

TAILS IN GENERALIZED JACKSON NETWORKS WITH SUBEXPONENTIAL SERVICE-TIME DISTRIBUTIONS

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Abstract

We give the exact asymptotics of the tail of the stationary maximal dater in generalized Jackson networks with subexponential service times. This maximal dater, which is an analogue of the workload in an isolated queue, gives the time taken to clear all customers present at some time t when stopping all arrivals that take place later than t . We use the property that a large deviation of the maximal dater is caused by a single large service time at a single station at some time in the distant past of t , in conjunction with fluid limits of generalized Jackson networks, to derive the relevant asymptotics in closed form.

Keywords: Generalized Jackson network; subexponential random variable; heavy tail; integrated tail; Veraverbeke's theorem; fluid limit

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1. Introduction

To the best of the authors' knowledge, the literature on generalized Jackson networks with heavy-tailed service times is limited to tandem queues. Exact asymptotics for these quantities were obtained in [4]. The present paper addresses the case of generalized Jackson networks of arbitrary topology. It focuses on a key state variable, namely the time to empty the network when stopping the arrival process, which has previously been used to determine the stability region of such networks [2], [3]. (This variable boils down to the virtual workload in an isolated queue or to the sojourn time for tandem queues.) The aim of the paper is to derive the exact asymptotics of the tail of this state variable in the stationary regime. The main ingredients for the derivation of this result are as follows.

- We require a generalization of the so-called 'single-big-event theorem', well known for isolated queues, to generalized Jackson networks of the sort we consider. This was established in [4]. In the GI/GI/1 queue, this theorem states that, in the case of subexponential service times, large workloads typically occur in the event that a single large service time has taken place in the distant past, while all the other service times were close to their means. Similarly, in generalized Jackson networks with subexponential

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service times, large maximal daters occur when a single large service time has taken place at one of the stations, while all the other service times are close to their means.

- We must identify the role played by fluid limits within the context of the single-big-event theorem for this class of networks.
- We must find the combination of these fluid limits and heavy-tailed calculus that allows us to derive the closed form formulae for the asymptotics.

Although this result sheds light on the way such a network experiences a deviation from its normal behaviour, it is in no way final, as the tail behaviour of other state variables, such as the stationary queue size, are still unknown. The derivation of the (more complex) asymptotic behaviour of these other state variables has already been obtained using a similar methodology in the particular case of tandem queues [4]. The extension of these queue-size asymptotics to generalized Jackson networks of arbitrary topology seems to require much more effort and will not be pursued in the present paper. The proposed method should, however, extend to other characteristics of stationary workload like, for instance, the sum of the residual service times of all customers present in the network at some given time.

The paper is structured as follows. In Section 2, we introduce generalized Jackson networks and show that the four assumptions required in order to use the single-big-event theorem (called (IA), (AA), (SE), and (H) in [4]) hold. The main result is then established in Section 3.

1.1. Notation

Here and later in the paper, for positive functions f and g , the equivalence $f(x) \sim dg(x)$ with $d > 0$ means that $f(x)/g(x) \rightarrow d$ as $x \rightarrow \infty$. By convention, the equivalence $f(x) \sim dg(x)$ with $d = 0$ means that $f(x)/g(x) \rightarrow 0$ as $x \rightarrow \infty$; this will be written $f(x) = o(g(x))$. We will also use the notation $f(x) = \mathcal{O}(g(x))$ to mean that both $\limsup_x f(x)/g(x) < \infty$ and $\liminf_x f(x)/g(x) > 0$.

In this paper, $\varepsilon(x)$ denotes a function such that $\varepsilon(x) \rightarrow 0$ as $x \rightarrow \infty$. The function ε may vary from place to place; for example, we write

$$\varepsilon(x) + \varepsilon(x) = \varepsilon(x), \quad \varepsilon(x)(1 + \varepsilon(x)) = \varepsilon(x),$$

and so on. Similarly, we will write $\varepsilon(x, y)$ for $\varepsilon(x) + \varepsilon(y)$, $\varepsilon(x)\varepsilon(y)$, etc.

2. Generalized Jackson networks

2.1. General framework and sample-path construction of the maximal dater

2.1.1. *Service time and routing sequences.* Here, we recall the notation introduced in [2] to describe a generalized Jackson network with K nodes. The networks we consider are characterized by the fact that service times and routing decisions are associated with stations and not with customers. This means that the j th service at station k takes $\sigma_j^{(k)}$ units of time, where $\{\sigma_j^{(k)}\}_{j \geq 1}$ is a predefined sequence. In the same way, when this service is completed, the departing customer is sent to station $v_j^{(k)}$ (or leaves the network, if $v_j^{(k)} = K + 1$) and is put at the end of the queue at this station, where $\{v_j^{(k)}\}_{j \geq 1}$ is also a predefined sequence, called the routing sequence. The sequences $\{\sigma_j^{(k)}\}_{j \geq 1}$ and $\{v_j^{(k)}\}_{j \geq 1}$, where k ranges over the set of stations, are called the driving sequences of the network. Node 0 models the external arrival of customers to the network and, therefore, the arrival time of the j th customer in the network takes place at $\sigma_1^{(0)} + \dots + \sigma_j^{(0)}$ and it joins the end of the queue at station $v_j^{(0)}$. Hence, $\sigma_j^{(0)}$ is the j th interarrival time.

The sample-path construction introduced here is that of [2]. The main interest of such a construction is that some monotonicity properties are preserved, as explained in [2]. As was shown in [4], these monotonicity properties are crucial for our asymptotics calculation.

A generalized Jackson network will be defined by

$$\mathbf{JN} = \{ \{ \sigma_j^{(k)} \}_{j \geq 1}, \{ v_j^{(k)} \}_{j \geq 0}, n^{(k)}, 0 \leq k \leq K \},$$

where $N = (n^{(0)}, n^{(1)}, \dots, n^{(K)})$ describes the initial condition. The interpretation is as follows: at time $t = 0$, in node $i, i \neq 0$, there are $n^{(i)}$ customers, with service times $\sigma_1^{(i)}, \dots, \sigma_{n^{(i)}}^{(i)}$ (if appropriate, $\sigma_1^{(i)}$ may be interpreted as a residual service time). On the other hand, the interpretation of $n^{(0)}$ (the $i = 0$ case) is as follows.

- If $n^{(0)} = 0$ there is no external arrival.
- If $\infty > n^{(0)} \geq 1$ then, for all $1 \leq j \leq n^{(0)}$, the arrival time of the j th customer in the network takes place at $\sigma_1^{(0)} + \dots + \sigma_j^{(0)}$. Note that, in this case, there may be a finite number of customers passing through a given station, so that the network is actually well defined once a finite sequence of routing decisions and service times is given for this station.
- If $n^{(0)} = \infty$ then when taking, for instance, the sequence $\{ \sigma_j^{(0)} \}_{j \geq 1}$ to be independent and identically distributed (i.i.d.), the arrival process is a renewal process.

