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JAS. ARCHIBALD, Esq., Vice-President, in the Chair.

A Symbolic Method in Geometrical Optics.

By EDWARD B. ROSS, M.A.

The formulæ given in Herman's Optics, pages 80, 82, 98, 111, and called Cotes's formulæ, are a little difficult to grasp, and do not lend themselves to manipulation. The notation explained below is useful as a mnemonic. I think it also renders the proofs simpler.

The symbol $\overline{AB}$ for the segment A to B is common; we may modify it slightly by turning the bar into a circumflex. The circumflex can then be split into two parts, an acute and a grave accent; these form convenient marks for the initial and final letters of a segment. $\acute{P}\grave{L}$, $\acute{M}\grave{P}$ are segments, $\grave{P}\acute{L}$ is not a segment.

Products are to be expanded by the distributive law, but the order of the letters must be preserved.

The special rules are only two.

(1) $\acute{P}\grave{Q} = \widehat{PQ}$ the length PQ.

(2) Units and meaningless letters are to be deleted; but if this rule would make a term vanish the value is 1 not 0; just as, in simplifying a fraction we can cancel factors in numerator and denumerator, provided we remember that it will not do if everything goes out to write 0 as the answer.

(In practice the accents may usually be dropped.)

This notation we apply to a system of n coaxial thin lenses at $A_1, \ldots, A_n$, of powers $\kappa_1, \ldots, \kappa_n$. From Q a ray issues making an angle α_0 with the axis and strikes the lenses at heights above the axis, $y_1, y_2, \ldots, y_n$; making after these refractions angles $\alpha_1, \ldots, \alpha_n$ with the axis.

Here we may give the formulæ, the brevity and similarity of which are the excuse for this paper.

The standard one is that for the apparent distance. The apparent distance is "the distance from the eye at which the object must be placed to subtend the same angle when viewed directly that it appears to subtend when viewed through the instrument."

Apparent distance of P from Q

$$=\acute{Q}(1+\grave{A}_1\kappa_1\acute{A}_1)(1+\grave{A}_2\kappa_2\acute{A}_2)\dots(1+\grave{A}_n\kappa_n\acute{A}_n)\grave{P}$$

or say $\acute{Q}\overset{n}{\underset{1}{\Pi}}\grave{P}$

where Π stands for a product of factors of the type $(1+\grave{A}_r\kappa_r\acute{A}_r)$ and the affixes are the values of r for the first and last factors.

If the first affix is 1, it will sometimes be dropped.

Angular magnification after lens n of object at $Q=\acute{Q}\overset{n}{\underset{1}{\Pi}}$. The power, K, of the system (which is the reciprocal of the focal length) increased by unity is $\overset{n}{\underset{1}{\Pi}}$.

The only apparatus we require for dealing with these symbols is a lemma with two corollaries.

Lemma. If m is a number

$$m\acute{R}\,.\,\grave{A}_n=m\acute{R}\,.\,\grave{A}_{n-1}+m\acute{R}\,.\,\acute{A}_{n-1}\grave{A}_n$$

Proof.

$$m\acute{R}\,.\,\grave{A}_n=m\widehat{RA}_n$$

$$=m\widehat{RA}_{n-1}\quad+m\acute{A}_{n-1}\grave{A}_n$$

$$=m\acute{R}\,.\,\grave{A}_{n-1}+m\acute{R}\acute{A}_{n-1}\grave{A}_n$$

where we avail ourselves of the second rule to introduce the meaningless letter $\acute{R}$.

Cor. 1. $$\acute{S}\overset{n-1}{\Pi}\grave{A}_n=\acute{S}\overset{n-2}{\Pi}\grave{A}_{n-1}+\acute{S}\overset{n-1}{\Pi}\acute{A}_{n-1}\grave{A}_n.$$

In $\acute{S}\overset{n-1}{\Pi}$ when expanded, every term ends with an *initial* letter, hence the lemma is applicable.

$$\therefore\quad\acute{S}\overset{n-1}{\Pi}\grave{A}_n=\acute{S}\overset{n-1}{\Pi}\grave{A}_{n-1}+\acute{S}\overset{n-1}{\Pi}\acute{A}_{n-1}\grave{A}_n$$

$$=\acute{S}\overset{n-2}{\Pi}(1+\grave{A}_{n-1}\kappa_n\acute{A}_{n-1})\grave{A}_{n-1}+\acute{S}\overset{n-1}{\Pi}\acute{A}_{n-1}\grave{A}_n$$

$$=\acute{S}\overset{n-2}{\Pi}\grave{A}_{n-1}+\acute{S}\overset{n-1}{\Pi}\acute{A}_{n-1}\grave{A}_n$$

$$\text{since}\quad\acute{A}_{n-1}\grave{A}_{n-1}=0.$$

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Cor. 2. $\overset{n-1}{\Pi}\grave{A}_n = \overset{n-2}{\Pi}\grave{A}_{n-1} + (\overset{n-1}{\Pi} - 1)\acute{A}_{n-1}\grave{A}_n.$

