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Invariant means and fixed point properties on completely regular spaces

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Two theorems are presented which characterize the existence of multiplicative left invariant means on a given algebra of unbounded continuous functions on a topological semigroup S in terms of certain common fixed point properties of actions of S on completely regular spaces. Also a lattice formulation of a related result of Theodore Mitchell for the case of bounded functions is shown to be equivalent to a certain common fixed point property on Bauer simplexes.

1. Introduction and preliminaries

Let S be a semigroup and H a translation invariant closed subalgebra of $m(S)$ that contains the constants. Mitchell has shown in [13, Theorem 1] that the existence of a multiplicative left invariant mean on H is equivalent to the pair S, H enjoying a common fixed point property with respect to certain actions of S on compact spaces. Given a topological semigroup S and a common fixed point property on compact spaces that S might possess, Mitchell's theorem (as well as Argabright's geometric analogue [1]) has proved to be an important tool for finding that function space on S whose extreme left amenability (left amenability) is equivalent to the given fixed point property (see [14] and for questions of existence [9]). In this note we obtain (using similar techniques) two results analogous to Mitchell's theorem, in the spirit of [10, Proposition 2], for function algebras of unbounded functions. The corresponding fixed

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point properties involve actions on completely regular spaces and actions on realcompact spaces. We also give a lattice formulation of a part of Mitchell's result and show that it is equivalent to a certain common fixed point property on Bauer simplexes.

Our notation and terminology is for the most part standard and we refer the reader to [13], [1], [9], and [10]. If X is a topological space, then $C(X)$ denotes all real-valued continuous functions on X (not just the bounded ones). For the purpose of this paper, a topological space X for which $C(X)$ separates points, is called a *separating space*. We refer to [9, p. 18] for the sense in which we use semitopological semigroup and topological semigroup. Let S be a semitopological semigroup, H a subset of $C(S)$ and X a topological space. We denote an action of S on X by (S, X) and employ a slightly addended version of Mitchell's notion of an E-representation [13] (*cf.* Day's concept of a slightly continuous action [4] as well as [10, p. 113]) which takes into account the topology of S and is equivalent to his definition when X is compact (see the remark after Theorem 1 below). An action (S, X) is an *E-representation of* S, H *on* X if there is an x in X such that the map $s \to sx$ is continuous and $Tx(C(X)) \subset H$ where $Txf(s) = f(sx)$.

2. Compact-open continuous means

The validity of the following lemma is indicated in [11, Remark 1.7, p. 164]. However, it seems to the author that a proof using Proposition 1.2 in [11, p. 160] (see also Remark 1.4) would require, in addition, that $C(X)$ be complete in the compact-open topology in order that $C(X)$ be the projective limit of the spaces $C(K)$, K a compact subset of X . We give an alternative proof that is free of this restriction.

LEMMA. *Let X be a separating space. If m is a non-zero multiplicative linear functional on $C(X)$ that is compact-open continuous, then there is a (unique) x_0 in X such that $m(f) = f(x_0)$ for all f in $C(X)$.*

Proof. If X is realcompact, then every non-zero multiplicative linear functional on $C(X)$ is evaluation at some point [6, p. 142]. If X is completely regular but not realcompact, then every non-zero multiplicative linear functional on $C(X)$ that is not an evaluation functional is not compact-open continuous (see the proof of Theorem 4.5 and Theorem 5.2 in [7]). Suppose $C(X)$ just separates points. Let X_T denote X equipped with the weak topology induced by $C(X)$. Then X_T is completely regular (see, for example, [6, p. 40]) and $C(X) = C(X_T)$. Consequently, if m is a non-zero compact-open continuous multiplicative linear functional on $C(X)$, then it is the same on $C(X_T)$ and therefore, evaluation at some x_0.