2.1.2. *Euler route, Euler network.* Consider a route $\mathbf{r} = (r_1, \dots, r_\phi)$ with $1 \leq r_i \leq K$ for $i = 2, \dots, \phi - 1$. Such a route is *successful* if $r_1 = 0$ and $r_\phi = K + 1$. To such a route, we associate a routing sequence $\nu = (\nu^{(0)}, \dots, \nu^{(K)})$ as follows (here, ‘ \oplus ’ means concatenation and \emptyset denotes the empty sequence).

Procedure (\mathbf{r}):

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- 1 -   for  $k = 0, \dots, K$  do
            $\nu^{(k)} := \emptyset;$ 
            $\phi^{(k)} := 0;$ 
       od
- 2 -   for  $i = 1, \dots, \phi - 1$  do
            $\nu^{(r_i)} := \nu^{(r_i)} \oplus r_{i+1};$ 
            $\phi^{(r_i)} := \phi^{(r_i)} + 1;$ 
       od
    
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Note that $\phi^{(j)}$ is the number of visits to node j in such a route.

A simple Euler network is a generalized Jackson network $E = \{ \sigma, \nu, N \}$, with $N = (1, 0, \dots, 0) = 1$, such that the routing sequence $\nu = \{ \nu_i^{(k)} \}_{i=1}^{\phi^{(k)}}$ is generated by a successful route and such that $\sigma = \{ \sigma_i^{(k)} \}_{i=1}^{\phi^{(k)}}$ is a sequence of real-valued nonnegative numbers, representing service times.

Consider a sequence of simple Euler networks, say $\{ E(n) \}_{n=-\infty}^0$, where $E(n) = \{ \sigma(n), \nu(n), 1 \}$. For $m \leq n \leq 0$, we define $\sigma_{[m,n]}$ and $\nu_{[m,n]}$ to be the concatenation of $\{ \sigma(k) \}_{m \leq k \leq n}$

and $\{v(k)\}_{m \leq k \leq n}$, and then define the following *composed* generalized Jackson network, with $N_{[m,n]} = (n - m + 1, 0, \dots, 0)$:

$$\mathbf{JN}_{[m,n]} = \{\sigma_{[m,n]}, v_{[m,n]}, N_{[m,n]}\}.$$

2.1.3. *Maximal dater.* As proved in [2], for all possible values of $v(p)$ and $\sigma(p)$ in the simple Euler networks, and for all integers $m \leq n$, the composed network $\mathbf{JN}_{[m,n]}$ stays empty forever after some finite time. We denote by $X_{[m,n]}$ the time taken to empty $\mathbf{JN}_{[m,n]}$ forever and, by

$$Z_{[m,n]} = X_{[m,n]} - \sum_{i=1}^{n-m+1} \sigma_{[m,n],i}^{(0)},$$

the associated maximal dater. The sequence $Z_{[-n,0]}$ is an increasing sequence. We define the maximal dater of the generalized Jackson network $\mathbf{JN} = \{\sigma, v, N\}$, where σ and v are the infinite concatenations of the $\{\sigma(n)\}_n$ and $\{v(n)\}_n$, and $N = (\infty, 0, \dots, 0)$, by

$$Z = \lim_{n \rightarrow \infty} Z_{[-n,0]}. \tag{2.1}$$

To all generalized Jackson networks $\mathbf{JN}_{[m,n]}$, we also associate the generalized Jackson network $\mathbf{JN}_{[m,n]}(0)$, in which the driving sequences are the same as in the original network except for the sequence $\{\sigma_j^{(0)}\}$, which is now $\sigma_j^{(0)} = 0$ for all j . Similarly, we define $Z_{[m,n]}(0)$ to be the time taken to empty the generalized Jackson network $\mathbf{JN}_{[m,n]}(0)$.

Let

$$Y_i^{(k)} = \sum_{j=1}^{\phi^{(k)}(i)} \sigma_j^{(k)}(i)$$

be the total load brought by (external) customer i to station k . In this expression, $\phi^{(k)}(i)$ is the number of visits of customer i to station k if he is alone in the network, and $\sigma_j^{(k)}(i)$ denotes his service time at this station. Note that

$$Z_i := Z_{[i,i]} = Y_i^{(1)} + \dots + Y_i^{(K)} \quad \text{for all } i,$$

$$Z_{[n,0]}(0) \geq \max_{j=1, \dots, K} \sum_{i=n}^0 Y_i^{(j)} \quad \text{for all } n \leq 0.$$

In addition, Lemma 4 of [3] implies that

$$\lim_{n \rightarrow \infty} \frac{Z_{[-n,0]}(0)}{n} =: b = \max_{1 \leq k \leq K} E[Y_1^{(k)}] \quad \text{almost surely.} \tag{2.2}$$

2.2. Stochastic assumptions

Assumption 2.1. (Independence of routing and service times.) *The sequences $\{v^{(k)}\}$ and $\{\sigma^{(k')}\}$ are mutually independent, for k and k' ranging over the set of stations.*

Assumption 2.2. (Independence of service times.) *We assume that the service times are independent for different stations and, in each station, i.i.d. with finite mean: $E[\sigma^{(j)}] = 1/\mu^{(j)} > 0$ for all $1 \leq j \leq K$.*

Assumption 2.3. (Routing.) We assume that each of the successful routes used to build v is obtained by a Markov chain on the state space $\{0, 1, \dots, K, K + 1\}$, with transition matrix

$$R = \begin{pmatrix} 0 & r_{0,1} & r_{0,2} & \cdots & r_{0,K} & 0 \\ 0 & r_{1,1} & r_{1,2} & \cdots & r_{1,K} & r_{1,K+1} \\ 0 & r_{2,1} & r_{2,2} & \cdots & r_{2,K} & r_{2,K+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & r_{K,1} & r_{K,2} & \cdots & r_{K,K} & r_{K,K+1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

This is equivalent to assuming that the routing decisions $\{v_j^{(k)}\}$ at station k are i.i.d. in j , independent of everything else, and such that the routing decision selects station i with probability $P[v^{(k)} = i] = r_{k,i}$. The fact that the routes built with this Markovian procedure are successful implies that state $K + 1$ is the only absorbing state of this chain and that all other states are transient; we then have the very same Markovian routing assumptions as in (exponential) Jackson networks. More generally, when denoting by E_k the law of the chain with initial condition k and, by V_j , the number of visits of this absorbing chain to state j , we define

$$E_0[V_k] = \pi_k, \quad P_0[V_k \geq 1] = p_k, \quad E_k[V_j] = \pi_{k,j}.$$

We will use the following notation:

$$b_j = \frac{\pi_j}{\mu^{(j)}}, \quad b_{j,i} = \frac{\pi_{j,i}}{\mu^{(i)}}, \quad B_j = \max_i b_{j,i}.$$

With this notation, we have $b = \max_i \pi_i / \mu^{(i)} = \max_i b_i$. Let $\lambda^{-1} = E[\sigma^0] = a$. Throughout this paper we will assume that the following stability condition holds:

$$\lambda b < 1. \tag{2.3}$$

Theorem 13 of [2] applies, meaning that if $\lambda b < 1$, then $Z < \infty$ almost surely; conversely, if $\lambda b > 1$ then $Z = \infty$ almost surely.

