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Linear congruence relations for 2-adic L-series at integers

AMBROŻY WÓJCIK
Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-950 Warszawa, Poland; e-mail: wojcik@impan.gov.pl

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Abstract. In the paper we find a further generalization of congruences of the K. Hardy and K. S. Williams [5] type which seems to be a full generalization of congruences of G. Gras [4]. Moreover we extend results of [5], [7], [8], [9] and in part of [6]. We apply ideas and methods of [2], [7] and [9].

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1. Notation

Let us recall the notation of the paper [9]. As usual let $\mathbb{C}_p$ (p-prime) stand for the completion of an algebraic closure of $\mathbb{Q}$ at some place above p. Let $L_p(k,\chi)$ denote the p-adic L-function defined in [10]. Here χ is a primitive Dirichlet character with values in $\mathbb{C}_p$. Following R. F. Coleman [2] we define p-adic multilogarithms by the formula

$$l_k^{(p)}(z) = l_k(z) - p^{-k} l_k(z^p),$$

where $l_k = l_{k,p}$ is a locally analytic function on $\mathbb{C}_p - \{1\}$ defined in [2]. We adopt the notation $\Sigma_{a=1}^{c}{}'$ to mean a sum taken over integers a coprime to c. Let A be a positive integer. For any Dirichlet character ψ modulo A, any integer k and $z \in \mathbb{C}_2$, we define

$$\mathcal{L}_{k,\psi}(z) = (-1)^{k+1} g(\overline{\psi}) A^{-1} \sum_{a=1}^{A}{}' \psi(a) l_k(\zeta_A^a z), \quad (z \neq \zeta_A^a),$$

if ψ is not trivial, and we set

$$\mathcal{L}_{k,\psi}(z) = (-1)^{k+1} l_k^{(2)}(z), \quad (z \neq \pm 1),$$

otherwise. Here ζ_A is a primitive Ath root of unity. Further on we shall adopt the same convention as in [9]. For any Dirichlet character χ modulo $M > 1$ and for any integer k we write

$$L_2^{[M]}(k, \chi\omega^{1-k}) = \prod_{p|M,\, p\text{-prime}} (1 - \chi(p)p^{1-k}) L_2(k, \chi\omega^{1-k}),$$

unless $k = 1$ and χ is trivial in which case we simply put $L_2^{[M]}(k, \chi\omega^{1-k}) = 0$. Here $\omega := \omega_p$ is the Teichmüller character at p. Let $\mathcal{T}_M$ denote the set of all fundamental discriminants dividing M. The set can be described as the set of square-free numbers of the form $4n + 1$ and 4 times square-free numbers not of this form. Let us denote by $\chi_d = \left(\frac{d}{\cdot}\right)$ the quadratic character (Kronecker symbol) associated with the fundamental discriminant d. It is convenient to denote by χ_1 the trivial character.

Let $\gamma_{e,l} = -1$ if $l \equiv 1, 2 \pmod 4$ and $e \in \mathcal{T}_8 - \mathcal{T}_4$, and let $\gamma_{e,l} = 1$ otherwise. Denote by K a finite set of integers. Let us consider a finite set of 2-adic integers $\{x_{k,e}\}_{k\in K,\, e\in\mathcal{T}_8}$. For $l \geqslant 0$ and $\varrho \in \{0, 1\}$ we define

$$t_{2l+\varrho} = 2^{\varrho} \sum_{k,e} (-1)^{l(k+1)} (2l+1)^{1-k} \gamma_{l,e} x_{k,e},$$

where the sum is taken over all $k \in K$, $e \in \mathcal{T}_8$ if $\varrho = 0$ and over $k \in K$, $e \in \mathcal{T}_8$ satisfying $\operatorname{sgn} e = (-1)^k$ if $\varrho = 1$.

Let us consider a sequence of the form

$$y_n = \sum_{\substack{k\in K,\\ e\in\mathcal{T}_8}} a_{k,e}(n) x_{k,e},$$

where $a_{k,e}(n)$ $(n \geqslant 0)$ are 2-adic integers. For this sequence, let $c := c(\{y_n\}) \geqslant 0$ denote an integer satisfying:

(i) there exits a sequence of 2-adic integers $\{x_{k,e}\}$ not all being even such that

$$y_n \equiv 0 (\operatorname{mod} 2^c),$$

(ii) if for a sequence of 2-adic integers $\{x_{k,e}\}$ we have $y_n \equiv 0 (\operatorname{mod} 2^{c+1})$ then all the numbers $x_{k,e}$ are even.

2. The main theorem

THEOREM. *Let $M > 1$ be a square-free odd natural number having r prime factors and let $\Psi\colon \mathbb{N} \to \mathbb{C}_2$ be a multiplicative function with odd values at divisors of M. Let K denote a finite set consisting of consecutive integers and write $m = \#K$. Let $\{x_{k,e}\}_{k\in K,\, e\in\mathcal{T}_8}$ be a sequence of 2-adic integers not all being even. Write*

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$\mathcal{J}_M = -(\log_2 M)/2$, *if M is a prime number and* $\mathcal{J}_M = 0$ *otherwise. Then the number*

$$\Lambda_2(x, M) := \sum_{\substack{e\in\mathcal{T}_8,\\ k\in K}} (-1)^{k+1} x_{k,e} \sum_{d\in\mathcal{T}_M} \Psi(|d|) L_2^{[M]}(k, \chi_{ed}\omega^{1-k}) \,+\, x_{1,1}\mathcal{J}_M$$

is a 2*-adic integer divisible by* $2^{r+\lambda}$, *where* 2^{λ} *is the greatest common divisor of* $2^{c(\{t_n\})}$ *and* t_n, $0 \leqslant n \leqslant 4m-1$. *Moreover we have*

$$c(\{t_n\}) = 4m - 1 - s_2(m) - \operatorname{ord}_2(m),$$

where $s_2(m)$ *denotes the sum of digits of the* 2*-adic expansion of* m.

