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ON CONVEX AND STARLIKE UNIVALENT FUNCTIONS

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In this paper we obtain some classical results by using the general integral operators which transform Jakubowski's class $K(m, M)$ into itself and $K(\mu) \times S(m, M)$ into $K(\mu)$. Our results generalize some recent known results due to Causey and Reade, Patil and Thakare.

1. Introduction

Let S denote the family of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are regular and univalent in the unit disc $D = \{z : |z| < 1\}$ with the normalization $f(0) = 0 = f'(0) - 1$. Let K and S^* be the subfamily of S whose members map D onto a domain which are respectively convex and starlike with respect to origin. Robertson [4] defined the convex and starlike functions of order μ for functions $f \in S$.

A function f of S belongs to the class $K(\mu)$, convex functions of order μ, $0 \le \mu < 1$, if

$$\text{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > \mu , \quad z \in D . \tag{1.1}$$

A function f of S belongs to the class $S^*(\mu)$, starlike functions of order μ, $0 \le \mu < 1$, if

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$$\text{(1.2)} \qquad \operatorname{Re}\left\{z\,\frac{f'(z)}{f(z)}\right\} > \mu\ ,\quad z \in D\ .$$

Jakubowski [2] defined the classes $K(m, M)$ and $S(m, M)$ of functions $f \in S$.

A function f of S belongs to the class $K(m, M)$ if the following condition is satisfied:

$$\text{(1.3)} \qquad \left|1 + \frac{zf''(z)}{f'(z)} - m\right| < M\ ,\quad z \in D\ ,\quad (m, M) \in E\ .$$

A function f of S belongs to the class $S(m, M)$ if the following condition is satisfied:

$$\text{(1.4)} \qquad \left|z\,\frac{f'(z)}{f(z)} - m\right| < M\ ,\quad z \in D\ ,\quad (m, M) \in E$$

where

$$\text{(1.5)} \qquad E = \{(m, M)\ :\ |m-1| < M \le m\}\ .$$

Evidently

$$\text{(1.6)} \qquad K(m, M) \subset K(m-M) \subset K(0) \subset S$$

and

$$\text{(1.7)} \qquad S(m, M) \subset S^*(m-M) \subset S^*(0) \subset S\ .$$

Recently Patil and Thakare [3] established the following result.

THEOREM A. *Let* γ *be a real number and* $f \in K(\mu)$ *; then the function* F *defined by*

$$\text{(1.8)} \qquad F(z) = \int_0^z \{f'(u)\}^{\gamma} du$$

belongs to $K(\eta)$ *for* $0 \le \gamma \le (1-\eta)/(1-\mu)$.

In 1982, Causey and Reade [1] established the following result.

THEOREM B. *Let* α, β *be real numbers and* $f, g \in K$ *; then the function* F *defined by*

$$\text{(1.9)} \qquad F(z) = \int_0^z \{f'(u)\}^{\alpha}\left\{\frac{g(u)}{u}\right\}^{\beta} du$$

belongs to K *only for those* (α, β) *in the closed convex hull of the points* $(0, 0)$, $(1, 0)$ *and* $(0, 2)$.

In this paper we study the integral operators of the form (1.8) and (1.9) which transform $K(m, M)$ into itself and $K(\mu) \times S(m, M)$ into $K(\mu)$ where β and γ are complex numbers. Powers in (1.8) and (1.9) are principal ones. Our results generalize the known results of Patil and Thakare [3], Causey and Reade [1].

2. Preliminary lemmas

LEMMA. *If $f \in S$ then f is in $K(m, M)$ if and only if there exists a regular function $w(z)$ in D, satisfying $w(0) = 0$ and $|w(z)| < 1$ for z in D, such that*

$$(2.1) \qquad 1 + \frac{zf''(z)}{f'(z)} = \frac{1+aw(z)}{1-bw(z)}, \quad z \in D,$$

where

$$(2.2) \qquad a = \frac{M^2-m^2+m}{M}, \quad b = \frac{m-1}{M} \text{ and } (m, M) \in E .$$

Proof. To prove our lemma we require the following result.

