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A SEGAL-LANGEVIN TYPE STOCHASTIC DIFFERENTIAL EQUATION ON A SPACE OF GENERALIZED FUNCTIONALS

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ABSTRACT. Let E' be the dual of a nuclear Fréchet space E and $L^*(t)$ the adjoint operator of a diffusion operator $L(t)$ of infinitely many variables, which has a formal expression:

$$L(t) = \sum_{i,j=1}^{\infty} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{\infty} b_i(t,x)\frac{\partial}{\partial x_i}.$$

A weak form of the stochastic differential equation

$$dX(t) = dW(t) + L^*(t)X(t)\,dt$$

is introduced and the existence of a unique solution is established. The solution process is a random linear functional (in the sense of I. E. Segal) on a space of generalized functionals on E'. The above is an appropriate model for the central limit theorem for an interacting system of spatially extended neurons. Applications to the latter problem are discussed.

1. **Introduction.** A class of stochastic equations (SDE's) governing nuclear space valued processes as a model for the behavior of single neurons was introduced in a recent paper by Kallianpur and Wolpert [12]. The present paper is motivated principally by the study of the asymptotic behavior of the voltage potentials of spatially extended neurons which are described by a system of n interacting SDE's of the type considered in [12]. The techniques developed in this paper enable us to prove a central limit theorem for empirical distributions of interacting dual nuclear space valued processes which is an infinite dimensional version of the fluctuation theorem for McKean's model of n-particle diffusions [9]. The result (Theorem 2) is derived in the last section as a consequence of Theorem 1 which is a general result whose proof occupies most of the rest of the paper. Theorem 2 is similar to the ones for mean-field interacting particle diffusions treated in a number of papers [2, 4, 9, 10, 17, 25]. However, the fact that the interesting SDE's represent infinite dimensional systems raises several technical difficulties. For instance, the interaction coefficient in the system of SDE's (5.4) is a function defined on infinite dimensional spaces so that conditions of smoothness *etc.* have to be given in terms of functional derivatives and one is required to introduce suitable distribution spaces on

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test functions of infinitely many variables. Consequently, the detailed proofs of some of the lemmas have acquired a forbidding aspect, a circumstance which seems unavoidable. Nevertheless, we think it a worthwhile effort to study this special example of a fluctuation theorem for infinite dimensional systems because the final result is not a routine extension of the finite dimensional case (the reasons for which are discussed below) and introduces in a natural way a SDE whose solution is best interpreted as a random linear process in the sense of I. E. Segal [24].

Before proceeding to the more technical part of the Introduction, we mention that another possible application of Theorem 1 is to the fluctuation theorem recently obtained by J. D. Deuschel for a system of lattice valued diffusion processes [5].

A natural setting for our problem is a nuclear space $C_0^\infty(E')$ of smooth functions on E'. However, in general $C_0^\infty(E')$ is not invariant under the operator $L(t)$ of interest to us. In other words, the range of $L(t)$ is not contained in $C_0^\infty(E')$. To remedy this situation we define a suitable space $\mathcal{D}_{E'}$ of test functions on E' as the completion of $C_0^\infty(E')$ in a sense different from that in the Malliavin calculus [29] to be made precise below which is invariant under the operator $L(t)$. To obtain the desired central limit theorem we also need to be able to regard Dirac measures on E' as generalized functions on E'. The identification problem of the limit measures leads us to study the stochastic analogue of the deterministic evolution equation

$$\frac{dX(t)}{dt} = L^*(t)X(t)$$

on the dual space $\mathcal{D}'_{E'}$ of $\mathcal{D}_{E'}$, $L^*(t)$ being the adjoint of $L(t)$. The corresponding SDE may formally be written as

$$(*) \qquad dX(t) = dW(t) + L^*(t)X(t)\,dt.$$

The work of the present paper differs from the by now familiar theory of stochastic evolution equations in duals of nuclear spaces in the following essential respect and also extends [21] to the case of infinitely many variables in a weak form. In the above SDE, the process $W(t)$ (the limiting Wiener process obtained in the course of our proof) lives in $C_0^\infty(E')'$ which is strictly larger than $\mathcal{D}'_{E'}$. We thus have to give a meaning to the SDE given above and this is done by introducing a weak form of $(*)$. We begin by explaining the setting more precisely: A stochastic process $X_F(t)$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ indexed by elements in $\mathcal{D}_{E'}$ is called an $\mathcal{L}(D_{E'})$-*process* if $X_F(t)$ is a real stochastic process for any fixed $F \in D_{E'}$ and $X_{\alpha F+\beta G}(t) = \alpha X_F(t) + \beta X_G(t)$ almost surely for real numbers α, β and elements of $F, G \in \mathcal{D}_{E'}$ and further $E[X_F(t)^2]$ is continuous with respect to F on $\mathcal{D}_{E'}$ [11]. $X_F(t)$ is called *continuous* if $\lim_{t\to s} E\big[\big(X_F(t) - X_F(s)\big)^2\big] = 0$ for each $F \in D_{E'}$. Let $W_F(t)$ be an $\mathcal{L}(D_{E'})$-Wiener process, *i.e.* such that for any fixed $F \in \mathcal{D}_{E'}$. $W_F(t)$ is a real continuous Gaussian additive process with mean 0.