3. Lemmas

The proof the theorem is divided into a sequence of lemmas. Some of these lemmas were proved in [9]. The others extend corresponding lemmas of [9].

LEMMA 1 ([9], cf. [7]). *Given any odd integer* M, *let* χ *be a primitive Dirichlet character modulo* M. *Suppose that* N *is an odd multiple of* M *such that* N/M *is square-free and relatively prime to M. Let* ψ *be a primitive Dirichlet character being either trivial or of even conductor coprime to* N. *Let* ω *denote the Teichmüler character at* $p = 2$ *and write* $\zeta_N = \zeta_M\zeta_{N/M}$. *Then for any integer* k *we have*

$$g(\overline{\chi})M^{-1}\sum_{a=1}^{N}{}' \chi(a)\mathcal{L}_{k,\psi}(\zeta_N^a)$$
$$= (-1)^{r(N/M)+k+1} \prod_{p|(N/M)} (1 - \overline{\chi}\overline{\psi}(p)p^{1-k}) L_2(k, \chi\psi\omega^{1-k}),$$

unless $k = 1$ *and both the characters* χ *and* ψ *are trivial, in which case we have*

$$\sum_{a=1}^{N}{}' \mathcal{L}_{k,\psi}(\zeta_N^a) = \begin{cases} -(\log_2 N)/2, & \text{if } N \text{ is a prime number},\\ 0, & \text{otherwise.}\end{cases}$$

Proof. The case if χ or ψ are not trivial for any k, and the case if $k \in \{-1, 0, 1, 2\}$ for any χ and ψ were considered in [9]. Let us assume that both the characters χ and ψ are trivial and $k \notin \{-1, 0, 1, 2\}$. In order to prove the lemma in this case we apply the methods of the proof of Lemma 1 of [9]. Then putting $N = nq$, where q is a prime number, we have

$$\begin{aligned}\sum_{a=1}^{N}{}' l_k(\zeta_N^a) &= \sum_{a=1}^{n}{}' \sum_{c=0}^{q-1} l_k(\zeta_{nq}^{cn+a}) - \sum_{b=1}^{n}{}' l_k(\zeta_{nq}^{bq}) \\ &= \sum_{a=1}^{n}{}' \sum_{c=0}^{q-1} l_k(\zeta_{nq}^{a}\zeta_q^c) - \sum_{a=1}^{n}{}' l_k(\zeta_n^a) \\ &= \sum_{a=1}^{n}{}' \frac{l_k(\zeta_n^a)}{q^{k-1}} - \sum_{a=1}^{n}{}' l_k(\zeta_n^a) \\ &= -(1-q^{1-k}) \sum_{a=1}^{n}{}' l_k(\zeta_n^a).\end{aligned}$$

Thus by induction on the number $r(N)$ of prime factors of N we get

$$\sum_{a=1}^{N}{}' l_k(\zeta_N^a) = (-1)^{r(N)-1} \prod_{p|(N/q)} (1-p^{1-k}) \sum_{a=1}^{q}{}' l_k(\zeta_q^a),$$

where the product is taken over primes dividing N/q.

On the other hand Corollary 7.1a [2] and formula (4), p. 2 [2] imply

$$\begin{aligned}\sum_{a=1}^{q}{}' l_k(\zeta_q^a) &= -(1-q^{1-k}) \lim_{z\to 1} l_k(z) \\ &= -(1-q^{1-k})(1-2^{-k})^{-1} L_2(k,\omega^{1-k})\end{aligned}$$

if $k \geqslant 2$. Therefore the lemma in this case follows easily from the obvious equation

$$\sum_{a=1}^{N}{}' \mathcal{L}_{k,\psi}(\zeta_N^a) = (-1)^{k+1}(1-2^{-k}) \sum_{a=1}^{N}{}' l_k(\zeta_N^a).$$

If $k \leqslant -1$ then we shall prove that

$$\sum_{a=1}^{q}{}' l_k(\zeta_q^a) = (1-q^{1-k}) \frac{B_{1-k,\chi_1}}{1-k}, \tag{1}$$

where B_n denotes the nth Bernoulli number. Further on we shall apply some polynomials $R_n \in \mathbb{Z}[z]$ introduced by Frobenius in [3] (see also formulas 2.6 and 2.14 in [1]) and defined by the formula

$$\frac{1-z}{\mathrm{e}^t - z} = \sum_{n=0}^{\infty} \frac{R_n(z)}{(1-z)^n} \frac{t^n}{n!}.$$

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We shall prove for that for $n \geqslant 0$ the following identity

$$l_{-n}(z) = -\frac{zR_n(z)}{(z-1)^{n+1}}$$

holds. By definition of l_k (see [2], p. 195) it suffices to check that the right-hand side, of the above equation, let us denote it by $r_n(z)$, satisfies

(i) $r_0(z) = \dfrac{z}{z-1}$,

(ii) $\dfrac{\mathrm{d}r_k(z)}{\mathrm{d}z} = \dfrac{r_{k-1}(z)}{z}$,

(iii) $\lim\limits_{z\to 0} r_k(z) = 0$.