Example 2.1. As an example, we will consider a network with the following routing matrix:

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & p & 1 - p \\ 0 & q & 0 & 1 - q \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case, we have

$$\pi_1 = 1 + q\pi_2, \quad \pi_2 = p\pi_1,$$

which imply that

$$\pi_1 = \frac{1}{1 - pq}, \quad \pi_2 = \frac{p}{1 - pq}.$$

Similarly, we have

$$\begin{aligned} \pi_{1,1} &= \pi_1, & \pi_{1,2} &= \pi_2, \\ \pi_{2,1} &= \frac{q}{1 - pq}, & \pi_{2,2} &= \frac{1}{1 - pq}, \end{aligned}$$

and

$$p_1 = 1, \quad p_2 = p.$$

Hence, we have

$$b = B_1 = \frac{1}{1 - pq} \max \left\{ \frac{1}{\mu^{(1)}}, \frac{p}{\mu^{(2)}} \right\} \quad \text{and} \quad B_2 = \frac{1}{1 - pq} \max \left\{ \frac{q}{\mu^{(1)}}, \frac{1}{\mu^{(2)}} \right\}.$$

In particular, for $p = 1$ and $q = 0$, we are dealing with the case of queues in tandem and we have $b = B_1 = \max(1/\mu^{(1)}, 1/\mu^{(2)})$ and $B_2 = 1/\mu^{(2)}$.

We now recall some definitions.

Definition 2.1. A distribution function F on \mathbb{R}_+ (with $\bar{F} = 1 - F$) is long tailed if, for any $y > 0$,

$$\bar{F}(x + y) \sim \bar{F}(x) \quad \text{as } x \rightarrow \infty.$$

We introduce a proper subset of the class of long-tailed distributions, namely the class of subexponential distributions, denoted by \mathcal{S} .

Definition 2.2. A distribution function F on \mathbb{R}_+ is called subexponential if $\overline{F^{*2}}(x) \sim 2\bar{F}(x)$, where $F^{*2}(x)$ is the tail of the two-fold convolution of F .

Definition 2.3. A positive measurable function f on $[0, \infty)$ is called regularly varying with index $\alpha \in \mathbb{R}$ (denoted $f \in \mathcal{R}(\alpha)$) if

$$\lim_{x \rightarrow \infty} \frac{f(tx)}{f(x)} = t^\alpha \quad \text{for all } t > 0.$$

Definition 2.4. A positive measurable function h on $[0, \infty)$ is called rapidly varying (denoted $h \in \mathcal{R}(-\infty)$) if

$$\lim_{x \rightarrow \infty} \frac{h(tx)}{h(x)} = 0 \quad \text{for all } t > 1.$$

For example, Weibull or lognormal random variables have tail distributions that are rapidly varying. For more on the matter, see [5] and [6].

Assumption 2.4. (Subexponentiality of service times.) *For a distribution function F on the positive real line, with finite first moment $M = \int_0^\infty \bar{F}(u) du$, $\bar{F}(u) = 1 - F(u)$ denotes the tail of F and F^s the integrated tail distribution, given by*

$$F^s(x) = 1 - \min \left\{ 1, \int_x^\infty \bar{F}(u) du \right\} =: 1 - \bar{F}^s(x).$$

The assumptions concerning service times are the following. There exists a distribution function F on \mathbb{R}_+ such that

1. F is subexponential, with finite first moment M ;
2. the integrated distribution F^s is subexponential; and
3. the following equivalence holds when x tends to ∞ :

$$P[\sigma_1^{(k)} > x] \sim c^{(k)} \bar{F}(x)$$

for all $k = 1, \dots, K$, with $\sum_{k=1}^K c^{(k)} =: c > 0$.

2.3. Main result

We first introduce some notation; the intuitive meanings of these quantities will be given later on. Let $f^j(\sigma, n)$ be the following piecewise-linear function of the nonnegative real numbers σ and n :

$$f^j(\sigma, n) = \mathbf{1}_{\{\sigma > na\}}(\sigma - na + np_j B_j) + \mathbf{1}_{\{\sigma \leq na\}} \max_k \left\{ p_j b_{j,k} \frac{\sigma}{a} + \left(\frac{b_k}{a} - 1 \right) (na - \sigma) \right\}^+. \tag{2.4}$$

In addition, for all positive real numbers x and all $j = 1, \dots, K$, let $\Delta^j(x)$ be the following domain:

$$\Delta^j(x) = \{(\sigma, t) \in \mathbb{R}_+^2 : f^j(\sigma, t) > x\}. \tag{2.5}$$

Remark 2.1. We may rewrite the function f^j as

$$f^j(\sigma, n) = \left\{ \sigma - na + \max_k \left\{ nb_k - \frac{\sigma}{a} (b_k - p_j b_{j,k}), np_j b_{j,k} \right\} \right\}^+.$$

This is due to the fact that $p_j b_{j,k} \leq b_k$. In particular, we see that

$$f^j(\sigma, n) \geq \sigma - na - np_j B_j. \tag{2.6}$$

Example 2.2. Continuing Example 2.1, we have

$$f^1(\sigma, n) = \left\{ \sigma - na + \frac{n}{1 - pq} \max \left\{ \frac{1}{\mu^{(1)}}, \frac{p}{\mu^{(2)}} \right\} \right\}^+,$$

$$f^2(\sigma, n) = \left\{ \sigma - na + \max \left\{ \frac{npq}{(1 - pq)\mu^{(1)}}, \frac{np}{(1 - pq)\mu^{(2)}}, \frac{n}{(1 - pq)\mu^{(1)}} - \frac{\sigma}{a\mu^{(1)}} \right\} \right\}^+.$$

In the specific case of queues in tandem ($p = 1$ and $q = 0$), these equations reduce to

$$f^1(\sigma, n) = \left\{ \sigma - na + n \max \left\{ \frac{1}{\mu^{(1)}}, \frac{1}{\mu^{(2)}} \right\} \right\}^+,$$

$$f^2(\sigma, n) = \left\{ \sigma - na + \frac{n}{\mu^{(2)}} \right\}^+,$$

and the corresponding domains are

$$\Delta^1(x) = \left\{ \sigma > x + t \left(a - \max \left\{ \frac{1}{\mu^{(1)}}, \frac{1}{\mu^{(2)}} \right\} \right) \right\}, \tag{2.7}$$

$$\Delta^2(x) = \left\{ \sigma > x + t \left(a - \frac{1}{\mu^{(2)}} \right) \right\}. \tag{2.8}$$

Theorem 2.1. Consider a stable generalized Jackson network with subexponential service-time distributions satisfying Assumptions 2.1–2.4. Let Z denote its stationary maximal dater at customer arrivals (see Section 2.1 for a formal definition of Z). When $x \rightarrow \infty$,

$$P[Z > x] \sim \sum_{j=1}^K \pi_j \iint_{(\sigma, t) \in \Delta^j(x)} P[\sigma^{(j)} \in d\sigma] dt. \tag{2.9}$$

This equation may be rewritten with the constants $\{\alpha_i^j, \beta_i^j, \gamma_i^j\}_{0 \leq i \leq l}$, which will be calculated in Lemma 3.3 below, as

$$P[Z > x] \sim \sum_{j=1}^K \pi_j \left\{ \sum_{i=0}^l \sum_{\alpha_i^j x \leq n < \alpha_{i+1}^j x} P\left[\sigma^{(j)} > \frac{x}{\beta_i^j} + n\gamma_i^j\right] \right\}$$

or, using $\delta_i^j = 1/\beta_i^j + \alpha_i^j \gamma_i^j$ and $d^{(j)} = \pi_j c^{(j)}$, as

$$P[Z > x] \sim \sum_{j=1}^K d^{(j)} \left\{ \sum_{i=0}^l \frac{1}{\gamma_i^j} [\bar{F}^s(\delta_i^j x) - \bar{F}^s(\delta_{i+1}^j x)] \right\}. \tag{2.10}$$

The following statements then hold, as x tends to ∞ .