The weak form of $(*)$ is an SDE of the form

$$(1.1) \qquad dX_F(t) = dW_F(t) + X_{L(t)F}\,dt$$

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with given initial value $X_F(0)$ and $F \in \mathcal{D}_{E'}$. Our aim is to show that (1.1) has a unique continuous $\mathcal{L}(\mathcal{D}_{E'})$-process solution $X_F(t)$. The process $X_F(t)$ is a random linear functional in $\mathcal{D}'_{E'}$ in the sense of I. E. Segal. Roughly speaking, if $L(t)$ generates the strongly continuous Kolmogorov evolution operator $U(t,s)$ from $\mathcal{D}_{E'}$ into itself, the unique solution for (1.1) can be given as follows:

$$X_F(t) = X_{U(t,0)F}(0) + W_F(t) + \int_0^t W_{L(s)U(t,s)F}(s)\, ds.$$

We now begin by giving the precise definitions of the operator $L(t)$ and the space $\mathcal{D}_{E'}$. Let E be a nuclear Fréchet space whose topology is defined by an increasing sequence of Hilbertian semi-norms $\|\cdot\|_0 \le \|\cdot\|_1 \le \|\cdot\|_2 \le \cdots \le \|\cdot\|_p \le \cdots$. As usual let E' be the dual space, E_p the completion of E by the p-th semi-norm $\|\cdot\|_p$, E'_p the dual space of E_p and $\|\cdot\|_{-p}$ the dual norm. Then we have

$$E = \bigcap_{p=0}^{\infty} E_p \text{ and } E' = \bigcup_{p=0}^{\infty} E'_p.$$

Let K be a separable Hilbert space with norm $\|\cdot\|_K$ and F a mapping from E' into K. Then F is said to be *E'_p-Fréchet differentiable* if for every $x \in E'$, we have a bounded linear operator $\mathcal{D}_p F(x)$ from E'_p into K such that

$$\lim_{t \to 0} \frac{F(x+th) - F(x)}{t} = \mathcal{D}_p F(x)(h), \quad h \in E'_p, \text{ in } K.$$

Suppose that F is E'_p-Fréchet differentiable for every integer $p \ge 0$. Then taking $E' = \bigcup_{p=0}^{\infty} E'_p$ and the strong topology of E', (which is equivalent to the inductive limit topology of E'_p ; $p = 0, 1, 2, \ldots$), into account, we have a continuous linear operator $DF(x)$ from E' equipped with the strong topology into K such that for any integer $p \ge 0$, $DF(x)(h) = \mathcal{D}_p F(x)(h)$ for $h \in E'_p$. Hence, if F is n-times E'_p-Fréchet differentiable for every integer $p \ge 0$, we have a continuous n-linear operator $D^n F(x)$ from $\underbrace{E' \times E' \times \cdots \times E'}_{n\text{-times}}$ into K such that the restriction of $D^n F(x)$ on $\underbrace{E'_p \times E'_p \times \cdots \times E'_p}_{n\text{-times}}$ = the n-th E'_p-Fréchet derivative $\mathcal{D}^n_p(F)(x)$. Then if F is infinitely many times E'_p-Fréchet differentiable for every integer $p \ge 0$, the Hilbert-Schmidt norm

$$\|D^n F(x)\|^{(p)}_{\mathrm{H.S.}} = \Big(\sum_{i_1, i_2, \ldots, i_n = 1}^{\infty} \|D^n F(x)(h^{(p)}_{i_1}, h^{(p)}_{i_2}, \ldots, h^{(p)}_{i_n})\|^2_K \Big)^{1/2}$$

is finite for each integer $n \ge 1$ and $p \ge 0$, where $(h^{(p)}_j)$ is a C.O.N.S., (complete orthonormal system), in E'_p [15].

From now on, we will use the conventional notation such that $\|D^0 F(x)\|^{(p)}_{\mathrm{H.S.}} = \|F(x)\|_K$.

The nuclear space E' will be assumed to satisfy the following basic assumptions.

ASSUMPTION 1. For any integer $p \geq 0$, there exists an integer $q > p$ such that $\|x\|_{-q} \leq \frac{1}{2}\|x\|_{-p}$.