By definition we have

$$r_0(z) = -\frac{zR_0(z)}{1-z} = \frac{z}{z-1}$$

and so (i) holds. (iii) is also obvious. As for (ii), we have to prove that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{zR_n(z)}{(1-z)^{n+1}}\right) = -\frac{R_{n+1}(z)}{(1-z)^{n+2}}. \tag{2}$$

By definition of the polynomials $R_n(z)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z}{e^t - z}\right) = \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{zR_n(z)}{(1-z)^{n+1}}\right)\frac{t^n}{n!}$$

and so we obtain

$$\frac{\mathrm{e}^t}{(\mathrm{e}^t - z)^2} = \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{zR_n(z)}{(1-z)^{n+1}}\right)\frac{t^n}{n!}. \tag{3}$$

On the other hand we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1-z}{\mathrm{e}^t - z}\right) = \frac{-\mathrm{e}^t(1-z)}{(\mathrm{e}^t - z)^2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{n=1}^{\infty}\frac{R_n(z)}{(1-z)^n}\frac{t^n}{(n)!}\right) = \sum_{n=1}^{\infty}\frac{R_n(z)}{(1-z)^n}\frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{R_{n+1}(z)}{(1-z)^{n+1}}\frac{t^n}{n!}.$$

Hence we find that

$$\frac{\mathrm{e}^t}{(\mathrm{e}^t - z)^2} = -\sum_{n=0}^{\infty}\frac{R_{n+1}(z)}{(1-z)^{n+2}}\frac{t^n}{n!}.$$

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This together with (3) gives the formula (2) at once. Consequently, we get

$$\frac{z}{\mathrm{e}^t - z} = \sum_{n=0}^{\infty} (-1)^n l_{-n}(z) \frac{t^n}{n!}. \tag{4}$$

Let p be a prime number and let $\zeta \neq 1$ be a pth root of unity. Then (4) implies

$$\begin{aligned} t \sum_{a=1}^{p}{}' \frac{\zeta^a}{\mathrm{e}^t - \zeta^a} &= \sum_{a=1}^{p-1} \sum_{n=0}^{\infty} (-1)^n l_{-n}(\zeta^a) \frac{t^{n+1}}{n!} \\ &= \sum_{n=1}^{\infty} \sum_{a=1}^{p-1} (-1)^{n-1} l_{1-n}(\zeta^a) n \frac{t^n}{n!}. \end{aligned} \tag{5}$$

On the other hand, we have formally

$$\begin{aligned} \sum_{a=1}^{p}{}' \frac{\zeta^a}{e^t - \zeta^a} &= -\sum_{n=0}^{\infty} \left(\sum_{a=1}^{p-1} \zeta^{-an} \right) \mathrm{e}^{tn} = \sum_{n \geqslant 0,\, p \nmid n} \mathrm{e}^{tn} - (p-1) \sum_{n \geqslant 0,\, p \mid n} \mathrm{e}^{tn} \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{tn} - p \sum_{n=0}^{\infty} \mathrm{e}^{tpn} = \frac{1}{1 - \mathrm{e}^t} - \frac{p}{1 - \mathrm{e}^{pt}} \end{aligned}$$

and in consequence by definition of Bernoulli numbers we get

$$t \sum_{a=1}^{p}{}' \frac{\zeta^a}{\mathrm{e}^t - \zeta^a} = -\sum_{n=0}^{\infty} B_n (1 - p^n) \frac{t^n}{n!}.$$

This together with (5) gives

$$\sum_{a=1}^{p-1} l_{1-n}(\zeta^a) = (-1)^n (1 - p^n) \frac{B_n}{n}$$

and so (1) holds as $B_n = 0$ if $n > 1$ is odd. Finally, we get

$$\begin{aligned} \sum_{a=1}^{N}{}' \mathcal{L}_{k,\psi}(\zeta_N^a) &= (-1)^{k+1} (1 - 2^{-k}) (-1)^{r(N)-1} \prod_{p \mid N} (1 - p^{1-k}) \frac{B_{1-k}}{1-k} \\ &= (-1)^{k+1} (-1)^{r(N)} \prod_{p \mid N} (1 - p^{1-k}) L_2(k, \omega^{1-k}) \end{aligned}$$

and the lemma is proved. □

Let $n \geqslant 0$ and k be integers. For any $e \in \mathcal{T}_8$ let us define

$$W_{k,e}(n) = \sum_{l=0}^{n} (-1)^{l(k+1)} (2l+1)^{1-k} \gamma_{e,l} \binom{2n+1}{n-l}. \tag{6}$$