A general bilinear transformation which maps the circular disc $|z| \le \rho$ into the circular disc $|w_1| \le \rho'$ can be put in the form

$$(2.3) \qquad w_1 = \rho\rho' e^{i\lambda} \frac{z-\alpha}{\bar{\alpha}z-\rho^2}, \quad (|\alpha| < \rho) .$$

Since $f \in K(m, M)$, $|1+z(f''(z)/f'(z))-m| < M$. Let us take

$$(2.4) \qquad p(z) = 1 + z\frac{f''(z)}{f'(z)} - m \text{ so that } |p(z)| < M ;$$

from (2.3) we have

$$(2.5) \quad p(z) = \rho\rho' e^{i\lambda} \frac{w_1(z)-\alpha}{\bar{\alpha}w_1(z)-\rho^2} = Me^{i\lambda} \frac{w_1(z)-\alpha}{\bar{\alpha}w_1(z)-1}, \quad \rho = 1, \quad \rho' = M .$$

From (2.4) and (2.5) we get

$$p(0) = (1-m) = Me^{i\lambda}(\alpha) \text{ or } \alpha = \big((1-m)/M\big)e^{-i\lambda}, \quad |\alpha| < 1 .$$

Substituting the value of α in (2.5) we have

$$(2.6) \qquad \left\{1+z\frac{f''(z)}{f'(z)}\right\} = m + \frac{Me^{i\lambda}w_1(z)-\big((1-m)/M\big)e^{i\lambda}}{\big((1-m)/M\big)e^{i\lambda}w_1(z)-1} .$$

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Rearranging (2.6) by using $w(z) = -e^{i\lambda}w_1(z)$, $w(0) = 0$, $|w(z)| < 1$ and (2.2), we get (2.1).

Conversely suppose that $f(z)$ satisfies (2.1). Then

$$(2.7) \qquad 1 + z\frac{f''(z)}{f'(z)} - m = M\frac{\{(1-m)/M\}+w(z)}{1+\{(1-m)/M\}w(z)} = Mh(z)$$

say. Since $|(1-m)/M| < 1$, the function h given by

$$h(z) = \frac{\{(1-m)/M\}+w(z)}{1+\{(1-m)/M\}w(z)}$$

satisfies $|h(z)| < 1$. Hence from (2.7) it follows now that the condition (2.1) is satisfied. This completes the proof of the lemma.

REMARK. Let us choose $m = N(1-\mu) + \mu$ and $M = N(1-\mu)$ where $N \geq 1$ and $0 \leq \mu < 1$. Then $|m-1| < M \leq m$, $a = \mu/N + (1-2\mu)$ 'and $b = 1 - 1/N$ also the condition $|1+z(f''(z)/f'(z))-m| < M$ can be written as

$$(2.8) \qquad \left|\frac{1+z(f''(z)/f'(z))-\mu}{1-\mu} - N\right| < N , \quad z \in D .$$

Now, as $N \to \infty$, $a \to (1-2\mu)$ and $b \to 1$, the condition (2.8) reduces to $\mathrm{Re}\{1+z(f''(z)/f'(z))\} > \mu$, $z \in D$. In this case the relation (2.1) becomes

$$1 + z\frac{f''(z)}{f'(z)} = \frac{1+(1-2\mu)w(z)}{1-w(z)} , \quad z \in D ,$$

which is a necessary and sufficient condition for f to be in $K(\mu)$.

3. Class preserving integral operator for $K(m, M)$

THEOREM. *Let* γ *be a complex number and* $f \in K(m, M)$; *then the function* F *defined by*

$$(3.1) \qquad F(z) = \int_0^z \{f'(u)\}^\gamma du$$

belongs to $K(m, M)$ *for* $0 \leq |\gamma| \leq \tfrac{1}{2}(1-b)$.