ASSUMPTION 2. $\|D^n F(x)\|^{(p)}_{\text{H.S.}} \leq \|D^n F(x)\|^{(q)}_{\text{H.S.}}$ for integers $n \geq 1$ and $p \leq q$.

These assumptions are satisfied for the nuclear spaces of interest to us. Here we give a brief verification of the assumptions for E' where $E = \Phi = \{\phi(x) = e(x)\varphi(x)\ ;\ \varphi \in \mathcal{S}(\mathbf{R})\}$, where $\mathcal{S}(\mathbf{R})$ is the Schwartz space of rapidly decreasing C^∞-functions on the 1-dimensional Euclidean space $\mathbf{R}$, $e(x) = 1/\pi(x)$, $\pi(x) = \int_R e^{-|y|}\rho(x-y)\,dy$,

$$\rho(x) = \begin{cases} c \cdot \exp\bigl(-1/(1-|x|^2)\bigr), & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

and c is the constant satisfying $\int_R \rho(x)\,dx = 1$ [9]. The nuclear Fréchet topology of Φ is equivalent to that metrized by $\|\phi\|_p = [\sum_{j=0}^\infty \alpha_j(p)^2(\varphi,\varphi_j)^2]^{1/2}$, $p = 0,1,2,\ldots$, where $(\cdot,\cdot)$ denotes the inner product of $L^2(\mathbf{R})$, $\{\varphi_j \in \mathcal{S}(\mathbf{R})\}$ is a C.O.N.S. in $L^2(\mathbf{R})$ and $\alpha_j(p) = (2+2j)^p$.

Since for any integer $p \geq 0$, there exists $q > p$ such that

$$2\|\phi\|_p \leq \|\phi\|_q,$$

we have

$$\|x\|_{-q} = \sup_{\|\phi\|_q \leq 1} |\langle x,\phi\rangle| \leq \sup_{2\|\phi\|_p \leq 1} |\langle x,\phi\rangle| \leq \frac{1}{2}\|x\|_{-p},$$

which asserts Assumption 1 for Φ'. Let $h_j \in L^2(\mathbf{R})' \subset \Phi'$ satisfy $\langle h_j,\varphi_i\rangle = \delta_{ij}$. Then Φ'_p has a C.O.N.S. $\{h_j^{(p)} = \alpha_j(p)h_j\}$. Since $\alpha_j(p) \leq \alpha_j(q)$ if $p \leq q$,

$$\begin{aligned}
&\|D^n F(x)\|^{(p)}_{\text{H.S.}} \\
&= \Bigl(\sum_{i_1,i_2,\ldots,i_n=1}^\infty \|D^n F(x)(h_{i_1}^{(p)},h_{i_2}^{(p)},\ldots,h_{i_n}^{(p)})\|_K^2\Bigr)^{1/2} \\
&= \Bigl(\sum_{i_1,i_2,\ldots,i_n=1}^\infty \Bigl\|D^n F(x)\bigl(\alpha_{i_1}(p)h_{i_1},\alpha_{i_2}(p)h_{i_2},\ldots,\alpha_{i_n}(p)h_{i_n}\bigr)\Bigr\|_K^2\Bigr)^{1/2} \\
&= \Bigl(\sum_{i_1,i_2,\ldots,i_n=1}^\infty \alpha_{i_1}(p)^2\alpha_{i_2}(p)^2\cdots\alpha_{i_n}(p)^2\|D^n F(x)(h_{i_1},h_{i_2},\ldots,h_{i_n})\|_K^2\Bigr)^{1/2} \\
&\leq \Bigl(\sum_{i_1,i_2,\ldots,i_n=1}^\infty \alpha_{i_1}(q)^2\alpha_{i_2}(q)^2\cdots\alpha_{i_n}(q)^2\|D^n F(x)(h_{i_1},h_{i_2},\ldots,h_{i_n})\|_K^2\Bigr)^{1/2} \\
&= \|D^n F(x)\|^{(q)}_{\text{H.S.}},
\end{aligned}$$

which implies Assumption 2 for Φ'.

Let $\beta(t)$ be the standard E'-Wiener process such that for any $\xi \in E$, $\langle\beta(t),\xi\rangle$ is a 1-dimensional Brownian motion with variance $E[\langle\beta(t),\xi\rangle^2] = t\|\xi\|_0^2$, where $\langle x,\xi\rangle$, $(x \in E', \xi \in E)$, denotes the canonical bilinear form on $E' \times E$. Without loss of generality, we assume $\beta(t)$ is an E'_1-valued Wiener process. [15].

DEFINITION OF $L(t)$. We need conditions which are infinite dimensional analogs to those given in [21]. For $t > 0$ and $x \in E'$, let $B(t,\cdot)$ be a continuous mapping from E' into itself such that the following conditions are satisfied.