We can apply the lemma here again to every term of $\overset{n-1}{\Pi}\grave{A}_n$ except the one $1.1.\ldots 1.\grave{A}_n$ which is 1. If the lemma held it would be represented by $\grave{A}_{n-1} + \acute{A}_{n-1}\grave{A}_n$. $\acute{A}_{n-1}\grave{A}_n$ must therefore be subtracted from the right-hand side.

The optical equations are

$$y_1 = a_0 Q A_1 \quad \text{- - - - - -} \quad (1)$$

$$a_1 - a_0 = y_1 \kappa_1 \quad \text{- - - - - -} \quad (2)$$

$$\ldots\ldots\ldots$$

$$y_r - y_{r-1} = a_{r-1} A_{r-1} A_r \quad \text{- - -} \quad (2r-1)$$

$$a_r - a_{r-1} = y_r \kappa_r \quad \text{- - -} \quad (2r)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

The formulæ to be established are

$$\frac{y_r}{a_0} = \acute{Q}\overset{r-1}{\Pi}\grave{A}_r \quad \text{- - - - - -} \quad \text{I.}$$

$$\frac{a_r}{a_0} = \acute{Q}\overset{r}{\Pi} \quad \text{- - - - - -} \quad \text{II.}$$

I. gives $\frac{y_1}{a_0} = \acute{Q}\grave{A}_1$ which agrees with (1).

II. *gives* $\frac{a_1}{a_0} = \acute{Q}(1 + \grave{A}_1\kappa_1\acute{A}_1) = 1 + \kappa_1 Q A_1.$

Assuming that $\frac{y_{r-1}}{a_0} = \acute{Q}\overset{r-2}{\Pi}\grave{A}_{r-1}$ and $\frac{a_{r-1}}{a_0} = \acute{Q}\overset{r-1}{\Pi}$:

from $(2r-1)$,
$$\frac{y_r}{a_0} = \frac{y_{r-1}}{a_0} + \frac{a_{r-1}}{a_0} A_{r-1} A_r$$

$$= \acute{Q}\overset{r-2}{\Pi}\grave{A}_{r-1} + \acute{Q}\overset{r-1}{\Pi}\acute{A}_{r-1}\grave{A}_r$$

$$= \acute{Q}\overset{r-1}{\Pi}\grave{A}_r \qquad \text{by Cor. 1;}$$

and from $(2r)$
$$\frac{a_r}{a_0} = \frac{a_{r-1}}{a_0} + \kappa_r \frac{y_r}{a_0}$$

$$= \acute{Q}\overset{r-1}{\Pi} + \acute{Q}\overset{r-1}{\Pi}\grave{A}_r\kappa_r$$

$$= \acute{Q}\overset{r}{\Pi}.$$

So the formulæ are established by the induction method.

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K is the coefficient of QA_1 in $\frac{a_n - a_0}{a_0}$

$$i.e., \quad K + 1 = \frac{\partial}{\partial Q}\acute{Q}\overset{n}{\Pi}$$

to use an obvious notation

$$= \overset{\prime\prime}{\Pi}.$$

The apparent distance from Q of a point P after the last lens might be worked out separately, but a simpler way is to take an extra lens at P.

$$\text{Apparent distance} \quad = \frac{y_{n+1}}{a_0} = \acute{Q}\overset{n}{\underset{1}{\Pi}}\grave{P}.$$

The symmetry of this expression is important.

If P is R, the image of Q, the apparent distance is 0; or, in other words, the height of a ray diverging from Q is zero at P.

$$0 = \acute{Q}\overset{n}{\underset{1}{\Pi}}\grave{R}.$$

To find the linear magnification, shift Q back a little, $\triangle Q$. $\triangle Q \,.\, a_0$ = height of object, $\triangle y_{n+1} = y_{n+1}$ is height of image.

$\therefore \quad \frac{\partial}{\partial Q}\left(\acute{Q}\overset{n}{\Pi}\grave{R}\right)$ is the linear magnification, *i.e.*, $\overset{n}{\Pi}\grave{R}$.