1. If $\bar{F}^s \in \mathcal{R}(-\alpha)$, with $\alpha > 0$, we can rewrite (2.10) as

$$\frac{P[Z > x]}{\bar{F}^s(x)} \rightarrow \sum_{j=1}^K d^{(j)} \left\{ \sum_{i=0}^l \frac{1}{\gamma_i^j} [(\delta_i^j)^{-\alpha} - (\delta_{i+1}^j)^{-\alpha}] \right\}.$$

2. If $\bar{F}^s \in \mathcal{R}(-\infty)$ then

$$\frac{P[Z > x]}{\bar{F}^s(x)} \rightarrow \sum_{j=1}^K \frac{d^{(j)}}{a - p_j B_j}. \tag{2.11}$$

Remark 2.2. In view of (2.6), we see that $\{\sigma > x + t(a - p_j B_j)\} \subset \Delta^{(j)}(x)$ and, hence,

$$P[Z > x] \geq \sum_{j=1}^K \pi_j \sum_{n \geq 0} P[\sigma^{(j)} > x + n(a - p_j B_j)] \sim \sum_{j=1}^K \frac{d^{(j)}}{a - p_j B_j} \bar{F}^s(x).$$

Hence, in general, the asymptotics described in (2.11) are a lower bound. In the rapidly varying case, this lower bound is tight. In the regularly varying case, the complete shape of the domain must be taken into consideration and, indeed, the part $\Delta^j(x) \setminus \{\sigma > x + t(a - p_j B_j)\}$ is no longer negligible, due to the scaling property of regularly varying functions. More insights into the shape of the domain will be given in Section 3.3.

Example 2.3. In the specific case of queues in tandem, from (2.7) and (2.8) we see that (2.9) reduces to

$$P[Z > x] \sim \sum_{n \geq 0} P\left[\sigma^{(1)} > x + n\left(a - \max\left\{\frac{1}{\mu^{(1)}}, \frac{1}{\mu^{(2)}}\right\}\right)\right] + \sum_{n \geq 0} P\left[\sigma^{(2)} > x + n\left(a - \frac{1}{\mu^{(2)}}\right)\right],$$

which corresponds to the exact asymptotics of Theorem 9 of [4].

2.4. Technical conditions

The properties (IA) and (AA) of [4] are as follows.

- (IA) The sequence of simple Euler networks $\{E(n)\}_{n=0}^\infty$ consists of i.i.d. random variables.

(AA) The random variables $\{Y_i^{(k)}\}$ are independent of the interarrival times and such that the sequence of random vectors $\{(Y_i^{(1)}, \dots, Y_i^{(K)})\}_{i \in \mathbb{N}}$ is i.i.d. (general dependences between the components of the vector $(Y_i^{(1)}, \dots, Y_i^{(K)})$ are allowed).

Under our Assumptions 2.1–2.3, both of these properties hold.

Under Assumption 2.1, the variable Z associated to $\mathbf{JN} = \{\sigma, \nu, N\}$ represents the stationary maximal dater of the generalized Jackson network, namely the time that it would take, in steady state, to clear the workload of all customers present in the system, when stopping future arrivals.

The assumptions (SE) and (H) of [4] are as follows.

(SE) For all $k = 1, \dots, K$,

$$P[Y_1^{(k)} > x] \sim \pi^k P[\sigma^{(k)} > x] \sim d^{(k)} \bar{F}(x),$$

with $d^{(k)} = c^{(k)} \pi_k$. Thus, $d := \sum_k d^{(k)} > 0$.

(H) We have

$$P\left[\sum_{k=1}^K Y_1^{(k)} > x\right] \sim P\left[\max_{1 \leq k \leq K} Y_1^{(k)} > x\right] \sim \sum_{k=1}^K P[Y_1^{(k)} > x] \sim d \bar{F}(x).$$

(See Sections 4.4.2 and 7.2 of [4].) Under our Assumption 2.4, both of these assumptions are satisfied. Also under Assumption 2.4, there exists a nondecreasing, integer-valued function N_x that tends to infinity as x tends to infinity and is such that, for all finite real numbers b ,

$$\sum_{n=0}^{N_x} \bar{F}(x + nb) = o(\bar{F}^s(x)), \quad x \rightarrow \infty$$

(see Section 4.1.2 of [4]).

3. Exact tail asymptotics

3.1. Single-big-event theorem

As already mentioned, one of the tools we will use within this setting is the ‘single-big-event theorem’ for generalized Jackson networks. More precisely, Theorems 7 and 8 of [4] give the following result.

Proposition 3.1. *Let Z be the stationary maximal dater of the generalized Jackson network defined in (2.1). For any x and for $j = 1, \dots, r$, let $\{K_{n,x}^j\}$ be a sequence of events such that,*

1. *for any n and j , the event $K_{n,x}^j$ and the random variable $Y_{-n}^{(j)} = \sum_{k=1}^{\phi^{(j)}(-n)} \sigma_k^{(j)}(-n)$ are independent; and*
2. *for any j , $P[K_{n,x}^j] \rightarrow 1$, uniformly in $n \geq N_x$, as $x \rightarrow \infty$.*

For all sequences $\varepsilon_n \rightarrow 0$, we write $x_n = x + n(a - b + \varepsilon_n)$. Then, as $x \rightarrow \infty$,

$$P[Z > x] \sim \sum_{j=1}^K \sum_{n \geq N_x} P[Z > x, Y_{-n}^{(j)} > x_n, K_{n,x}^j]$$

and $P[Z > x] = \mathcal{O}(\bar{F}^s(x))$.

Proposition 3.1 leads to the following, more convenient, result.

Corollary 3.1. *Take any sequence of events $\{K_n^j\}$ such that, for any j , K_n^j and the random variable $Y_{-n}^{(j)}$ are independent, and $P(K_n^j) \rightarrow 1$ as $n \rightarrow \infty$. Take $z_x \rightarrow \infty$, with $z_x = o(x)$, such that $\bar{F}^s(x \pm z_x) \sim \bar{F}^s(x)$, and let*

$$G(x) = \sum_{j=1}^K \sum_{n \geq N_x} P[Z_{[-n,0]} > x, K_n^j, Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L].$$

Then we have

$$(1 + \varepsilon(x))G(x) \leq P[Z > x] \leq (1 + \varepsilon(x))G(x - z_x) + \varepsilon(L, x)\bar{F}^s(x). \tag{3.1}$$

Furthermore, if G is long tailed we have

$$P[Z > x] \sim G(x) \text{ as } x \rightarrow \infty.$$

The proof can be found in Appendix A.