(H1) There exists an integer $p_0 \geq 1$ such that $B(t,\cdot)$ maps E' into E'_{p_0} and for each $T > 0$,

$$\sup_{\substack{x\in E' \\ 0\leq t\leq T}} \|B(t,x)\|_{-p_0} < \infty.$$

(H2) $B(t,x)$ is infinitely many times E'_p-Fréchet differentiable for every integer $p \geq 0$ such that for any $T > 0$ and any integer $n \geq 1$,

$$\sup_{\substack{x\in E' \\ 0\leq t\leq T}} \|D^n B(t,x)\|^{(p)}_{\mathrm{H.S.}} < \infty,$$

where $\|D^n B(t,x)\|^{(p)}_{\mathrm{H.S.}} = (\sum_{i_1,i_2,\ldots,i_n=1}^{\infty} \|D^n B(t,x)(h^{(p)}_{i_1}, h^{(p)}_{i_2}, \ldots, h^{(p)}_{i_n})\|^2_{-p_0})^{1/2}$.

(H3) For any integer $n \geq 0$ and any $T > 0$, there exist $\lambda(n,p,T) > 0$ and $\lambda_1(n,p,T) > 0$ such that

$$\sup_{\substack{x\in E' \\ 0\leq k\leq n}} \|D^k B(t,x) - D^k B(s,x)\|^{(p)}_{\mathrm{H.S.}} \leq \lambda_1(n,p,T)|t-s|^{\lambda(n,p,T)}, \quad 0 \leq s,\ t \leq T.$$

For simplicity, let $\mathcal{K}(t)$ be a continuous linear operator from E' to itself and generate the strongly continuous evolution operator $V(t,s)$ from E' to itself such that for any integer p and any $T > 0$, there exist integers $m(p,T) \geq p$ and $n(p,T) \geq p$ satisfying

(V1)
$$\left\|\big(\mathcal{K}(t) - \mathcal{K}(s)\big)x\right\|_{-m(p,T)} \leq C_1|t-s|\,\|x\|_{-p},$$
$$\sup_{0\leq s\leq t\leq T} \|V(t,s)x\|_{-n(p,T)} \leq \|x\|_{-p},$$
$$\|V(t',s')x - V(t,s)x\|_{-n(p,T)} \leq \|x\|_{-p}\{|t-t'| + |s-s'|\}.$$

Without loss of generality, we assume $m(p,T) \leq m(q,T)$ and $n(p,T) \leq n(q,T)$ if $p \leq q$. Here and in the sequel, we denote positive constants by C_i or, if necessary, by $C_i(\tau_1,\tau_2,\ldots)$, $i = 1,2,\ldots$, in case they depend on the parameters $\tau_1,\tau_2,\ldots$.

Then for any twice E'_p-Fréchet differentiable real valued functional F on E' for every $p \geq 0$, we put

$$\big(L(t)F\big)(x) = \frac{1}{2}\,\mathrm{trace}_{E_0} D^2F(x) + DF(x)\big(B(t,x) + \mathcal{K}(t)x\big),$$

where

$$\mathrm{trace}_{E_0} D^2F(x) = \sum_{j=1}^{\infty} D^2F(x)(h^{(0)}_j, h^{(0)}_j).$$

DEFINITION OF $\mathcal{D}_{E'}$. We will extend the weighted Schwartz space Φ introduced in [9], [21] to the case of infinitely many variables. For a real valued infinitely many times

E'_p-Fréchet differentiable functional F on E' for every integer $p \geq 0$, we define the following semi-norms:

$$\|F\|_{p,q,n} = \sum_{k=0}^{n} \|F\|_{p,k}^{(q)},$$

where $p \geq 0$, $q \geq 0$ and $n \geq 0$ are integers and

$$\|F\|_{p,k}^{(q)} = \sup_{x \in E'_p} e^{-\|x\|_{-p}} \|D^k F(x)\|_{\mathrm{H.S.}}^{(q)}.$$

For any natural number n, define

$$\Phi(\mathbf{R}^n) = \{ \phi(x) = h(\mathbf{x})\varphi(\mathbf{x})\ ;\ \varphi \in \mathcal{S}(\mathbf{R}^\mathbf{n}) \},$$

where $h(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, is a weight function such that $h(\mathbf{x}) = 1/g(\mathbf{x})$, $g(\mathbf{x}) = \prod_{i=1}^{n} g_0(x_i)$, $g_0(x_i) = \exp\Big(-\sqrt{\int_R |y| \rho(x_i - y)\, dy}\Big)$. Let $\{ \overset{\circ}{\xi}_j; j = 1, 2, \ldots \}$ be a countable dense subset of E. Define