Proof. Let us choose a function w such that

$$(3.2) \qquad 1 + z\frac{F''(z)}{F'(z)} = \frac{1+aw(z)}{1-bw(z)}$$

where $w(0) = 0$ and w is either regular or meromorphic in D . From

(3.1) we have

$$F'(z) = \{f'(z)\}^{\gamma} . \tag{3.3}$$

Differentiating logarithmically (3.3) with respect to z and using (3.2) we get

$$\frac{zf''(z)}{f'(z)} = \frac{(a+b)w(z)}{\gamma\{1-bw(z)\}} \tag{3.4}$$

or

$$1 + z\frac{f''(z)}{f'(z)} - m = \frac{(1-m)\gamma+[a+b\{1-(1-m)\gamma\}]w(z)}{\gamma\{1-bw(z)\}} . \tag{3.5}$$

Let z_1 with $|z_1| = r_0$ be the nearest pole of $w(z)$ in D. Hence $w(z)$ is regular in $|z| < r_0 < 1$. Thus for $|z|$ $r < r_0$ there is a point z_0 for which

$$1 + z_0\frac{f''(z_0)}{f'(z_0)} - m = \frac{(1-m)\gamma+[a+b\{1-(1-m)\gamma\}]w(z_0)}{\gamma\{1-bw(z_0)\}} \equiv \frac{N(z_0)}{D(z_0)} \tag{3.6}$$

where

$$N(z_0) = (1-m)\gamma + [a+b\{1-(1-m)\gamma\}]w(z_0) , \tag{3.7}$$

$$D(z_0) = \gamma(1-bw(z_0)) . \tag{3.8}$$

Now suppose that it were possible to have $M(r, w) = \max_{|z|=r} w(z_0) = 1$ for some $r < r_0 < 1$. At the point z_0 where this occurred we would have $|w(z_0)| = 1$.

CASE 1. When $\text{Re}(\gamma) \geq 0$, $\text{Im}(\gamma) \geq 0$ and $\text{Re}(\gamma) \geq 0$, $\text{Im}(\gamma) < 0$,

$$|N(z_0)|^2 = (a+b)^2 + (1+b^2)M^2|\gamma|^2 - 2(a+b)M\,\text{Re}\{\gamma w(z_0)\} - 2bM^2|\gamma|^2\text{Re}(w(z_0)) + 2bM(a+b)\text{Re}(\gamma) , \tag{3.9}$$

$$|D(z_0)|^2 = (1+b^2)|\gamma|^2 - 2b|\gamma|^2\text{Re}(w(z_0)) . \tag{3.10}$$

CASE 2. When $\text{Re}(\gamma) < 0$, $\text{Im}(\gamma) < 0$ and $\text{Re}(\gamma) < 0$, $\text{Im}(\gamma) \geq 0$,

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$$(3.11)\quad |N(z_0)|^2 = (a+b)^2 + (1+b^2)M^2|\gamma|^2 + 2(a+b)M\,\mathrm{Re}\{\gamma w(z_0)\} - 2bM^2|\gamma|^2\mathrm{Re}\{w(z_0)\} - 2bM(a+b)\mathrm{Re}(\gamma)$$

and

$$(3.12)\quad |D(z_0)|^2 = (1+b^2)|\gamma|^2 - 2b|\gamma|^2\mathrm{Re}\{w(z_0)\} .$$

Now for each case

$$|N(z_0)|^2 - M^2|D(z_0)|^2 \geq (a+b)^2 - 2M(a+b)(1+b)|\gamma| \geq 0 \quad \text{for} \quad |\gamma| \leq \tfrac{1}{2}(1-b) .$$

Thus from (3.6) it follows that

$$\left|1+z_0\frac{f''(z_0)}{f'(z_0)} - m\right| > M \quad \text{for} \quad |\gamma| \leq \tfrac{1}{2}(1-b) .$$

But this is contrary to the fact that $f \in K(m, M)$. So we cannot have $M(r, w) = 1$. Thus $|w(z)| \neq 1$ in $|z| < r_0$. Since $|w(0)| = 0$, $|w(z)|$ is continuous and $|w(z)| \neq 1$ in $|z| < r_0$, $w(z)$ cannot have a pole at $|z_1| = r_0$. Therefore $w(z)$ is regular and $|w(z)| < 1$ for z in D .

Hence $F \in K(m, M)$.

4. Integral operator that maps $K(\mu) \times S(m, M)$ into $K(\mu)$

THEOREM. *Let α be a non zero positive real number and β be a complex number such that $0 \leq |\beta| \leq -(\alpha-1)/2M$, $(m, M) \in E$.*

Let $f \in K(\mu)$ and $g \in S(m, M)$; then the function F defined by

$$(4.1)\quad F(z) = \int_0^z \{f'(u)\}^\alpha\left\{\frac{g(u)}{u}\right\}^\beta du$$

belongs to $K(\mu)$.