Let us notice that the numbers $W_{k,e}(n)$ are 2-adic integers. Moreover if $k=0$ and $e\in\mathcal{T}_4$ then we have $W_{k,e}(0)=1$ and $W_{k,e}(n)=0$ if $n\geqslant 1$. If $k=0$ and $e\notin\mathcal{T}_4$ then we have $W_{k,e}(n)=(2n+1)2^n$. If $k=1$ then we have $W_{k,e}(n)=4^n$ if $e\in\mathcal{T}_4$ and $W_{k,e}(n)=2^n$ if $e\notin\mathcal{T}_4$. Furthermore for $e\in\mathcal{T}_8$ we have

$$W_{k-2,e}(n)=(2n+1)^2W_{k,e}(n)-8n(2n+1)W_{k,e}(n-1). \tag{7}$$

Indeed, by the definition the right-hand side of (7) is equal to

$$\begin{aligned}
&(2n+1)^2\sum_{l=0}^{n}(-1)^{l(k+1)}(2l+1)^{1-k}\gamma_{e,l}\binom{2n+1}{n-l}\\
&\quad -8n(2n+1)\sum_{l=0}^{n-1}(-1)^{l(k+1)}(2l+1)^{1-k}\gamma_{e,l}\binom{2n-1}{n-1-l}\\
&\quad =\sum_{l=0}^{n-1}(-1)^{l(k+1)}(2l+1)^{1-k}\gamma_{e,l}\frac{(2n+1)!}{(n-l-1)!(n+l)!}\\
&\qquad \times\left(\frac{(2n+1)^2}{(n-l)(n+l+1)}-4\right)\\
&\qquad +(-1)^{n(k+1)}(2n+1)^{1-(k-2)}\gamma_{n,e}\\
&\quad =\sum_{l=0}^{n}(-1)^{l(k-1)}(2l+1)^{1-(k-2)}\gamma_{e,l}\binom{2n+1}{n-l}.
\end{aligned}$$

Moreover, by a simple induction on n we can deduce from (7) that

$$W_{k,e}(n)=\frac{2^{4n}}{2n+1}\binom{2n}{n}^{-1}\sum_{l=0}^{n}\frac{1}{2l+1}\binom{2l}{l}2^{-4l}W_{k-2,e}(l). \tag{8}$$

Using the numbers $W_{k,e}(n)$ we shall extend Lemma 3 [9], and next Lemmas 6 and 7 of [7] and Lemma 4 [9].

LEMMA 2. *Let* $n\geqslant 0$ *be an integer. Set* $\gamma_n=-1$, *if* $n\equiv 1,2 \pmod 4$, *and* $\gamma_n=1$, *otherwise. Then for the numbers* $W_{m,e}(n)$ *defined above we have*

$$\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{2k+1}W_{m,e}(k)=\frac{(-1)^{nm}\gamma_n}{(2n+1)^m}.$$

Proof. First we shall prove (using Granville's ideas similarly as in the proof of Lemma 3 [9]) the following identity

$$(2n+1)^2\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{2k+1}W_{m,e}(k)$$
$$=\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{2k+1}W_{m-2,e}(k). \tag{9}$$

Let us denote

$$\lambda_k=(-1)^k 2^{4k}\binom{n+k}{n-k}\binom{2k}{k}^{-1}.$$

Then for all $k\geqslant 0$ we have

$$\lambda_k-\lambda_{k+1}=\left(\frac{2n+1}{2k+1}\right)^2\lambda_k$$

and so we get

$$(2n+1)^2\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{2k+1}W_{m,e}(k)$$
$$=(2n+1)^2\sum_{k=0}^{n}\frac{\lambda_k}{(2k+1)^2}\sum_{l=0}^{k}\frac{1}{2l+1}\binom{2l}{l}2^{-4l}W_{m-2,e}(l)$$
$$=\sum_{k=0}^{n}(\lambda_k-\lambda_{k+1})\sum_{l=0}^{k}\frac{1}{2l+1}\binom{2l}{l}2^{-4l}W_{m-2,e}(l)$$
$$=\sum_{k=0}^{n}\frac{\lambda_k}{2k+1}\binom{2k}{k}2^{-4k}W_{m-2,e}(k)$$
$$=\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{2k+1}W_{m-2,e}(k).$$

Now the lemma follows from (9) by induction on m in virtue of the identities

$$\sum_{k=0}^{n}\binom{n+k}{n-k}(-2)^k=\gamma_n;\qquad \sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k W_{0,1}(k)}{2k+1}=1,$$
$$\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-4)^k}{2k+1}=\frac{(-1)^n}{2n+1};\qquad \sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-2)^k}{2k+1}=\frac{(-1)^n\gamma_n}{2n+1}.$$

To evaluate the first identity, note that

$$\binom{n+k}{n-k}$$

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is the coefficient of t^n in

$$\frac{t^k}{(1-t)^{2k+1}}.$$

Thus the left-hand side of the identity is the coefficient of t^n in

$$\sum_{k\geqslant 0}\frac{(-2t)^k}{(1-t)^{2k+1}} = \frac{1}{1-t}\frac{1}{1-((-2t)/(1-t)^2)}$$

$$= \frac{1-t}{1+t^2} = \frac{1-t-t^2+t^3}{1-t^4}$$

and so equals -1, if $n \equiv 1, 2 \pmod 2$ and 1, otherwise. See the proof of Lemma 3 [9]. The identity with the numbers $W_{0,1}(k)$ follows immediately from the definition of these numbers. Two remaining identities follow from the obvious formula

$$\frac{2n+1}{2k+1}\binom{n+k}{n-k} = \binom{n+k+1}{n-k} + \binom{n+k}{n-k-1}.$$

It suffices to notice that

$$\binom{n+k+1}{n-k}\ \left(\text{resp.}\binom{n+k}{n-k-1}\right)$$

is the coefficient of t^n (resp. t^{n+1}) in

$$\frac{t^k}{(1-t)^{2k+2}}.$$

The rest of the proof runs as above. □

In the next lemmas let $\xi \neq 1$ be an Nth root of unity, where N is an odd natural number.