3.2. Fluid limit

We have to find sequences of events $\{K_n^j\}$ allowing us to calculate the sum

$$\sum_{n \geq N_x} P[Z_{[-n,0]} > x, K_n^j, Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L],$$

where, as above, $x_n = x + n(a - b + \varepsilon_n)$. The events in question will be based on the piecewise-linear functions $f^j(\sigma, n)$ defined in (2.4). Let us describe the intuitive reason for introducing this function. Assume that the ‘big’ service time is equal to σ and takes place at station j within the set of service times of the simple Euler network $E(-n)$. Let us look at the maximal dater $Z_{[-n,0]}$ in the fluid scale suggested by the almost-sure limit of (2.2).

- If $\sigma > na$ then the number of customers queued at station j at time σ is of order np_j , whereas the number of customers at the other stations is small. So, according to (2.2), the time taken to empty the network, from time σ on, should be of order $np_j B_j$. Hence, in this case, the maximal dater in question should indeed be of order $f^j(\sigma, n)$.
- If $\sigma < na$ then, at time σ , the number of customers blocked at station j is of order $p_j \sigma / a$, and the other stations have few customers; from time σ to the time of the last arrival (which is of order na), station k has to serve the approximate load $p_j(\sigma/a)b_{j,k}$ generated by these blocked customers plus the load $(na - \sigma)b_k/a$ generated by the external arrivals in the time interval from σ to the last arrival. In this time interval, the service capacity is of order $(na - \sigma)$. Hence, the maximal dater should again be of order $f^j(\sigma, n)$.

Particularly for the last case, a more thorough understanding of the fluid limit is clearly necessary. This is provided by the results of [7]. We now return to a more rigorous treatment.

Consider a generalized Jackson network built from the i.i.d. sequence of simple Euler networks $\{E(k)\}$. To all simple Euler networks E and all positive integers n , we associate the network $\mathbf{JN}^n(E)$ with input $\{\tilde{E}(k)\}_{k=-n}^\infty$, where $\tilde{E}(k) = E(k)$ for all $k > -n$ and $\tilde{E}(-n) = E$.

We allow for a specific initial condition for $k = -n$ and then take the original sequence. That is, if we denote by $\sigma^{(k),n}$ and $\nu^{(k),n}$ the concatenations

$$(\{\sigma^{(k)}(E)\}, \{\sigma^{(k)}(-n + 1)\}, \dots, \{\sigma^{(k)}(0)\}, \dots)$$

and

$$(\{\nu^{(k)}(E)\}, \{\nu^{(k)}(-n + 1)\}, \dots, \{\nu^{(k)}(0)\}, \dots),$$

respectively, then

$$\mathbf{JN}^n(E) = \{\sigma^n(E), \nu^n(E), N^n\}, \quad \text{with } N^n = (n, 0, \dots, 0).$$

The maximal dater of order $[-n, 0]$ in this network will be denoted by $\tilde{Z}^n(E)$; of course $\tilde{Z}^n(E(n)) = Z_{[-n,0]}$. More generally, we will add the superscript ‘ n ’ to any other function associated to the network $\mathbf{JN}^n(E)$.

For all simple Euler networks $E = (\sigma, \nu, 1)$, let $Y^{(j)}(E) = \sum_{u=1}^{\phi^{(j)}} \sigma_u^{(j)}$. We are now in a position to state the main result pertaining to the fluid limit. Let ε_n and z_n be some sequences of positive real numbers. We define

$$U^j(n) = \{E \text{ is a simple Euler network such that } Y^{(k)}(E) \leq z_n \text{ for all } k \neq j\},$$

$$V^j(n) = \{E \in U^j(n), Y^{(j)}(E) \geq n(a - b), \phi^{(j)} \leq L\},$$

$$K_n^j = \left\{ \sup_{E \in V^j(n)} \left| \frac{\tilde{Z}^n(E) - f^j(Y^{(j)}(E), n)}{n} \right| \leq \varepsilon_n \right\},$$

$$\bar{K}_n^j = K_n^j \cap \{E(-n) \in U^j(n)\}.$$

We first recall a result that derives directly from Proposition 5.1 and Remark 5.1 of [7].

Proposition 3.2. *Under the previous assumptions, there exists a sequence $z_n \rightarrow \infty$, with $z_n/n \rightarrow 0$, such that*

$$\sup_{E \in V^j(n)} \left| \frac{\tilde{Z}^n(E) - f^j(Y^{(j)}(E), n)}{n} \right| \xrightarrow{n \rightarrow \infty} 0 \quad \text{almost surely.}$$

Lemma 3.1. *We have, as $x \rightarrow \infty$,*

$$G(x) = \sum_{j=1}^K \sum_{n \geq N_x} \mathbb{P}[Z_{[-n,0]} > x, \bar{K}_n^j, Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L] + o(\bar{F}^s(x)). \quad (3.2)$$

Proof. Note that, owing to Proposition 3.2, the sequence of events $\{K_n^j\}$ satisfies the assumptions of Corollary 3.1. Since $\bar{K}_n^j \subset K_n^j$, we have

$$G(x) \geq \sum_{j=1}^K \sum_{n \geq N_x} \mathbb{P}[Z_{[-n,0]} > x, \bar{K}_n^j, Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L].$$

To find an upper bound, note that

$$\begin{aligned} & \mathbb{P}[Z_{[-n,0]} > x, K_n^j, Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L] \\ & \leq \mathbb{P}[Z_{[-n,0]} > x, \bar{K}_n^j, Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L] \\ & \quad + \mathbb{P}\left[Y_{-n}^{(j)} > x_n, \sum_{i \neq j} Y_{-n}^{(i)} > z_n \right]. \end{aligned}$$

Therefore, Lemma 3.1 will follow from our next result.

Lemma 3.2.

$$\sum_{n \geq N_x} \mathbb{P} \left[Y_{-n}^{(j)} > x_n, \sum_{i \neq j} Y_{-n}^{(i)} > z_n \right] = o(\bar{F}^s(x)).$$

Proof. We use the same notation as in Appendix 7.2 of [2]. We have (with $z_x = z_{N_x}$)

$$\frac{\mathbb{P}[Y_{-n}^{(j)} > x_n, \sum_{i \neq j} Y_{-n}^{(i)} > z_n \mid \nu]}{\bar{F}(x_n)\bar{F}(z_x)} = \frac{\mathbb{P}[Y_{-n}^{(j)} > x_n \mid \nu]}{\bar{F}(x_n)} \frac{\mathbb{P}[\sum_{i \neq j} Y_{-n}^{(i)} > z_n \mid \nu]}{\bar{F}(z_x)} \leq \tilde{d}_\nu^{(j)} \sum_{i \neq j} \tilde{d}_\nu^{(i)}.$$

Taking the sum over n and then the expectation, we have

$$\sum_{n \geq N_x} \mathbb{P} \left[Y_{-n}^{(j)} > x_n, \sum_{i \neq j} Y_{-n}^{(i)} > z_n \right] \leq \mathbb{E} \left[\tilde{d}_\nu^{(j)} \sum_{i \neq j} \tilde{d}_\nu^{(i)} \right] \bar{F}(z_x) \sum_{n \geq N_x} \bar{F}(x_n),$$

which gives the result, completing the proofs of Lemma 3.2 and, hence, Lemma 3.1.

