ON A PROBLEM OF CHEVALLEY

HIDEO KUNIYOSHI

In the present note we wish to deal with the same problem as the preceding paper [1] for the case of modular fields.

Let k be a field of characteristic $p \neq 0$ and $K = k(x_1, \ldots, x_p)$ a purely transcendental extension of k. Let S be the automorphism of K which is induced by the cyclic permutation $(x_1, \ldots, x_p)$ and L the fixed subfield of S. Then *L is a purely transcendental extension field over k.*

Proof. We put

$$u_1 = x_1 + x_2 + x_3 + \ldots + x_p,$$
$$u_2 = (1x_2 + 2x_3 + \ldots + (p-1)x_p)/u_1,$$
$$u_3 = (1^2 x_2 + 2^2 x_3 + \ldots + (p-1)^2 x_p)/u_1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$u_p = (1^{p-1} x_2 + 2^{p-1} x_3 + \ldots + (p-1)^{p-1} x_p)/u_1.$$

Since $u_1, u_2 u_1, u_3 u_1, \ldots, u_p u_1$ are linear forms in $x_1, \ldots, x_p$ and their determinant is $\prod_{p > i > j \geqq 0} (i-j) \neq 0$,

$$K = k(x_1, \ldots, x_p) = k(u_1, u_2 u_1, \ldots, u_p u_1) = k(u_1, u_2, \ldots, u_p).$$

To see the effect of S on u_i, we compute $S^{-1}u_i - u_i (= \varDelta u_i)$ intstead of Su_i.

$$\varDelta u_1 = 0,$$
$$\varDelta u_2 = 1,$$
$$\varDelta u_3 = 2u_2 + 1,$$
$$\ldots\ldots\ldots$$
$$\varDelta u_{i+1} = \binom{i}{1} u_i + \binom{i}{2} u_i + \ldots + \binom{i}{i-1} u_2 + 1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

From these u_i we now construct new elements $v_2 (= u_2), v_3, \ldots, v_p \in K$ such that

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$$\Delta v_i = 1,$$
$$v_i = u_i + f_i(v_2, \ldots, v_{i-1}),$$

where f_i is a linear form of v_j^e, $j = 2, \ldots, i-1$, $e = 0, \ldots, i-1$, with coefficients in the prime field. We take, at first, $v_2 = u_2$, $v_3 = u_3 - u_2^2 + u_2$ and construct them by induction. If we get first $i-2$ terms $v_2, \ldots, v_{i-1}$, then v_i is obtained as follows:

$$\begin{aligned} \Delta u_i &= \binom{i-1}{1} u_{i-1} + \binom{i-1}{2} u_{i-2} + \ldots + 1 \\ &= \binom{i-1}{1}(v_{i-1} - f_{i-1}(v_2, \ldots, v_{i-2})) \\ &\quad + \binom{i-1}{2}(v_{i-2} - f_{i-2}(v_2, \ldots, v_{i-3})) + \ldots + 1. \end{aligned} \tag{1}$$

The right side of this relation is a linear form of v_j^e, $j = 2, \ldots, i-2$, $e = 0, \ldots, i-2$. We compute Δv_j^2, using the inductive assumption $\Delta v_j = 1$,

$$\begin{aligned} &\Delta v_j = 1, \\ &\Delta v_j^2 = 1 + 2v_j, \\ &\Delta v_j^3 = 1 + 3v_j + 3v_j^2, \\ &\ldots\ldots\ldots\ldots\ldots \\ &\Delta v_j^e = 1 + \binom{e}{1} v_j + \binom{e}{2} v_j^2 + \ldots + \binom{e}{e-1} v_j^{e-1}. \end{aligned}$$

From these relations we solve v_j^e in a linear form of $\Delta v_j^{e'}$.

$$v_j^2 = h_j(\Delta v_j, \Delta v_j^2, \ldots, \Delta v_j^{e+1}) = \Delta h_j(v_j, v_j^2, \ldots, v_j^{e+1}), \quad 1 \leqq e \leqq i-2 < p, \tag{2}$$

where h_j is a linear form in its arguments. We put (2) into (1), then

$$\Delta u_i = \Delta g_i(v_2, \ldots, v_{i-1}),$$

where g_i is a linear form of v_j^e, $j = 2, \ldots, i-1$, $e = 0, \ldots, i-1$. Since

$$\Delta [u_i - g_i(v_2, \ldots, v_{i-1})] = 0,$$

the element

$$u_i - g_i(v_2, \ldots, v_{i-1}) + v_2$$

satisfies the inductive assumption and we may take it as v_i.

Now, we construct algebraically independent generators of L over k. We put

$$w_1 = u_1,$$
$$w_2 = v_2^p - v_2,$$
$$w_i = v_i - u_2, \qquad i = 3, \dots, p.$$

Then $$\Delta w_i = 0, \qquad i = 1, \dots, p,$$

hence $$k(w_1, \dots, w_p) < L.$$

On the other hand

$$[k(w_1, \dots, w_p, v_2) : k(w_1, \dots, w_p)] \leqq p,$$
$$k(w_1, \dots, w_p, v_2) \supset k(u_i, \dots, u_p) = K,$$

and $[K : L] = p$. Therefore

$$L = k(w_1, \dots, w_p).$$

Since L is an extension field of dimension (degree of transcendency) p over k, we see that $w_1, \dots, w_p$ are algebraically independent over k.

Reference

[1] K. Masuda: On a problem of Chevalley, this journal.

Mathematical Institute
Tôhoku University, Sendai

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