LEMMA 3. *For any* $e \in \mathcal{T}_8$ *and* $m \in \mathbb{Z}$ *write* $\alpha = (-1)^{m+1}\operatorname{sgn}(e)$ *and let*

$$w_\alpha = \frac{\alpha\xi}{1+\alpha\xi^2}.$$

Then

$$\mathcal{L}_{m,e}(\xi) = \sum_{k=0}^{\infty}\frac{\alpha^k W_{m,e}(k)}{2k+1}w_\alpha^{2k+1}.$$

Proof. First let us observe that the 2-adic series on the right-hand side of the above equation converges. In fact it is easy to see that $\operatorname{ord}_2(W_{m,e}(n)) \geqslant n$.

We can prove it by induction on m. It is obvious that $\operatorname{ord}_2(W_{0,e}(n)) \geqslant n$ and $\operatorname{ord}_2(W_{1,e}(n)) \geqslant n$. If $m \leqslant 0$ we can apply formula (7). If $m \geqslant 1$ then it follows from (8) at once since

$$\begin{aligned}\operatorname{ord}_2(W_{m,e}(n)) &= \operatorname{ord}_2\left(\frac{2^{4n}}{2n+1}\binom{2n}{n}^{-1}\sum_{l=0}^{n}\frac{1}{2l+1}\binom{2l}{l}2^{4l}W_{m-2,e}(l)\right)\\ &= \operatorname{ord}_2\left(\sum_{l=0}^{n}\frac{n!(2l+1)!!}{l!(2l+1)(2n+1)!!}2^{3(n-l)}W_{m-2,e}(l)\right),\end{aligned}$$

where $r!!$ is the product of all odd integers $\leqslant r$.

Now write $\gamma^2 = \alpha$. On the open unit ball in $\mathbb{C}_2$ we have

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{\alpha^k W_{m,e}(k)}{2k+1}\left(\frac{\alpha x}{1+\alpha x^2}\right)^{2k+1}\\ &\quad= -i\gamma\sum_{k=0}^{\infty}\frac{(-1)^k W_{m,e}(k)}{2k+1}\left(\sum_{l=0}^{\infty}(i\gamma x)^{2l+1}\right)^{2k+1}\\ &\quad= -i\gamma\sum_{k=0}^{\infty}\frac{(-1)^k W_{m,e}(k)}{2k+1}\sum_{l=0}^{\infty}\binom{2k+l}{l}(i\gamma x)^{2(k+l)+1}\\ &\quad= -i\gamma\sum_{l=0}^{\infty}\sum_{k=0}^{l}\frac{(-1)^k W_{m,e}(k)}{2k+1}\binom{l+k}{l-k}(i\gamma x)^{2l+1}\\ &\quad= -i\gamma\sum_{l=0}^{\infty}(i\gamma x)^{2l+1}\sum_{k=0}^{l}\binom{l+k}{l-k}\frac{(-1)^k W_{m,e}(k)}{2k+1}.\end{aligned}$$

Therefore the lemma follows immediately from Lemma 2, Theorem 5.11 [2] and the uniqueness principle (see p. 176, [2]). □

In order to prove the key lemma of the paper we will need the following elementary fact

LEMMA 4 (cf. Lemma 5.19 and 5.21 [10]). *For integers $b, p \geqslant 0$ we have*

$$\sum_{a=0}^{b} a^p(-1)^a\binom{b}{a} = \begin{cases} 0 & \text{if } p < b,\\ (-1)^b b! & \text{if } p = b,\\ b! \times \text{integer} & \text{if } p > b.\end{cases}$$

Proof. The first and the third identites are proved in [10] (see Lemmas 5.19 and 5.21 respectively). As for the second identity it follows by the same manner as the

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third one (for details, see the proof of Lemma 5.21 [10]). □

LEMMA 5. *Let $m \geqslant 1$ be an integer and let $K = \{-m+2, -m+3, \dots, 0, 1\}$. Then for the sequence $\{t_n\}_{n\geqslant 0}$ defined in the Notation we have*

$$c(\{t_n\}) = 3m - 1 + \mathrm{ord}_2((m-1)!).$$

Proof. In what follows, let $r = 3m - 1 + \mathrm{ord}_2((m-1)!)$. Let us consider the infinite system of congruences

$$t_{2n+1} \equiv 0 (\mathrm{mod}\, 2^{r+1}), \quad n \geqslant 0. \tag{10}$$

We shall prove that the above congruences with $n \leqslant 2m-1$ imply $x_{k,e} \equiv 0 (\mathrm{mod}\, 2)$. Substituting in (10)

$$x_k = x_{k,e} + x_{k,e'} \quad \text{and} \quad y_k = x_{k,e} - x_{k,e'},$$

where $e \in \mathcal{T}_4$ and $e' \notin \mathcal{T}_4$, we can rewrite it in the form of two subsystems of congruences