$$C_{0,n}^{\infty}(E') = \{ F(x) = \phi(\langle x, \overset{\circ}{\xi}_1 \rangle, \langle x, \overset{\circ}{\xi}_2 \rangle, \ldots, \langle x, \overset{\circ}{\xi}_n \rangle)\ ;\ \phi \in \Phi(\mathbf{R}^n) \}$$

and introduce the nuclear Fréchet topology on this space by the countably many semi-norms;

$$\|F\|_p = \sup_{\substack{x \in R^n \\ 0 \leq k \leq p}} (1 + |\mathbf{x}|^2)^p \Big| \Big(\frac{d}{d\mathbf{x}}\Big)^k \big(g(\mathbf{x})\big)\phi(\mathbf{x})\big) \Big|, \quad p = 0, 1, 2 \ldots,$$

where $(\frac{d}{d\mathbf{x}})^k = \sum_{k_1 + k_2 + \cdots + k_n = k} \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}}$. Then we have a fundamental space $C_0^{\infty}(E') = \bigcup_{n=1}^{\infty} C_{0,n}^{\infty}(E')$ which is the strict inductive limit of nuclear Fréchet spaces $C_{0,n}^{\infty}(E')$.

For any integers $p \geq 0$, $q \geq 0$ and $n \geq 0$, let $\mathcal{D}_{p,q,n}$ be the completion of $C_0^{\infty}(E')$ by the semi-norm $\| \cdot \|_{p,q,n}$. We define $\mathcal{D}_{E'} = \bigcap_{p,q,n} \mathcal{D}_{p,q,n}$ and introduce a topology on $\mathcal{D}_{E'}$ by the countably many semi-norms $\| \cdot \|_{p,q,n}$, $p \geq 0$, $q \geq 0$ and $n \geq 0$.

Then $\mathcal{D}_{E'}$ becomes a complete separable metric space [7].

REMARK 1. The definition of $\mathcal{D}_{E'}$ is independent of the way of choosing a countable dense subset of E. We call a real valued functional $F(x) = \phi(\langle x, \xi_1 \rangle, \langle x, \xi_1 \rangle, \ldots, \langle x, \xi_n \rangle)$ where n is a natural number, $\xi_i \in E$, $i = 1, 2, \ldots, n$, and $\phi \in \Phi(\mathbf{R}^n)$ a weighted Schwartz functional. Let $\mathcal{P}$ be the set of all weighted Schwartz functionals, $\mathcal{P}_{p,q,n}$ the completion of $\mathcal{P}$ by $\| \cdot \|_{p,q,n}$ and $\mathcal{D} = \bigcap_{p,q,n} \mathcal{P}_{p,q,n}$ where $p \geq 0$, $q \geq 0$ and $n \geq 0$ are integers. Then

$$\mathcal{D} = \mathcal{D}_{E'}.$$

PROOF. It is enough to show that $F(x) = \phi(\langle x, \xi_1 \rangle, \langle x, \xi_2 \rangle, \ldots, \langle x, \xi_m \rangle)$, $\xi_i \in E$, $\phi \in \Phi(\mathbf{R}^m)$, belongs to $\mathcal{D}_{p,q,n}$. By the nuclearity of E, we have a natural number $r > \max\{p, q\}$ such that

$$\sum_{j=1}^{\infty} \|h_j^{(q)}\|_{-r}^2 < \infty \tag{1.2}$$

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and since $\{\overset{\circ}{\xi}_j\}$ is dense in E, for each i, there exists a sequence $\{\overset{\circ}{\xi}_{i,k}\}$, $\overset{\circ}{\xi}_{i,k} \in \{\overset{\circ}{\xi}_j\}$ such that

$$\lim_{k\to\infty} \|\xi_i - \overset{\circ}{\xi}_{i,k}\|_r = 0. \tag{1.3}$$

On the other hand, $D^n F(x)(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \ldots, h_{i_n}^{(q)})$ is a finite sum of terms;

$$\begin{aligned}(1.4)\qquad &\frac{\partial^n}{\partial x_1^{k_1}\partial x_2^{k_2}\cdots\partial x_m^{k_m}}\phi(\langle x,\xi_1\rangle,\langle x,\xi_2\rangle,\ldots,\langle x,\xi_m\rangle)\\ &\langle h_{j_1^{(1)}}^{(q)},\xi_1\rangle\langle h_{j_2^{(1)}}^{(q)},\xi\rangle\cdots\langle h_{j_{k_1}^{(1)}}^{(q)},\xi_1\rangle\\ &\langle h_{j_1^{(2)}}^{(q)},\xi_2\rangle\langle h_{j_2^{(2)}}^{(q)},\xi_2\rangle\cdots\langle h_{j_{k_2}^{(2)}}^{(q)},\xi_2\rangle\cdots\\ &\langle h_{j_1^{(m)}}^{(q)},\xi_m\rangle\langle h_{j_2^{(m)}}^{(q)},\xi_m\rangle\cdots\langle h_{j_{k_m}^{(m)}}^{(q)},\xi_m\rangle,\end{aligned}$$

