f_1, x_0\otimes f_2} V^*_{x_0\otimes f_1, x_0\otimes f_2} J(x_0 \otimes f_1)^* J V_{x_0\otimes f_1, x_0\otimes f_2} = Dx_0 \otimes Cf_1$$

and

$$\phi(x_0 \otimes f_2) = U_{x_0\otimes f_1, x_0\otimes f_2} V^*_{x_0\otimes f_1, x_0\otimes f_2} J(x_0 \otimes f_2)^* J V_{x_0\otimes f_1, x_0\otimes f_2} = Dx_0 \otimes Cf_2.$$

It follows that $f_1$ and $f_2$ are linear dependent, leading to a contradiction.

Thus, for any $f_1 \neq 0, f_1 \in H$, there exist $x_0 \neq 0, x_0 \in (0)_+$ and $f_2 \neq 0, f_2 \in H$ such that

$$\phi(x_0 \otimes f_1) = U_{x_0\otimes f_1, x_0\otimes f_2} V^*_{x_0\otimes f_1, x_0\otimes f_2} (x_0 \otimes f_1) V_{x_0\otimes f_1, x_0\otimes f_2} = Dx_0 \otimes Cf_1.$$

Hence, $Dx_0 = \lambda_{x_0\otimes f_1, x_0\otimes f_2} U_{x_0\otimes f_1, x_0\otimes f_2} V^*_{x_0\otimes f_1, x_0\otimes f_2} x_0$ and $Cf_1 = V^*_{x_0\otimes f_1, x_0\otimes f_2} f_1 / \overline{\lambda}_{x_0\otimes f_1, x_0\otimes f_2}$ for some $\lambda_{x_0\otimes f_1, x_0\otimes f_2} \in \mathbb{C}$ on the unit circle. By the arbitrariness of $f_1$ and $V^*_{x_0\otimes f_1, x_0\otimes f_2} \in \operatorname{Alg}\mathcal{N} \cap \operatorname{Alg}\mathcal{N}^\perp$, we obtain $C \in \operatorname{Alg}\mathcal{N} \cap \operatorname{Alg}\mathcal{N}^\perp$.

Similarly, for any $E \in \mathcal{N}$ with $\dim E > 1$, fix $f_0 \in E_-^\perp$. Let $x_1, x_2 \in E$ be any linearly independent elements. It is impossible for $\phi_{x_1\otimes f_0, x_2\otimes f_0}$ to be in the form of Lemma 2.2(2). Thus, for any $x_1 \neq 0, x_1 \in H$, there exist $f_0 \neq 0, f_0 \in H_-^\perp$ and $x_2 \neq 0, x_2 \in H$ such that

$$\phi(x_1 \otimes f_0) = U_{x_1\otimes f_0, x_2\otimes f_0} V^*_{x_1\otimes f_0, x_2\otimes f_0} (x_1 \otimes f_0) V_{x_1\otimes f_0, x_2\otimes f_0} = Dx_1 \otimes Cf_0.$$

It follows that $D \in \operatorname{Alg}\mathcal{N} \cap \operatorname{Alg}\mathcal{N}^\perp$.

For any $T \in \operatorname{Alg}\mathcal{N}$, denote $G := C^*\phi(T)^*D - T^*$. Using a similar method to that in Lemma 2.10, we see that $G$ maps $E_+ \ominus E = E_+ \cap E^\perp$ into $E \cap E^\perp$ for any $E \neq H$, $E \in \mathcal{N}$, which yields $G = 0$ and $\phi(T) = DTC^*$.

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*Case 2.* Suppose that Lemma 2.6(2) holds, that is, $\phi(x\otimes f)=Df\otimes Cx$ for every $x\otimes f\in \operatorname{Alg}\mathcal{N}$ where $C,D$ are conjugate linear operators such that $CJ,DJ\in B(H)$ are unitary operators.

In this case, for any $E\in\mathcal{J}(\mathcal{N})$ with $\dim E_-^\perp>1$, fix $x_0\in E$. For any linearly independent $f_1,f_2\in E_-^\perp$, $x_0\otimes f_1, x_0\otimes f_2$ are in $\operatorname{Alg}\mathcal{N}$. It is impossible for $\phi_{x_0\otimes f_1,x_0\otimes f_2}$ to be in the form of Proposition 2.2(1). Otherwise,

$$\phi(x_0\otimes f_1)=U_{x_0\otimes f_1,x_0\otimes f_2}V^*_{x_0\otimes f_1,x_0\otimes f_2}x_0\otimes V^*_{x_0\otimes f_1,x_0\otimes f_2}f_1=Df_1\otimes Cx_0$$

and

$$\phi(x_0\otimes f_2)=U_{x_0\otimes f_1,x_0\otimes f_2}V^*_{x_0\otimes f_1,x_0\otimes f_2}x_0\otimes V^*_{x_0\otimes f_1,x_0\otimes f_2}f_2=Df_2\otimes Cx_0,$$

implying that $f_1,f_2$ are linear dependent, which leads to a contradiction.