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & -7 & (-7)^2 & \dots & (-7)^{m-1} \\ 1 & 9 & 9^2 & \dots & 9^{m-1} \\ \vdots & & & \vdots & \vdots \\ 1 & s_1 & s_1^2 & \dots & s_1^{m-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_0 \\ x_{-1} \\ \vdots \\ x_{-m+2} \end{pmatrix} \equiv 0 (\mathrm{mod}\, 2^r) \tag{11}$$

(in this subsystem we consider the congruences of (10) with $n \equiv 0$ or $3 (\mathrm{mod}\, 4)$, here $s_1 = (-1)^{m+1}(4m-2) - 1$) and

$$\begin{pmatrix} 1 & -3 & (-3)^2 & \dots & (-3)^{m-1} \\ 1 & 5 & 5^2 & \dots & 5^{m-1} \\ 1 & -11 & (-11)^2 & \dots & (-11)^{m-1} \\ \vdots & & & \vdots & \vdots \\ 1 & s_2 & s_2^2 & \dots & s_2^{m-1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_0 \\ y_{-1} \\ \vdots \\ y_{-m+2} \end{pmatrix} \equiv 0 (\mathrm{mod}\, 2^r) \tag{12}$$

(in this subsystem we consider the congruences of (10) with $n \equiv 1$ or $2 (\mathrm{mod}\, 4)$, here $s_2 = (-1)^m(4m-2) - 1$). We have

$$x_{k,e} = \frac{x_k + y_k}{2} \quad \text{and} \quad x_{k,e'} = \frac{x_k - y_k}{2}. \tag{13}$$

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Let $0 \leqslant b \leqslant m-1$ be a fixed integer. Let us notice that for any $1 \leqslant k \leqslant b+1$ (resp. any $0 \leqslant a \leqslant b$) there exists $0 \leqslant a \leqslant b$ (resp. $1 \leqslant k \leqslant b+1$) such that

$$2(-1)^{k+1}(2k-1)-1 = 8([b/2]-a)+1. \tag{14}$$

It suffices to take $a = [b/2] + (-1)^k[k/2]$ (resp. $k = 2a - 2[b/2]$ if $a \geqslant [b/2]$ and $k = 2[b/2] - 2a + 1$ if $a \leqslant [b/2] - 1$). Similarly, for any $1 \leqslant k \leqslant b+1$ (resp. any $0 \leqslant a \leqslant b$) there exists $0 \leqslant a \leqslant b$ (resp. $1 \leqslant k \leqslant b+1$) such that

$$2(-1)^{k+1}(2k-1)-1 = 8([b/2]-a)-3. \tag{15}$$

It suffices to take $a = [b/2]] + (-1)^{k+1}[k/2]$ (resp. $k = 2[b/2] - 2a$ if $a \leqslant [b/2] - 1$ and $k = 2a - 2[b/2] + 1$ if $a \geqslant [b/2]$). In other words, there are two one-one correspondences between integers $k \in [1, b+1]$ and $a \in [0, b]$ satisfying (14) or (15) respectively. Using these correspondences and identity

$$\sum_{a=0}^{b}(8(c-a)+f)^p(-1)^a\binom{b}{a} = \begin{cases} 0, & \text{if } p < b, \\ 8^b b!, & \text{if } p = b, \\ 8^b b! \times \text{integer}, & \text{if } p > b, \end{cases} \tag{16}$$

with $b \leqslant m-1$, $p \leqslant m-1$, $c = [b/2]$ and $f = 1, -3$, we can rewrite the above systems in the equivalent triangular forms

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & -8 & 8 \times \text{integer} & \dots & 8 \times \text{integer} \\ 0 & 0 & 128 & \dots & 128 \times \text{integer} \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & (-8)^{m-1}(m-1)! \end{pmatrix} \begin{pmatrix} x_1 \\ x_0 \\ x_{-1} \\ \vdots \\ x_{-m+2} \end{pmatrix} \equiv 0 (\operatorname{mod} 2^r), \tag{17}$$

$$\begin{pmatrix} 1 & -3 & (-3)^2 & \dots & (-3)^{m-1} \\ 0 & 8 & 8 \times \text{integer} & \dots & 8 \times \text{integer} \\ 0 & 0 & 128 & \dots & 128 \times \text{integer} \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & 8^{m-1}(m-1)! \end{pmatrix} \begin{pmatrix} y_1 \\ y_0 \\ y_{-1} \\ \vdots \\ y_{-m+2} \end{pmatrix} \equiv 0 (\operatorname{mod} 2^r), \tag{18}$$

To construct the ith congruence of (17) or (18) one can multiply the kth congruence of (11) (resp. of (12)) for $1 \leqslant k \leqslant i$ through by $(-1)^a\binom{b}{a}$, where $b = i-1$ and $a = [a/2] + (-1)^k[k/2]$ (resp. $a = [b/2] + (-1)^{k+1}[k/2]$) and next add up the first i congruences of each of the systems using identity (16). Therefore $x_{-m+2} \equiv$

$0 \pmod 4$ at once and then the congruences $x_{-m+3} \equiv \cdots \equiv x_1 \equiv 0 \pmod 4$ follow by induction. Similarly we obtain the congruences $y_{-m+2} \equiv \cdots \equiv y_1 \equiv 0 \pmod 4$ and the parity of $x_{k,e}$ with $\operatorname{sgn} e = (-1)^k$ follows from (13) immediately.