where $k_1 + k_2 + \cdots + k_m = n$. Since

$$\left|\frac{\partial_n}{\partial x_1^{k_1}\partial x_2^{k_2}\cdots\partial x_m^{k_m}}h(\mathbf{x})\right| \le C_2 \exp\Big(\sum_{i=1}^m \sqrt{|x_i|}\Big), \tag{1.5}$$

noticing $\phi(\mathbf{x}) = h(\mathbf{x})\varphi(\mathbf{x})$, $\varphi \in \mathcal{S}(\mathbf{R^m})$ and (1.2) and setting $F^{(k)}(x) = \phi(\langle x, \overset{\circ}{\xi}_{1,k}\rangle, \langle x, \overset{\circ}{\xi}_{2,k}\rangle, \ldots, \langle x, \overset{\circ}{\xi}_{m,k}\rangle)$, we have

$$\lim_{k\to\infty} \|F - F^{(k)}\|_{p,q,n} = 0,$$

which completes the proof.

Before proceeding to the discussion of equation (1.1), the following remarks on the $\mathcal{L}(\mathcal{D}_{E'})$-Wiener process are in order. Taking the continuity of $W_F(t)$ and $E[W_F(t)^2]$ with respect to the parameters t and F into account, we note that $\sup_{0\le t\le T} E[W_F(t)^2] < \infty$ and $\sup_{0\le t\le T} E[W_F(t)^2]$ is lower semi-continuous on $\mathcal{D}_{E'}$. Since $\mathcal{D}_{E'}$ is a complete metric space, by the Baire category theorem there exist positive integers p_1, q_1 and m_1 such that

$$\sup_{0\le t\le T} E[W_F(t)^2] \le C_3(T)\|F\|^2_{p_1,q_1,m_1}. \tag{1.6}$$

2. **Existence and uniqueness of solutions of the** SDE. First we need to define the term, "approximated by bounded smooth functionals", which comes from dealing with an infinitely many variables version of [21]. Let K be a separable Hilbert space. We call a K-valued functional $G(x) = g(\langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \ldots, \langle x,\xi_n\rangle)$, $\xi_1, \xi_2, \ldots, \xi_n \in E$ a smooth functional if $g(x)\colon \mathbf{R}^n \longrightarrow K$ is a C^∞-function. Further we call $G(x)$ a bounded smooth functional if $g(x)$ itself and all the derivatives of $g(x)$ are bounded. The coefficient $B(t,x)$ is said to be approximated by bounded smooth functionals on E' if for any integers, $p \geqq p_0$, $q \ge 0$ and $n \ge 0$, there exists a sequence of bounded smooth functionals

$$B_m(t,x) = b_m(t, \langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \ldots, \langle x,\xi_{k_m}\rangle)$$

such that the following conditions are satisfied:

I. $B_m(t,x)$ satisfies the conditions (H1), (H2) and (H3),

II. For any $T > 0$,

$$\lim_{m\to\infty} \sup_{\substack{x\in E'\\ 0\le t\le T}} \|B(t,x) - B_m(t,x)\|_{-p_0} = 0,$$

$$\lim_{m\to\infty} \sup_{\substack{x\in E'\\ 0\le t\le T}} \|D^k B(t,x) - D^k B_m(t,x)\|^{(q)}_{\mathrm{H.S.}} = 0, \quad k = 1, 2, \ldots, n.$$

Then we have

THEOREM 1. *Suppose that the coefficient $B(t,x)$ satisfies the conditions (H1)–(H3) and is approximated by bounded smooth functionals on E'. Then $L(t)$ generates the Kolmogorov evolution operator $U(t,s)$ from $\mathcal{D}_{E'}$ into itself. Further the continuous $\mathcal{L}(\mathcal{D}_{E'})$-process solution of (1.1) such that for some $0 < \alpha < 1$, $E[|X_F(0)|^{2+\alpha}] < \infty$ is uniquely given as follows:*

$$X_F(t) = X_{U(t,0)F}(0) + W_F(t) + \int_0^t W_{L(s)U(t,s)F}(s)\, ds.$$

PROOF. As in [21], [22], we carry out the proof via the stochastic method. Let $\eta_{s,t}(x)$ be a solution of the following stochastic differential equation:

$$\eta_{s,t}(x) = V(t,s)x + \int_s^t V(t,r)\, d\beta(r) + \int_s^t V(t,r) B\big(r, \eta_{s,r}(x)\big)\, dr.$$

By the assumptions (H1) and (H2), if $p \ge p_0$ and $x \in E'_p$, then the solution of the above equation is uniquely obtained by the usual method successive approximations in $E'_{n(p,T)}$.