Proof. Let us choose a function w such that

$$(4.2)\quad 1 + z\frac{f''(z)}{f'(z)} = \frac{1+(2\mu-1)w(z)}{1+w(z)}$$

where $w(0) = 0$ and w is either regular or meromorphic in D.

Differentiating (4.1) with respect to z we have

$$(4.3) \qquad F'(z) = \{f'(z)\}^{\alpha}\left\{\frac{g(z)}{z}\right\}^{\beta} .$$

Differentiating logarithmically (4.3) with respect to z and using (4.2) we have

$$(4.4) \quad \left\{1+z\,\frac{f''(z)}{f'(z)}\right\} = \frac{\alpha+(1-m)\beta}{\alpha} - \frac{\beta}{\alpha}\left\{z\,\frac{g'(z)}{g(z)} - m\right\} + \frac{2(\mu-1)}{\alpha}\,\frac{w(z)}{1+w(z)} .$$

Let z_1 with $|z_1| = r_0$ be the nearest pole of $w(z)$ in D. Hence $w(z)$ is regular in $|z| < r_0 < 1$. Thus for $|z| \le r < r_0$ there is a point z_0 for which

$$(4.5) \quad \left\{1+z_0\,\frac{f''(z_0)}{f'(z_0)}\right\} = \frac{\alpha-(m-1)\beta}{\alpha} - \frac{\beta}{\alpha}\left\{z_0\,\frac{g'(z_0)}{g(z_0)} - m\right\} + \frac{2(\mu-1)}{\alpha}\,\frac{w(z_0)}{1+w(z_0)} ,$$

or

$$(4.6) \quad \operatorname{Re}\left\{1+z_0\,\frac{f''(z_0)}{f'(z_0)}\right\} \le \frac{\alpha+(m-1)|\beta|}{\alpha} + \frac{|\beta|}{\alpha}\left|z_0\,\frac{g'(z_0)}{g(z_0)} - m\right| + \frac{2(\mu-1)}{\alpha}\,\frac{\operatorname{Re}w(z_0)+|w(z_0)|^2}{1+2\operatorname{Re}w(z_0)+|w(z_0)|^2} .$$

Now suppose that it were possible to have $M(r, w) = \max_{|z|=r} |w(z_0)| = 1$ for some $r < r_0 < 1$. At the point z_0 where this occurred we would have

$$(4.7) \qquad \operatorname{Re}\left\{1+z_0\,\frac{f''(z_0)}{f'(z_0)}\right\} < \frac{\alpha+M|\beta|}{\alpha} + \frac{|\beta|}{\alpha}\,M + \frac{(\mu-1)}{\alpha}$$

$$\le \mu \quad \text{for} \quad |\beta| \le -\frac{(\alpha-1)}{2M} .$$

But this is contrary to the fact that $f \in K(\mu)$. So we cannot have $M(r, w) = 1$. Thus $|w(z)| \ne 1$ in $|z| < r_0$. Since $w(0) = 0$, $|w(z)|$ is continuous in $|z| < r_0$ and $|w(z)| \ne 1$ where $w(z)$ cannot have a pole at $|z_1| = r_0$. Therefore $|w(z)| < 1$ and $w(z)$ is regular in D.

Hence from (4.2) it follows that $F \in K(\mu)$.

APPLICATIONS. By using the same techniques, we can also study the following types of integral operators of the forms

$$\text{(i)} \quad F(z) = \int_0^z \{f(u)/u\}^\beta du ,$$

$$\text{(ii)} \quad F(z) = \int_0^z \{f'(u)\}^\alpha \{g'(u)\}^\beta du , \text{ and}$$

$$\text{(iii)} \quad F(z) = \int_0^z \{f(u)/u\}^\alpha \{g(u)/u\}^\beta du ,$$

which transform $S(m, M)$ into $K(m, M)$, $K(\mu) \times K(m, M)$ into $K(\mu)$ and $S^*(\mu) \times S(m, M)$ into $K(\mu)$ respectively where α is a real number and β is a complex number.

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