3.3. Computation of the exact asymptotics

In this section, we show that we can calculate the sum in (3.2). Its use in Corollary 3.1 and Lemma 3.1 will give the exact asymptotics of $\mathbb{P}[Z > x]$ in Theorem 2.1. Before stating this result, we must introduce some additional notation.

In the event $\bar{K}_n^j \cap \{Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L\}$, we have

$$Z_{[-n,0]} = f^j(Y_{-n}^{(j)}, n) + n\eta_n, \quad \text{with } \eta_n \text{ a random variable such that } |\eta_n| \leq \varepsilon_n.$$

Then, $\{Z_{[-n,0]} > x\} = \{f^j(Y_{-n}^{(j)}, n) > x - n\eta_n\}$. In order to prove the equivalence (2.9), we will first give an explicit form for the domains Δ^j defined in (2.5).

Lemma 3.3. *There exist constants $\{\alpha_i^j, \beta_i^j, \gamma_i^j\}_{0 \leq i \leq l}$ (given in closed form, in the proof of the present lemma, as functions of the quantities p_j and $b_{j,k}$ defined in Section 2.2), with $0 = \alpha_0^j \leq \alpha_1^j \leq \dots \leq \alpha_l^j$, such that*

$$\Delta^j(z) = \bigcup_{i=0}^l \left\{ \alpha_i^j z \leq t < \alpha_{i+1}^j z, \sigma > \frac{z}{\beta_i^j} + t\gamma_i^j \right\},$$

with the convention that $\alpha_{l+1}^j = \infty$. Moreover, we have

$$\alpha_0^j = 0, \quad \alpha_1^j = \frac{1}{p_j B_j}, \quad \beta_0^j = 1, \quad \gamma_0^j = a - p_j B_j,$$

for all j . In addition, $\beta_i^j \leq 1$ for all i and j , and the following inclusion holds:

$$\{\sigma \geq z + t(a - p_j B_j)\} \subset \Delta^j(z). \tag{3.3}$$

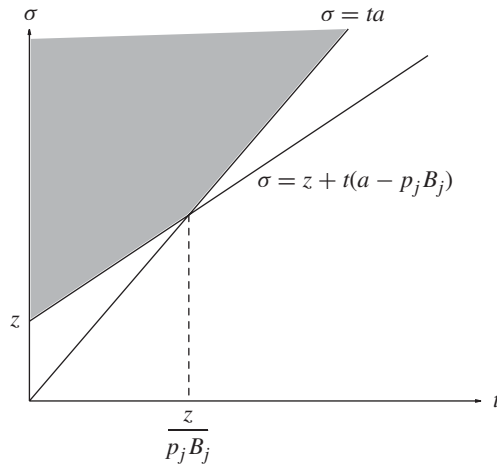


FIGURE 1: The first part of $\Delta^j(z)$.

Proof. The domain Δ^j may be divided into two parts, as follows:

$$\begin{aligned} \Delta^j(z) &= \{(\sigma, t) : f^j(\sigma, t) > z\}. \\ &= \{\sigma > ta, \sigma > z + t(a - p_j B_j)\} \cup \left\{ \sigma \leq ta, \sigma > a \min_k \frac{z + t(a - b_k)}{a - b_k + p_j b_{j,k}} \right\}. \end{aligned}$$

For the first part, we have (see Figure 1)

$$\begin{aligned} &\{\sigma > ta, \sigma > z + t(a - p_j B_j)\} \\ &= \left\{ 0 \leq t < \frac{z}{p_j B_j}, \sigma > z + t(a - p_j B_j) \right\} \cup \left\{ \frac{z}{p_j B_j} \leq t, \sigma > ta \right\}. \end{aligned}$$

For the second part, we have (see Figure 2)

$$\left\{ \sigma \leq ta, \sigma > a \min_k \frac{z + t(a - b_k)}{a - b_k + p_j b_{j,k}} \right\} = \bigcup_k \left\{ \frac{z}{p_j b_{j,k}} \leq t, a \frac{z + t(a - b_k)}{a - b_k + p_j b_{j,k}} < \sigma \leq ta \right\}.$$

From this, it is easy to see that the lemma holds (see Figure 3).

The inequality on the β_i^j follows directly from the fact that $p_j b_{j,k} \leq b_k$, from which we have

$$\frac{a}{a - b_k + p_j b_{j,k}} \geq 1.$$

Lemma 3.4. Let X be a random variable such that $\bar{F}^s \in \mathcal{S}$, let $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, and let $a(x)$ and $b(x)$ be real-valued functions such that, as $x \rightarrow \infty$,

$$\frac{a(x)}{x} \rightarrow a, \quad \frac{b(x)}{x} \rightarrow b, \quad \text{with } 0 < a < b.$$

If $\bar{F}(x) = P[X > x]$ then, for $\alpha \geq 1$ and $\beta > 0$, we have

$$\sum_{a(x) \leq n < b(x)} P[X > \alpha x + n(\beta + \varepsilon_n)] - \sum_{ax \leq n < bx} P[X > \alpha x + n\beta] = o(\bar{F}^s(x)) \quad \text{as } x \rightarrow \infty.$$

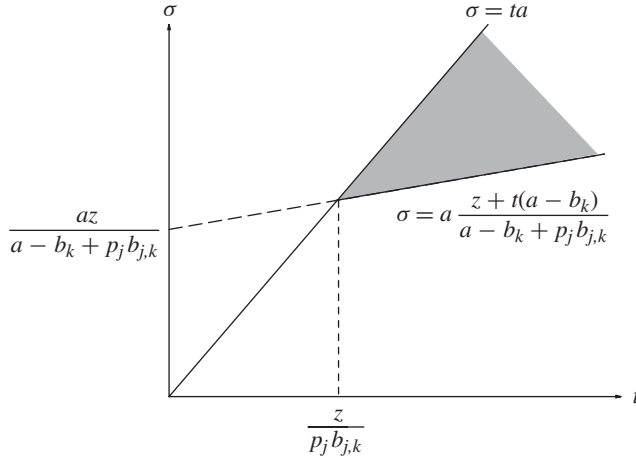


FIGURE 2: Construction of the second part of $\Delta^j(z)$.