For any F in $\mathcal{D}_{E'}$, we set

$$\big(U(t,s)F\big)(x) = E[F\big(\eta_{s,t}(x)\big)].$$

The proof of Theorem 1 is based on several lemmas whose proof will be given in Sections 3 and 4. We begin by using the following lemmas which will be proved later.

LEMMA 1. *Suppose that the coefficient $B(t,x)$ is approximated by bounded smooth functionals on E'. Then if $F \in \mathcal{D}_{E'}$, $U(t,s)F \in \mathcal{D}_{E'}$ and $L(t)F \in \mathcal{D}_{E'}$.*

LEMMA 2. *Under the same assumptions as in Theorem 1, $L(t)$ generates the Kolmogorov evolution operator $U(t,s)$ from $\mathcal{D}_{E'}$ into itself such that*

(1) $U(t,s)$ is a continuous linear operator from $\mathcal{D}_{E'}$ into itself,

(2) for any $F \in \mathcal{D}_{E'}$, $U(t,s)F$ is continuous from $\{(t,s)\,;\, 0 \le s \le t\}$ into $\mathcal{D}_{E'}$,

(3) $U(t,t) = U(s,s) =$ identity operator,

(4) $\frac{d}{dt}U(t,s)F = U(t,s)L(t)F, \quad 0 \le s \le t$ on $\mathcal{D}_{E'}$,

(5) $\frac{d}{ds}U(t,s)F = -L(s)U(t,s)F, \quad 0 \le s \le t \quad t > 0$ on $\mathcal{D}_{E'}$.

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Further, for any integers $p \geq 0$, $q \geq 0$, $n \geq 0$, $j \geq 1$ and any $T > 0$ and $F \in \mathcal{D}_{E'}$, we have

$$\| U(t,s)F\|_{p,q,n} \leq C_4 \| F\|_{\hat{p},\hat{q},n}, \tag{2.1}$$

$$\| U(t',s')F - U(t,s)F\|^{2j}_{p,q,n} \leq C_5(T,F,p,q,n)\{ |t-t'|^j + |s-s'|^j\}, \quad 0 \leq s,t,\ s',t' \leq T,$$

where $\hat{p}$, $\hat{q}$ are integers given as n_3 in (3.15) later.

First we will verify that the integral in Theorem 1 is well defined by showing that for any fixed $F \in \mathcal{D}_{E'}$, $W_{L(s)U(t,s)F}(s)$ is continuous in (t,s). Since $W_F(t)$ is a Gaussian additive process with mean 0 and variance $V_t(F)$, we get for any integer $n \geq 1$,

$$E[|W_F(t_1) - W_F(t_2)|^{2n}] \leq C_6(T)\big(V_{t_1}(F) - V_{t_2}(F)\big)^n, \quad 0 \leq t_1,\ t_2 \leq T. \tag{2.2}$$

We choose an integer $k > 2$ such that $2k\lambda(m_1,q_1,T) > 2$, where m_1 and q_1 are the numbers which appeared in (1.6) and $\lambda(m_1,q_1,T)$ is the number in (H3). By Assumptions 1 and 2, we may assume $q_1 > p_1$ and $\|x\|_{-q_1} \leq \frac{1}{2}\|x\|_{-p_1}$. For $0 \leq s,t,\ s',t' \leq T$, the inequalities (2.1) and (2.2) yield, together with (VI), (H3) and the nuclearity of E,

$$\begin{aligned} E\Big[|W_{L(s)U(t,s)F}(s') - W_{L(s)U(t,s)F}(s)|^{2k}\Big] \\ \leq C_7(T)\Big(V_{s'}\big(L(s)U(t,s)F\big) - V_s\big(L(s)U(t,s)F\big)\Big)^k \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} E\Big[|W_{L(s')U(t',s')F}(s') - W_{L(s)U(t,s)F}(s')|^{2k}\Big] &\leq C_8(T)\|L(s')U(t',s')F - L(s)U(t,s)F\|^{2k}_{p_1,q_1,m_1} \\ &\leq C_9(T)\Big\{\|U(t',s')F - U(t,s)F\|^{2k}_{p_1,q_1,m_1+1} \\ &\quad + \|U(t',s')F - U(t,s)F\|^{2k}_{p_1,q_1,m_1+2} \\ &\quad + |s'-s|^{2k\lambda(m_1,q_1,T)} + |s'-s|^{2k}\Big\} \\ &\leq C_{10}(T)\{|t-t'|^k + |s-s'|^k + |s'-s|^{2k\lambda(m_1,q_1,T)}\}. \end{aligned} \tag{2.4}$$

The inequalities (2.3) and (2.4) are sufficient for the Kolmogorov-Totoki criterion [27] for continuity in (t,s). The continuity of $W_{L(s)U(t,s)L(t)F}(s)$ in (t,s) can be proved similarly.