In order to prove the same for $x_{k,e}$ with $\operatorname{sgn} e \neq (-1)^k$, let us notice that

$$t_{2n} = t_{2n+1} + \tilde{t}_{2n+1},$$

where $\tilde{t}_{2l+1}$ comes into t_{2l+1} by substituting $x_{k,1}$ (resp. $x_{k,-4}$, $x_{k,8}$ or $x_{k,-8}$) instead of $x_{k,-4}$ (resp. $x_{k,1}$, $x_{k,-8}$ or $x_{k,8}$). Then the divisibility $2^{r+1} \mid t_{2l}, t_{2l+1}$, leads to $2^{r+1} \mid \tilde{t}_{2l+1}$ and by the same reasoning as in the case of $\operatorname{sgn} e = (-1)^k$ we get $x_{k,e} \equiv 0 \pmod 2$ in the case under consideration. We have showed that $c(\{t_n\}) \leqslant r$.

In order to prove the lemma completely we should find a sequence of 2-adic integers $\{x_{k,e}\}$ not all being even such that the congruences

$$t_s \equiv 0 \pmod{2^r}, \quad s \geqslant 0 \tag{19}$$

hold.

We begin by putting $x_{k,1} = -x_{k,-4}$ and $x_{k,8} = -x_{k,-8}$ which implies $t_s = 0$ for s even and next we will find $x_{k,-4}, x_{k,-8}$ not all being even satisfying $t_s \equiv 0 \pmod{2^r}$ for s odd. It is obvious that the systems (17) and (18) have a solutions such that $x_{-m+2} = 2$ and $y_{-m+2} = 0$ and hence that the system (19) with odd $s \leqslant 4m-1$ has a solution such that $x_{-m+2,-4} = 1$, and $x_{-m+2,-8} = 1$. Now we have to prove that $t_s \equiv 0 \pmod{2^r}$ with the above r if $s \geqslant 4m$. It will follow from the identity (16). □

LEMMA 6. *Let $m \geqslant 1$ be an integer and let $K = \{-m+2, -m+3, \ldots, 1\}$. Let $\{x_{k,e}\}_{k \in K, e \in \mathcal{T}_8}$ be a sequence of integers in $\mathbb{C}_2$ not all being even. Then we have:*

(i)

$$\sum_{\substack{k \in K,\\ e \in \mathcal{T}_8}} x_{k,e} \mathcal{L}_{k,e}(\xi) \equiv 0 \pmod{2^\lambda}, \tag{20}$$

where 2^λ is the greatest common divisor of t_n, $0 \leqslant n \leqslant 4m-1$ and $2^{c(\{t_n\})}$.

(ii) *For any integer l we get*

$$\sum_{\substack{k \in K,\\ e \in \mathcal{T}_8}} x_{k,e} \mathcal{L}_{k+l,e}(\xi) \equiv 0 \pmod{2^\lambda}.$$

Proof. Lemma 3 and formula (6) imply

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$$\sum_{\substack{k\in K,\\ e\in\mathcal{T}_8}} x_{k,e}\mathcal{L}_{k,e}(\xi) = \sum_{\substack{k\in K,\\ e\in\mathcal{T}_8}} x_{k,e}\sum_{n=0}^{\infty}\frac{\alpha^n W_{k,e}(n)}{2n+1}\omega_\alpha^{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{1}{2n+1}\sum_{\substack{k\in L,\\ e\in\mathcal{T}_8}} x_{k,e}\sum_{l=0}^{n}\alpha^n(-1)^{l(k+1)}$$

$$\times(2l+1)^{1-k}\gamma_{e,l}\binom{2n+1}{n-l}\omega_\alpha^{2n+1}.$$

Consequently, putting for $n\geqslant 0$ and $\varrho\in\{0,1\}$

$$z_{2n+\varrho} = \sum_{l=0}^{n}\frac{1}{2n+1}\binom{2n+1}{n-l}t_{2l+\varrho}$$

and

$$v_{2n+1} = \tfrac{1}{2}((-1)^n\omega_{-1}^{2n+1} - w_1^{2n+1}),$$

we have

$$\sum_{\substack{k\in K,\\ e\in\mathcal{T}_8}} x_{k,e}\mathcal{L}_{k,e}(\xi) = \sum_{n=0}^{\infty} z_{2n}\omega_1^{2n+1} + \sum_{n=0}^{\infty} z_{2n+1}v_{2n+1}. \tag{21}$$

The numbers $z_{2n+\varrho}$ and v_{2n+1} are 2-adic integers. Write $c = c(\{z_s\})$ and $\tilde{c} = c(\{t_s\})$. Let us notice that

$$c = \tilde{c}.$$

Indeed, if for a sequence of 2-adic integers $\{x_{k,e}\}$ not all being even we have $t_s\equiv 0(\operatorname{mod} 2^{\tilde{c}})$ then by the definition of $\{z_s\}$ we have $z_s\equiv 0(\operatorname{mod} 2^{\tilde{c}})$ and so $c\geqslant\tilde{c}$. Let us prove the inequality $c\leqslant\tilde{c}$. By definition there exists a sequence of 2-adic integers $\{x_{k,e}\}$ such that $z_s\equiv 0(\operatorname{mod} 2^c)$. Then the congruences $t_s\equiv 0(\operatorname{mod} 2^c)$ follow by induction on s from the obvious identity

$$t_s = (s-\varrho+1)z_s - \sum_{l=0}^{(s-\varrho-2)/2}\binom{s-\varrho-1}{\frac{s-\varrho-2l}{2}}t_{2l+\varrho}.$$

Here $\varrho = 0$ if n is even and $\varrho = 1$ if n is odd.