Proof. For simplicity of notation, we assume that $a(x) \leq ax$ for all x . We have

$$\begin{aligned} \sum_{a(x) \leq n \leq ax} \mathbb{P}[X > \alpha x + n(\beta + \varepsilon_n)] &= \frac{1 + \varepsilon(x)}{\beta} \int_{\alpha x + a(x)\beta}^{\alpha x + ax\beta} \bar{F}(u) \, du \\ &\leq \frac{1 + \varepsilon(x)}{\beta} \frac{ax - a(x)}{ax} \bar{F}^s(\alpha x), \end{aligned}$$

since $\bar{F}(x)$ is nonincreasing. Hence, we have only to prove the lemma for $a(x) = ax$ and $b(x) = bx$. We have the following bound, with $\delta_x = \sup_{n \geq ax} \varepsilon_n$:

$$\begin{aligned} &\sum_{ax \leq n < bx} \mathbb{P}[X > \alpha x + n(\beta + \varepsilon_n)] - \mathbb{P}[X > \alpha x + n\beta] \\ &\leq \sum_n \mathbb{P}[X \in (\alpha x + n\beta, \alpha x + n(\beta + \delta_x))] \\ &= (1 + \varepsilon(x)) \bar{F}^s(\alpha x) \left(\frac{1}{\beta} - \frac{1}{\beta + \delta_x} \right) \\ &= o(\bar{F}^s(\alpha x)) = o(\bar{F}^s(x)). \end{aligned}$$

Proof of Theorem 2.1. From Corollary 3.1 and Lemma 3.1, we know that the tail asymptotics of the maximal dater is linked to the quantity $S(j)$, defined by

$$S(j) = \sum_{n \geq N_x} \mathbb{P}[Z_{[-n,0]} > x, \bar{K}_n^j, Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L].$$

In the event $A_{n,x}^j = \bar{K}_n^j \cap \{Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L\}$, we have

$$\{Z_{[-n,0]} > x\} = \{f^j(Y_{-n}^{(j)}, n) > x - n\eta_n\} = \{(Y_{-n}^{(j)}, n) \in \Delta^j(x - n\eta_n)\}.$$

Clearly, $\Delta^j(z)$ is a nonincreasing function of z , and we define

$$D_-^j =: \Delta^j(x - n\varepsilon_n) \supset \Delta^j(x - n\eta_n) \supset \Delta^j(x + n\varepsilon_n) =: D_+^j.$$

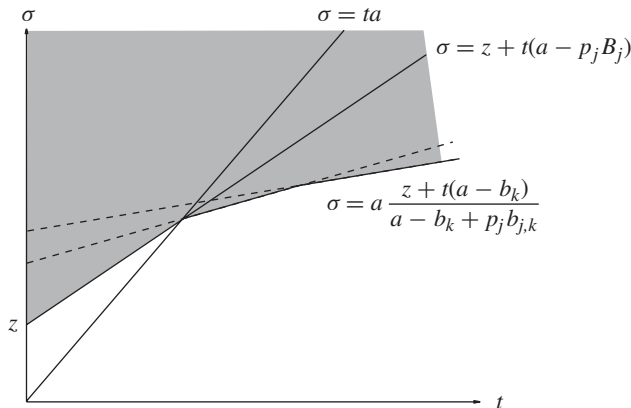


FIGURE 3: Domain $\Delta^j(z)$.

For simplicity of notation, we write $Y^{(j)} = Y_{-n}^{(j)}$ and $\phi^{(j)} = \phi^{(j)}(-n)$. We assume, without loss of generality, that ε_n is a decreasing sequence; hence, $\varepsilon_n \leq \varepsilon_{N_x} = \varepsilon(x)$ for $n \geq N_x$ and, for $n \geq N_x$, we have

$$\begin{aligned} A_+(n) &:= \mathbb{P}[(Y^{(j)}, n) \in D_+^j] \\ &= \sum_{i=0}^l \mathbf{1}_{\{\alpha_i^j(x+n\varepsilon_n) \leq n < \alpha_{i+1}^j(x+n\varepsilon_n)\}} \mathbb{P}\left[Y^{(j)} > \frac{x}{\beta_i^j} + n\gamma_i^j + \frac{n\varepsilon_n}{\beta_i^j}\right] \\ &\leq \sum_{i=0}^l \mathbf{1}_{\{\alpha_i^j x \leq n < \alpha_{i+1}^j(x+n\varepsilon(x))\}} \mathbb{P}\left[Y^{(j)} > \frac{x}{\beta_i^j} + n\gamma_i^j + \frac{n\varepsilon_n}{\beta_i^j}\right]. \end{aligned}$$

Then,

$$\begin{aligned} \sum_{n \geq N_x} A_+(n) &\leq \sum_{N_x \leq n < \alpha_1^j x(1+\varepsilon(x))} \mathbb{P}[Y^{(j)} > x + n(a - p_j B_j) + n\varepsilon_n] \\ &\quad + \sum_{i=1}^l \sum_{\alpha_i^j x \leq n < \alpha_{i+1}^j x(1+\varepsilon(x))} \mathbb{P}\left[Y^{(j)} > \frac{x}{\beta_i^j} + n\gamma_i^j + \frac{n\varepsilon_n}{\beta_i^j}\right]. \end{aligned}$$

From Assumption 2.2, we know that $Y^{(j)}$ has an integrated tail that belongs to \mathcal{S} ; therefore, as a corollary of Lemma 3.4, we have

$$\begin{aligned} \sum_{n \geq N_x} A_+(n) &= \sum_{0 \leq n < \alpha_1^j x} \mathbb{P}[Y^{(j)} > x + n(a - p_j B_j)] \\ &\quad + \sum_{i=1}^l \sum_{\alpha_i^j x \leq n < \alpha_{i+1}^j x} \mathbb{P}\left[Y^{(j)} > \frac{x}{\beta_i^j} + n\gamma_i^j\right] + \varepsilon(x) \bar{F}^s(x) \\ &= (1 + \varepsilon(x)) \sum_{i=0}^l \sum_{\alpha_i^j x \leq n < \alpha_{i+1}^j x} \mathbb{E}[\phi^{(j)}] \mathbb{P}\left[\sigma^{(j)} > \frac{x}{\beta_i^j} + n\gamma_i^j\right] + \varepsilon(x) \bar{F}^s(x), \end{aligned}$$

where the last equality follows from assumption (SE) of [4]. However, we have

$$S(j) \leq \sum_{n \geq N_x} A_+(n).$$

We now look at the lower bound. By the same arguments as above, we easily find that

$$\sum_{n \geq N_x} A_-(n) = \sum_{n \geq N_x} A_+(n) + \varepsilon(x) \bar{F}^s(x), \quad \text{with } A_-(n) := \mathbb{P}[(Y^{(j)}, n) \in D_-^j].$$

We now show that

$$\sum_{n \geq N_x} A_-(n) = \sum_{n \geq N_x} \mathbb{P}[(Y^{(j)}, n) \in D_-^j, A_{n,x}^j] + \varepsilon(x, L) \bar{F}^s(x).$$

Consider the difference

$$\begin{aligned} A_-(n) - \mathbb{P}[(Y^{(j)}, n) \in D_-^j, A_{n,x}^j] &\leq \mathbb{P}[(Y^{(j)}, n) \in D_-^j, \phi^{(j)}(-n) > L] \\ &\leq \mathbb{P}[Y^{(j)} \geq x + n(a - p_j B_j - \varepsilon_n), \phi^{(j)}(-n) > L], \end{aligned}$$

where the last inequality follows from inclusion (3.3) of Lemma 3.3. By the same kind of argument as in Corollary 3.1, we have

$$\sum_{n \geq N_x} A_-(n) - \mathbb{P}[(Y^{(j)}, n) \in D_-^j, A_{n,x}^j] \leq \varepsilon(x, L) \bar{F}^s(x).$$

Hence, we have proved that, when $x \rightarrow \infty$,

$$S(j) \sim \sum_{i=0}^l \sum_{\alpha_i^j x \leq n < \alpha_{i+1}^j x} \mathbb{E}[\phi^{(j)}] \mathbb{P}\left[\sigma^{(j)} > \frac{x}{\beta_i^j} + n\gamma_i^j\right].$$

Now, since this quantity is long tailed, we can use Corollary 3.1 to derive the required asymptotics for $\mathbb{P}[Z > x]$. This completes the proof.