REMARK. Jonathan Lewin has indicated to the author a simple non-measure theoretic proof of the above lemma for the case X completely regular. The following adaptation of his argument shows that the lemma is, in fact, valid for any topological space X (of course, x_0 then need not be unique). Let $E : X \to R^{C(X)}$ be the evaluation map and equip $R^{C(X)}$ with the product topology. Then the set of non-zero multiplicative linear functionals on $C(X)$ coincides with Y, the closure of $E(X)$ ($E : X \to Y$ enjoys the universal property of a realcompactification, but E is not an embedding unless X is completely regular). Fix $m \in Y - E(X)$ and for each compact $K \subset X$, let $f_K \in C(Y)$ be such that f_K is 1 on $E(K)$ and $f_K(m) = 0$ (Y is completely regular). Let $g_K = f_K \circ E$ and consider the net $\{g_K\}$ where K ranges over all compact subspaces of X upwardly directed. Then $\{g_K\}$ converges in the compact-open topology of $C(X)$ to the function identically one on X but $m(g_K) = f_K(m) = 0$ for all K (since $m \in Y$). Thus, m is not compact-open continuous.

If S is a semigroup and H a set of functions on S, then we say an $s_0 \in S$ is an *H-right zero* if for all $s \in S$ and $h \in H$, $h(ss_0) = h(s_0)$ (that is, evaluation on H at s_0 is a left invariant mean when H is left translation invariant).

THEOREM 1. *If S is a semitopological semigroup and H is a left translation invariant subalgebra of $C(S)$ that contains the constants, then each of the conditions below implies the next one. If, in addition, S is a topological semigroup, H is right translation invariant and compact-open closed in $C(S)$, then all four statements are equivalent.*

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(1) *S possesses an H-right zero.*

(2) *H admits a compact-open continuous multiplicative left invariant mean.*

(3) *S, H has the common fixed point property on separating spaces with respect to E-representations.*

(4) *S, H has the common fixed point property on completely regular spaces with respect to E-representations.*

Proof. (1) $\to$ (2). If s_0 is an H-right zero, then the evaluation functional on H at s_0 is certainly compact-open continuous.

(2) $\to$ (3). This implication is proved exactly as in the proof of Proposition 1 in [*10*] making use of the above lemma.

(3) $\to$ (4). Condition (3) is formally stronger than (4).

(4) $\to$ (1). Suppose S and H have the additional properties of the theorem and that (4) holds. Consider the canonical map $E : S \to H^*$ (the dual of H where H has the compact-open topology) where $E(s)$ is the evaluation functional on H at s. If we equip $Y = E(S)$ with the restriction of the w^*-topology, then Y is completely regular (Y is uniformizable). For each h in H, let $\hat{h} \in C(Y)$ be defined by $\hat{h}(\mu) = \mu(h)$ for all $\mu \in Y$. Then $\hat{H} = \{\hat{h} \mid h \in H\}$ is a subalgebra of $C(Y)$ that contains the constants and separates points. Since H is compact-open closed in $C(S)$, it is easy to check that $\hat{H}$ is compact-open closed in $C(Y)$ (since E is continuous). By the Stone-Weierstrass theorem (see, for example, [5, p. 282] for the compact-open setting), $\hat{H} = C(Y)$. If we consider the canonical action (S, Y) where $s \cdot \mu(h) = \mu\left(h_s\right)$ for all $h \in H$, then it follows exactly as in [*13*, p. 120] (from the right translation invariance of H) that (S, Y) is an E-representation (in fact, an A-representation) of the pair S, H on Y. Consequently, there is a common fixed point μ_0 for the action (S, Y). Any s_0 such that $E\left(s_0\right) = \mu_0$ is an H-right zero.

REMARKS. Condition (1) implies directly a slightly stronger version of (3). For if S satisfies (1) and (S, Y) is an action of S on Y such that there is a y_0 with $Ty_0\left(C(Y)\right) \subset H$, then $C(Y)$ separating

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implies that s_0y_0 is a common fixed point whenever s_0 is an H-right zero. We also note that for completely regular Y, the condition $Ty_0(C(Y)) \subset H$ implies that the map $s \to sy_0$ is continuous (since the topology induced by $C(Y)$ coincides with the given topology).