Similarly $\acute{Q}\overset{n}{\Pi}$ is the reciprocal.

This is $a_n/_{a_0}$ in accordance with Helmholtz's Theorem.

$$\text{So we have} \qquad \left[\acute{Q}\overset{n}{\Pi}\right]\left[\overset{n}{\Pi}\grave{R}\right] = 1$$

where the square bracket means arithmetical value.

This may be written

$$\left[\acute{Q}\grave{A}_1\left(\overset{n}{\underset{1}{\Pi}} - 1\right) + \acute{A}_1\overset{n}{\underset{2}{\Pi}}\right]\left[\overset{n-1}{\underset{1}{\Pi}}\grave{A}_n + \left(\overset{n}{\Pi} - 1\right)\acute{A}_n\grave{R}\right] = 1$$

$$\text{or} \qquad \left[QA \,.\, K \quad + \frac{\partial K}{\partial \kappa_1}\right]\left[\frac{\partial K}{\partial \kappa_n} \quad + KA_nR\right] = 1.$$

The other equation is

$$\acute{Q}\overset{n}{\underset{1}{\Pi}}\grave{R} = 0$$

$$i.e., \quad \acute{Q}\overset{n-1}{\underset{1}{\Pi}}\grave{A}_n + \acute{Q}\overset{n}{\underset{1}{\Pi}} \,.\, \acute{A}_nR = 0$$

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$$\text{or}\quad \acute{Q}\grave{A}_1 \prod_1^{n-1} + \acute{A}_1 \prod_2^{n-1} \grave{A}_n + \acute{Q}\grave{A}_1\left(\prod_1^{n} - 1\right)\acute{A}_n R$$
$$+ \acute{A}_1 \prod_2^{n} \acute{A}_n \grave{R}$$

$$\text{or}\quad K\,.\,QA_1\,.\,A_1R + QA\,.\,\frac{\partial K}{\partial \kappa_n} + AR\,.\,\frac{\partial K}{\partial \kappa_1} + \frac{\partial^2 K}{\partial \kappa_1 \partial \kappa_n} = 0.$$

Comparing these equations we derive the identity

$$\frac{\partial K}{\partial \kappa_1}\,.\,\frac{\partial K}{\partial \kappa_n} - K\frac{\partial^2 K}{\partial \kappa_1 \partial \kappa_n} = 1.$$

This we may verify directly. Call the quantity on the left c_n.

$$c_n = \left[\prod_1^{n-1} \grave{A}_n\right]\left[\acute{A}_1 \prod_2^{n}\right] - \left[\prod_1^{n} - 1\right]\left[\acute{A}_1 \prod_2^{n-1} \grave{A}_n\right]$$

(separate κ_n)

$$= \left[\prod_1^{n-1} \grave{A}_n\right]\left[\acute{A}_1 \prod_2^{n}\right] - \left[\prod_1^{n-1} - 1\right]\left[\acute{A}_1 \prod_2^{n-1} \grave{A}_n\right]$$
$$+ \left[\prod_1^{n-1} \grave{A}_n\right]\left[\acute{A}_1 \prod_2^{n} \grave{A}_n \kappa_n\right] - \left[\prod_1^{n-1} \grave{A}_n \kappa_n\right]\left[\acute{A} \prod_2^{n-1} \grave{A}_n\right]$$

where the second line is zero

$$= \left[\prod_1^{n-2} \grave{A}_{n-1} + \left(\prod_1^{n-1} - 1\right)\acute{A}_{n-1}\grave{A}_n\right]\left[\acute{A}_1 \prod_2^{n-1}\right]$$
$$- \left[\prod_1^{n-1} - 1\right]\left[\acute{A}_1 \prod_2^{n-2} \grave{A}_{n-1} + \acute{A}_1 \prod_2^{n-1} \acute{A}_{n-1}\grave{A}_n\right]$$

by corollaries to lemma,

$$= c_{n-1}.$$

Now
$$c_2 = (1 + A_1\kappa_1A_1)A_2\,.\,A_1(1 + A_2\kappa_2A_2)$$
$$- [(1 + A_1\kappa_1A_1)(1 + A_2\kappa_2A_2) - 1]A_1A_2$$
$$= (1 + \kappa_1A_1A_2)1 + A_1A_2\kappa_2)$$
$$- (\kappa_1 + \kappa_2 + \kappa_1A_1A_2\kappa_2)A_1A_2$$
$$= 1.$$

So the identity is verified.

Thick lenses may be treated in the usual way by reduced distances.