Appendix A. Proof of Corollary 3.1

The proof is based on Proposition 3.1, which shows that

$$\mathbb{P}[Z > x] = (1 + \varepsilon(x)) \left(\sum_{j=1}^K \sum_{n \geq N_x} \mathbb{P}[Z > x, Y_{-n}^{(j)} > x_n, K_n^j] \right).$$

Since $Z \geq Z_{[-n,0]}$, we have

$$\begin{aligned} \mathbb{P}[Z > x, Y_{-n}^{(j)} > x_n, K_n^j] &\geq \mathbb{P}[Z_{[-n,0]} > x, Y_{-n}^{(j)} > x_n, K_n^j] \\ &\geq \mathbb{P}[Z_{[-n,0]} > x, Y_{-n}^{(j)} > x_n, K_n^j, \phi^{(j)}(-n) \leq L] \end{aligned}$$

and, hence,

$$\mathbb{P}[Z > x] \geq (1 + \varepsilon(x)) \left(\sum_{j=1}^K \sum_{n \geq N_x} \mathbb{P}[Z_{[-n,0]} > x, Y_{-n}^{(j)} > x_n, K_n^j, \phi^{(j)}(-n) \leq L] \right).$$

We now derive the upper bound. Take $z_x \rightarrow \infty$ such that $\bar{F}^s(x + z_x) \sim \bar{F}^s(x)$. Then, when $x \rightarrow \infty$, we have

$$P[Z_{[-\infty, -n-1]} < z_x] = P[Z < z_x] \rightarrow 1.$$

We define $\tilde{x} = x + z_x$ and

$$\tilde{K}_{n,x}^j = K_n^j \cap \{Z_{[-\infty, -n-1]} \leq z_x\}.$$

Observe that $\tilde{K}_{n,x}^j$ satisfies the assumptions of Proposition 3.1. By subadditivity, we have $Z \leq Z_{[-\infty, -n-1]} + Z_{[-n, 0]}$ (see [2]) and, hence,

$$\begin{aligned} P[Z > \tilde{x}, \tilde{K}_{n,x}^j, Y_{-n}^{(j)} > \tilde{x}_n] &\leq P[Z_{[-\infty, -n-1]} + Z_{[-n, 0]} > \tilde{x}, \tilde{K}_{n,x}^j, Y_{-n}^{(j)} > \tilde{x}_n] \\ &\leq P[Z_{[-n, 0]} > x, \tilde{K}_{n,x}^j, Y_{-n}^{(j)} > x_n] \\ &\leq P[Z_{[-n, 0]} > x, K_n^j, Y_{-n}^{(j)} > x_n]. \end{aligned}$$

We now make a partition of ϕ :

$$\begin{aligned} A(n) &= P[Z_{[-n, 0]} > x, K_n^j, Y_{-n}^{(j)} > x_n] \\ &\leq P[Z_{[-n, 0]} > x, K_n^j, Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L] + P[Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) > L] \\ &= P[Z_{[-n, 0]} > x, K_n^j, Y_{-n}^{(j)}(E(-n)) > x_n, \phi^{(j)}(-n) \leq L] + B(n). \end{aligned}$$

We will use the following result due to Kesten (for a proof, see [1]).

Lemma A.1. *Let $X \in \mathcal{S}$ and let S_n be the sum of n independent copies of X . Then, for every $\epsilon > 0$, there exists a $K(\epsilon) > 0$ such that*

$$\sup_{x \geq 0} \frac{P[S_n > x]}{P[X > x]} \leq K(\epsilon)(1 + \epsilon)^n, \quad n = 1, 2, \dots$$

Recall that $P[\phi^{(j)}(0) = l] = \delta^l(1 - \delta)$ for some $0 < \delta < 1$. Hence, if we take an ϵ such that $(1 + \epsilon)\delta < 1$, then

$$\begin{aligned} B(n) &= \sum_{l \geq L+1} P[\phi^{(j)}(-n) = l] P\left[\sum_{k=1}^l \sigma_k^{(j)}(-n) > x_n\right] \\ &\leq \sum_{l \geq L+1} \delta^l(1 - \delta)K(\epsilon)(1 + \epsilon)^l P[\sigma^{(j)} > x_n] \\ &\leq (1 - \delta)K(\epsilon) P[\sigma^{(j)} > x_n] \frac{((1 + \epsilon)\delta)^{L+1}}{1 - (1 + \epsilon)\delta}, \end{aligned}$$

and we have

$$\sum_{n \geq N_x} A(n) \leq \sum_{n \geq N_x} P[Z > x, K_n^j, Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L] + \epsilon(x, L)\bar{F}^s(x).$$

Since $\tilde{K}_{n,x}^j$ satisfies the assumptions of Proposition 3.1, we have

$$\begin{aligned} \mathbb{P}(Z > \tilde{x}) &= (1 + \varepsilon(\tilde{x})) \sum_{j=1}^K \sum_{n \geq N_{\tilde{x}}} \mathbb{P}[Z > \tilde{x}, \tilde{K}_{n,x}^j, Y_{-n}^{(j)} > \tilde{x}_n] \\ &\leq (1 + \varepsilon(x)) \sum_{j=1}^K \sum_{n \geq N_x} \mathbb{P}[Z_{[-n,0]} > x, K_n^j, Y_{-n}^{(j)} > x_n] \\ &\leq (1 + \varepsilon(x)) \sum_{j=1}^K \sum_{n \geq N_x} \mathbb{P}[Z_{[-n,0]} > x, K_n^j, Y_{-n}^{(j)} > x_n, \phi^{(j)}(-n) \leq L] \\ &\quad + \varepsilon(x, L) \bar{F}^s(x). \end{aligned}$$

Hence, we have shown, in the notation of Lemma A.1, that

$$\begin{aligned} \mathbb{P}[Z > x] &\geq (1 + \varepsilon(x))G(x), \\ \mathbb{P}[Z > x + z_x] &\leq (1 + \varepsilon(x))G(x) + \varepsilon(x, L)\bar{F}^s(x). \end{aligned}$$

From these inequalities, we can directly derive (3.1). If G is long tailed, we can choose $z_x \rightarrow \infty$ such that $G(x + z_x) \sim G(x)$ and $\bar{F}^s(x + z_x) \sim \bar{F}^s(x)$, and the last statement of Corollary 3.1 follows.

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