The following corollary gives completely regular analogues of Proposition 4.1 and Corollary 4.2 in [9]. Some of the implications in the corollary follow from the proof of Theorem 1 rather than directly from the statement of Theorem 1. For the implication (3) → (2) see the remark after the above lemma.

COROLLARY 1. *The following implications hold for the conditions listed below. If S is a semitopological semigroup, then each condition implies the next one and* (2) *and* (3) *are equivalent. If S is semitopological and left multiplication is slightly continuous, then* (2), (3), *and* (4) *are equivalent. If S is a separating semitopological semigroup, then* (1), (2), *and* (3) *are equivalent. If S is a topological semigroup, then* (2), (3), (4), *and* (5) *are equivalent.*

(1) *S has a right zero.*

(2) *S has a $C(S)$-right zero.*

(3) *$C(S)$ admits a compact-open continuous multiplicative left invariant mean.*

(4) *S has the common fixed point property on completely regular spaces with respect to slightly continuous actions.*

(5) *S has the common fixed point property on completely regular spaces with respect to separately continuous actions.*

REMARK. In [9, Corollary 4.4] it was shown that for discrete S of non-measurable cardinal, S has a right zero if and only if R^S (all real-valued functions on S) has a multiplicative left invariant mean. The above shows, in particular, that for arbitrary discrete S, S has a right zero if and only if R^S has a pointwise continuous multiplicative left invariant mean.

The proof of (2) → (3) in the corollary below is a slight modification of the proof of (P1) → (F1) of Theorem 1 in [14] making use of (2) → (4) of

Theorem 1 above. (See [*10*, p. 111] for the definition of LCC(S) .)

COROLLARY 2. *If* S *is a topological semigroup, then each of the following conditions implies the next.*

(1) S *has a* LCC(S)*-right zero.*

(2) LCC(S) *admits a compact-open continuous multiplicative left invariant mean.*

(3) S *has the common fixed point property on completely regular spaces with respect to jointly continuous actions.*

REMARK. It is of interest to note here a result of Granirer and Lau [*8*]. Namely, if S is a subsemigroup of a locally compact group and LUC(S) (see, for example, [*14*] or [*8*] for the definition) admits a multiplicative left invariant mean, then $S = \{e\}$.

3. Related results

The theorem stated below is a realcompact analogue of Theorems 1 and 2 in [*13*]. The implication (1) → (2) is implicit in the proof of Proposition 4.1 in [9]. One verifies the implication (3) → (1) (respectively, (4) → (1)) exactly as in the proof of (4) → (1) of Theorem 1 above choosing Y to be the closure of the set of evaluation functionals on H (respectively, all multiplicative means on H) in the product topology (Y is then realcompact [*6*, p. 119]).

THEOREM 2. *If* S *is a semitopological semigroup and* H *is a left translation invariant subalgebra of* $C(S)$ *that contains the constants, then statement* (1) *implies statement* (2) (*and of course,* (2) *implies* (3) *and* (3) *implies* (4)). *If, in addition,* S *is a topological semigroup,* H *is right translation invariant and compact-open closed in* $C(S)$, *then* (1), (2), *and* (3) *are equivalent.*

(1) H *admits a multiplicative left invariant mean.*

(2) S, H *has the common fixed point property on realcompact spaces with respect to* E*-representations.*

(3) S, H *has the common fixed point property on realcompact spaces with respect to* D*-representations.*

Furthermore, if H *is also left* M*-introverted (that is, for every*

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$h \in H$ and multiplicative mean m on H, $m(h_s)$ as a function of s lies in H), then (1), (2), *and* (3) *are equivalent to*

(4) *S, H has the common fixed point property on realcompact spaces with respect to A-representations.*