Thus, for any $f_1\neq 0, f_1\in H$, there exist $x_0\neq 0, x_0\in(0)_+$ and $f_2\neq 0, f_2\in H$ such that

$$\phi(x_0\otimes f_1)=U_{x_0\otimes f_1,x_0\otimes f_2}V^*_{x_0\otimes f_1,x_0\otimes f_2}J(x_0\otimes f_1)^*JV_{x_0\otimes f_1,x_0\otimes f_2}=Df_1\otimes Cx_0.$$

So $Df_1=\lambda_{x_0\otimes f_1,x_0\otimes f_2}U_{x_0\otimes f_1,x_0\otimes f_2}V^*_{x_0\otimes f_1,x_0\otimes f_2}Jf_1$ and $Cx_0=V^*_{x_0\otimes f_1,x_0\otimes f_2}Jx_0/\overline{\lambda}_{x_0\otimes f_1,x_0\otimes f_2}$ for some $\lambda_{x_0\otimes f_1,x_0\otimes f_2}\in\mathbb{C}$ on the unit circle. According to Lemma 2.9, $V^*_{x_0\otimes f_1,x_0\otimes f_2}$ and $V_{x_0\otimes f_1,x_0\otimes f_2}$ both map $E_i$ onto $I-E_{n-i}$ for $0\le i\le n$. Since $EJ=JE$ for any $E\in\mathcal{N}$, by the arbitrariness of $f_1$ and $U_{x_0\otimes f_1,x_0\otimes f_2}\in\operatorname{Alg}\mathcal{N}\cap\operatorname{Alg}\mathcal{N}^\perp$, we see that $D$ maps $E_i$ into $I-E_{n-i}$ and $I-E_i$ into $E_{n-i}$ for $0\le i\le n$, respectively.

Similarly, for any $E\in\mathcal{N}$ with $\dim E>1$, fix $f_0\in E_-^\perp$. For any linearly independent $x_1,x_2\in E$, $x_1\otimes f_0, x_2\otimes f_0$ are in $\operatorname{Alg}\mathcal{N}$. It is impossible for $\phi_{x_1\otimes f_0,x_2\otimes f_0}$ to be in the form of Proposition 2.2(1). Thus, for any $x_1\neq 0, x_1\in H$, there exist $f_0\neq 0, f_0\in H_-^\perp$ and $x_2\neq 0, x_2\in H$ such that

$$\phi(x_1\otimes f_0)=U_{x_1\otimes f_0,x_2\otimes f_0}V^*_{x_1\otimes f_0,x_2\otimes f_0}J(x_1\otimes f_0)^*JV_{x_1\otimes f_0,x_2\otimes f_0}=Df_0\otimes Cx_1.$$

So $Df_0=\lambda_{x_1\otimes f_0,x_2\otimes f_0}U_{x_1\otimes f_0,x_2\otimes f_0}V^*_{x_1\otimes f_0,x_2\otimes f_0}Jf_0$ and $Cx_1=V^*_{x_1\otimes f_0,x_2\otimes f_0}Jx_1/\overline{\lambda}_{x_1\otimes f_0,x_2\otimes f_0}$ for some $\lambda_{x_1\otimes f_0,x_2\otimes f_0}\in\mathbb{C}$ on the unit circle. Since $V^*_{x_1\otimes f_0,x_2\otimes f_0}, V_{x_1\otimes f_0,x_2\otimes f_0}$ both map $E_i$ onto $I-E_{n-i}$ for any $0\le i\le n$ and $EJ=JE$ for any $E\in\mathcal{N}$, by the arbitrariness of $x_1$, we see that $C$ maps $E_i$ into $I-E_{n-i}$ and $I-E_i$ into $E_{n-i}$ for all $0\le i\le n$, respectively.

By (2.2), $(JP^*J)((CJ)^*\phi(T)^*(DJ)-(JTJ))(JP^*J)=0$ for any $T\in\operatorname{Alg}\mathcal{N}$ and any $P=x\otimes f\in\operatorname{Alg}\mathcal{N}$. So $\langle((CJ)^*\phi(T)^*(DJ)-JTJ)Jf,Jx\rangle=0$ for all $P=x\otimes f\in\operatorname{Alg}\mathcal{N}$ which means that $((CJ)^*\phi(T)^*(DJ)-JTJ)$ maps $(E_i)_-^\perp$ into $(E_i)^\perp$.

Moreover, for any $E_i\in\mathcal{N}$,

$$E_i\xrightarrow{DJ}I-E_{n-i}\xrightarrow{\phi(T)^*}I-E_{n-i}\xrightarrow{(CJ)^*}E_i,$$

and $JTJ$ maps $E_i$ into $E_i$. It follows that $((CJ)^*\phi(T)^*(DJ)-JTJ)$ maps $E_i\cap(E_i)_-^\perp$ into $E_i\cap E_i^\perp=\{0\}$. So $((CJ)^*\phi(T)^*(DJ)-JTJ)=0$, which implies that $\phi(T)=(DJ)JT^*J(CJ)^*$ for any $T\in\operatorname{Alg}\mathcal{N}$. It is easy to check that $\phi(T)$ is a surjective linear isometry. □

Combining Lemmas 2.10 and 2.11 completes the proof of Theorem 2.1.

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