Thus in order to prove the part (i) of the lemma it suffices to use the fact that the divisibility $2^r|t_s$ for $s\leqslant 4m-1$ implies the same for $s>4m-1$ which was already proved for odd s in the previous lemma. If s is even then we apply the formula

$$t_{2n} = t_{2n+1} + \tilde{t}_{2n+1}.$$

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Since $2^r|t_{2n},\ t_{2n+1}$ for $2n+1<4m$ we deduce that $2^r|\tilde{t}_{2n+1}$ then. The proof that $2^r|\tilde{t}_{2n+1}$ for $2n+1>4m$ is the same as for t_{2n+1} in the previous lemma and we get $2^r|t_{2n}$ for any $n \geqslant 0$.

In order to prove the part (ii) let us notice that by the definition the numbers t_s defined on the set $K+l$ are equal to t_s defined on K multiplied by the factor $(2s+1)^l$ and since the factor is odd the second part of the lemma follows at once. □

4. Proof of the theorem

By Lemma 1 we have

$$
\begin{aligned}
&\Lambda_2(x,M)\\
&\quad=(-1)^r\sum_{\substack{k\in K,\\ e\in\mathcal{T}_8}}(-1)^{k+1}(-1)^{k+1}x_{k,e}\sum_{d\in\mathcal{T}_M}\Psi(|d|)\mu(d)g(\chi_d)|d|^{-1}\\
&\qquad\times{\sum_{a=1}^{M}}'\chi_d(a)\mathcal{L}_{k,\chi_e}(\zeta_M^a)\\
&\quad=(-1)^r{\sum_{a=1}^{M}}'\sum_{\substack{k\in K,\\ e\in\mathcal{T}_8}}x_{k,e}\mathcal{L}_{k,\chi_e}(\zeta_M^a)\sum_{d\in\mathcal{T}_M}\Psi(|d|)\mu(d)g(\chi_d)|d|^{-1}\chi_d(a)\\
&\quad=(-1)^r\sum_{a=1}^{M}\left(\sum_{\substack{k\in K,\\ e\in\mathcal{T}_8}}x_{k,e}\mathcal{L}_{k,\chi_e}(\zeta_M^a)\right)\\
&\qquad\times\left(\prod_{p|M}(1-\Psi(p)g(\chi_{p^*})|p|^{-1}\chi_{p^*}(a))\right),
\end{aligned}
$$

where $p^*=(-1)^{(p-1)/2}p$. Therefore it follows from Lemma 3 that the numbers $\Lambda_2(x,M)$ are 2-adic integers and since

$$\Psi(p)g(\chi_{p^*})|p|^{-1}\chi_{p^*}(a)-1\ \equiv\ 1+\zeta_p+\cdots\zeta_p^{p-1}\ \equiv\ 0(\operatorname{mod}2),$$

the theorem follows from Lemmas 5 and 6 easily. Here we have used the obvious identity $\mathrm{ord}_2((m-1)!)=m-1-s_2(m-1)=m-s_2(m)-\mathrm{ord}_2(m)$. □

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References

1. Carlitz, L.: Arithmetic properties of generalized Bernoulli numbers, *J. Reine Angew. Math*. 202 (1959) 174–182.
2. Coleman, R. F.: Dilogarithms, Regulators and p-adic L-functions, *Invent. Math.* 69 (1982) 171–208.
3. Frobenius, G.: Über die Bernoullischen Zahlen und die Eulerschen Polynome, *Sitzungsber. Preuss. Akad. Wissenschaften* 2 (1910) 809–847, or Mathematische Werke III, Berlin, Heidelberg, New York, 1968, 440–478.
4. Gras, G.: Relations congruentielles linéaires entre nombres de classes de corps quadratiques, *Acta Arith.* 52 (1989) 147–162.
5. Hardy, K. and Williams, K. S.: A congruence relating the class numbers of complex quadratic fields, *Acta Arith.* 47 (1986) 263–276.
6. Szmidt, J., Urbanowicz, J. and Zagier, D.: Congruences among generalized Bernoulli numbers, *Acta Arith.* 71 (1995) 273–278.
7. Uehara, T.: On linear congruences between class numbers of quadratic fields, *J. Number Theory* 34 (1990) 362–392.
8. Urbanowicz, J.: Connections between $B_{2,\chi}$ for even quadratic Dirichlet characters χ and class numbers of appropriate imaginary quadratic fields, II, *Composito Math.* 75 (1990) 271–285, *Corrigendum: ibidem* 75 (1991) 119–125.
9. Urbanowicz, J. and Wójcik, A.: On linear congruence relations related to 2-adic dilogarithms, *Publ. Math. Fac. Sci. Besançon*, Théorie des Nombres, Années 1995/1996 (to appear).
10. Washington, L. C.: Introduction to cyclotomic fields, Springer Verlag, New York, 1982.