The following theorem is essentially a lattice reformulation of a part of Mitchell's Theorem [13, Theorem 1]. One proof of the equivalence of conditions (1) and (3) below follows exactly as in [13] relying on Kakutani's Theorem on abstract M-spaces. (In the proof of (3) → (1) of Theorem 1 in [13] one can apply directly (as was done in the proof of Theorem 1 above) the Stone-Weierstrass Theorem in algebra form rather than Kakutani's Theorem.) We give an alternative proof which passes through a geometric fixed point property (condition (2) below) and uses the theory of Bauer simplexes [2] (a compact Choquet simplex whose set of extreme points is closed). $A(K)$ denotes the space of all real continuous affine functions on the compact convex K and ex K the set of extreme points of K. It should be noted that a space of bounded real functions on a set X can be a vector lattice with respect to the pointwise ordering without being a sublattice of $m(X)$ (for example, the space of real continuous functions on the closed unit disc that are harmonic inside) [2].

We recall that a compact convex set K is a Bauer simplex if and only if $A(K)$ is a vector lattice [2, p. 120]. The proof of (1) → (2) below relies on the fact (see [2], [16]) that if $A(K)$ is a vector lattice, then the lattice preserving means on $A(K)$ are precisely the evaluation functionals at extreme points. An immediate proof of this fact can be obtained by considering the canonical map $E : K \to Y$ where Y is the space of means on $A(K)$ in the w^*-topology and $E(x)$ is the evaluation functional at x. It is well-known that E is a surjective affine homeomorphism (for example, since barycenters exist [2, p. 122]) and since ex Y is precisely the set of lattice preserving means on $A(K)$ (see the proof of (2) → (1) below) we are finished.

THEOREM 3. *Let S be a semigroup and H a uniformly closed left translation invariant linear subspace of $m(S)$ that contains the constants. Suppose H is a lattice with respect to the pointwise ordering; for all $s \in S$, the left translation operator $l_s : H \to H$ is* *lattice preserving and there exists a lattice preserving mean* μ_0 *on* H *such that* $\mu_0(h_s)$ *as a function of* s *lies in* H *for every* h *in* H. *Then the following conditions are equivalent:*

(1) H *admits a lattice preserving left invariant mean;*

(2) *if* (S, K) *is an affine action of* S *on the Bauer simplex* K *such that for all* $s \in S$ *and* $x \in \text{ex } K$, $sx \in \text{ex } K$, *and there is an* $x_0 \in \text{ex } K$ *with* $Tx_0(A(K)) \subset H$, *then there exists a common fixed point for* (S, K) *that is an extreme point;*

(3) S, H *has the common fixed point property on compacta with respect to E-representations.*

Proof. (1) → (2). Suppose H admits the lattice preserving left invariant mean μ. Let (S, K) be an affine action on the Bauer simplex K as in (2) with $x_0 \in \text{ex } K$ such that $Tx_0(A(K)) \subset H$. If $f, g \in A(K)$ and $s \in S$, then $Tx_0(f \vee g)(s) = (f \vee g)(sx_0) = f(sx_0) \vee g(sx_0)$ since $sx_0 \in \text{ex } K$ and K is a Bauer simplex (the evaluation functional at sx_0 is lattice preserving). Thus $Tx_0(f \vee g)$ is the pointwise sup of $Tx_0 f$ and $Tx_0 g$ and therefore, also the sup relative to H. Consequently, Tx_0 is lattice preserving so that $\nu = \mu \circ Tx_0$ is a lattice preserving mean on $A(K)$ that is invariant under the action of S. Then ν is evaluation on $A(K)$ at some $y \in \text{ex } K$ and y is the desired common fixed point. (H need only be a lattice for this implication.)

(2) → (1). Choose K to be the space of means on H in the w^*-topology and (S, K) the canonical action where $s \cdot \mu(h) = \mu(h_s)$ for all h in H. Then $\text{ex } K$ is precisely the set of lattice preserving means on H (for example, [12, p. 238]). It is well-known that $\hat{H}$ (defined as in Theorem 1 above) coincides with $A(K)$ and $h \to \hat{h}$ is a linear order-preserving isometry (see, for example, [16, p. 142]). Consequently, $T\mu_0(A(K)) \subset H$ and K is a Bauer simplex since $A(K)$ is a lattice. The action of S is extreme point preserving, for if $\mu \in \text{ex } K$ and $s \in S$, then $s \cdot \mu = \mu \circ l_s$ is lattice preserving since both μ and

l_S are. It follows from (2) that there exists $\nu \in \text{ex } K$ which is a common fixed point for (S, K) . ν is then a lattice preserving left invariant mean.

Since every Bauer simplex K is affinely homeomorphic to the space of probability measures (in the w^*-topology) on a compact Hausdorff space (namely, ex K) and every such space is a Bauer simplex [2], statements (2) and (3) are formally equivalent.

REMARK. The equivalence of (1) and (2) of Theorem 1 in [13] is a particular case of the above theorem. If H is a closed subalgebra of $m(S)$, then H is a sublattice of $m(S)$ (see, for example, [15, p. 150]). Furthermore, it follows from the two classical identities

$$fg = \tfrac{1}{4}|(f+g)^2-(f-g)^2| \quad \text{and} \quad f \vee g = \tfrac{1}{2}[(f+g)+|f-g|] ,$$

that the multiplicative linear functionals on H coincide with the lattice preserving linear functionals on H . The latter two well-known statements are implicit in the use of Kakutani's Theorem in the proof of (3) implies (1) of Theorem 1 in [13].

References

[1] L.N. Argabright, "Invariant means and fixed points; a sequel to Mitchell's paper", *Trans. Amer. Math. Soc.* 130 (1968), 127-130.

[2] Heinz Bauer, "Šilovscher Rand und Dirichletsches Problem", *Ann. Inst. Fourier (Grenoble)* 11 (1961), 89-136.

[3] Mahlon M. Day, "Fixed-point theorems for compact convex sets", *Illinois J. Math.* 5 (1961), 585-590.

[4] Mahlon Marsh Day, "Correction to my paper 'Fixed-point theorems for compact convex sets'", *Illinois J. Math.* 8 (1964), 713.

[5] James Dugundji, *Topology* (Allyn and Bacon, Boston, 1966).

[6] Leonard Gillman and Meyer Jerison, *Rings of continuous functions* (Van Nostrand, Princeton, New Jersey; Toronto; London; New York; 1960).

[7] G.G. Gould and M. Mahowald, "Measures on completely regular spaces", *J. London Math. Soc.* **37** (1962), 103-111.

[8] E. Granirer and Anthony T. Lau, "Invariant means on locally compact groups", *Illinois J. Math.* **15** (1971), 249-257.

[9] Marvin W. Grossman, "A categorical approach to invariant means and fixed point properties", *Semigroup Forum* **5** (1972), 14-44.

[10] Marvin W. Grossman, "Uniqueness of invariant means on certain introverted spaces", *Bull. Austral. Math. Soc.* **9** (1973), 109-120.

[11] A. Guichardet, *Special topics in topological algebras* (Gordon and Breach, New York, London, Paris, 1968).

[12] J.L. Kelley, Isaac Namioka and W.F. Donoghue, Jr, Kenneth R. Lucas, B.J. Pettis, Ebbe Thue Poulsen, G. Baley Price, Wendy Robertson, W.R. Scott, Kennan T. Smith, *Linear topological spaces* (Van Nostrand, Princeton, New Jersey; Toronto; New York; London; 1963).

[13] Theodore Mitchell, "Function algebras, means, and fixed points", *Trans. Amer. Math. Soc.* **130** (1968), 117-126.

[14] Theodore Mitchell, "Topological semigroups and fixed points", *Illinois J. Math.* **14** (1970), 630-641.

[15] H.L. Royden, *Real analysis* (Macmillan, New York; Collier-Macmillan, London; 1963).

[16] Z. Semadeni, "Free compact convex sets", *Bull. Acad. Sci. Polon. Sér. Sci. Math. Astronom. Phys.* **13** (1965), 141-146.

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