Now we proceed to the proof of the existence of solutions for (1.1). Taking the relation $U(t,s)F = F + \int_s^t U(\tau,s)L(\tau)F\,d\tau$, the continuity of $W_{L(s)U(\tau,s)L(\tau)F}(s)$ in τ, the linearity of $W_\bullet(s)$ and the L^2-continuity of $W_\bullet(s)$, into account, we have

$$\begin{aligned} W_{L(s)U(t,s)F}(s) &= W_{L(s)F}(s) + W_{L(s)\int_s^t U(\tau,s)L(\tau)F\,d\tau}(s) \\ &= W_{L(s)F}(s) + \int_s^t W_{L(s)U(\tau,s)L(\tau)F}(s)\,d\tau, \end{aligned}$$

so that by making the use of the continuity of $W_{L(s)U(\tau,s)L(\tau)F}(s)$ in (τ,s) again, we get

$$\begin{aligned} \int_0^t W_{L(s)U(t,s)F}(s)\,ds &= \int_0^t W_{L(s)F}(s)\,ds + \int_0^t\Big(\int_s^t W_{L(s)U(\tau,s)L(\tau)F}(s)\,d\tau\Big)\,ds \\ &= \int_0^t W_{L(s)F}(s)\,ds + \int_0^t\Big(\int_0^\tau W_{L(s)U(\tau,s)L(\tau)F}(s)\,ds\Big)\,d\tau \\ &= \int_0^t\Big(W_{L(\tau)F}(\tau) + \int_0^\tau W_{L(s)U(\tau,s)L(\tau)F}(s)\,ds\Big)\,d\tau \\ &= \int_0^t\big(X_{L(\tau)F}(\tau) - X_{U(\tau,0)L(\tau)F}(0)\big)\,d\tau. \end{aligned} \tag{2.5}$$

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Combining the L^2-continuity of $X_F(0)$ in the definition of $\mathcal{L}(\mathcal{D}_{E'})$-process and the Jensen inequality such that $E[|X_F(0)|^{2+\alpha}] \leq E[|X_F(0)|^2]^\alpha$, we get that $E[|X_F(0)|^{2+\alpha}]$ is continuous in $\mathcal{D}_{E'}$. Hence there exist positive integers $p_2 \geq p_0, q_2$ and m_2 such that

$$E[|X_F(0)|^{2+\alpha}] \leq C_{11}\|F\|^{2+\alpha}_{p_2,q_2,m_2}. \tag{2.6}$$

Therefore the Kolmogorov criterion for continuity, together with (H3), (V1), the nuclearity of E and the inequalities (2.1) in Lemma 1 and (2.6), yields the continuity of $X_{U(\tau,0)L(\tau)F}(0)$ in τ. Hence it follows that

$$\int_0^t X_{U(\tau,0)L(\tau)F}(0)\,d\tau = X_{U(t,0)F}(0) - X_F(0). \tag{2.7}$$

The equalities (2.5) and (2.7) show that $X_F(t)$ is a solution of the equation (1.1).

Following H. Komatsu [13], we now prove the uniqueness of L^2-continuous solutions for the equation (1.1). Let $Y_1(t,F)$ and $Y_2(t,F)$ be two continuous $\mathcal{L}(\mathcal{D}_{E'})$-process solutions for the equation (1.1). First we remark by the Baire category theorem that for each $T > 0$, we have natural numbers $p_3 \geq p_0, q_3$ and m_3 such that

$$\max_{i=1,2}\sup_{0\leq t\leq T} E[Y_i(t,F)^2] \leq C_{12}(T)\|F\|_{p_3,q_3,m_3}. \tag{2.8}$$

Define $v(t,F) = Y_1(t,F) - Y_2(t,F)$. Then for any $a > 0$, we will prove $\frac{d}{dt}E\Big[v\big(t,U(a,t)F\big)^2\Big] = 0$ for $t \in (0,a]$. The inequality (2.8) and the strong continuity of $U(t,s)$, ((2) in Lemma 2), yield

$$E\left[\left|\frac{v\big(s,U(a,s)F\big)^2 - v\big(t,U(a,t)F\big)^2}{s-t}\right|\right]$$
$$\leq C_{13}(T,F)E\left[\left(\frac{v\big(s,U(a,s)F\big) - v\big(t,U(a,t)F\big)}{s-t}\right)^2\right]^{1/2}, \quad s,t \in (0,a] \subset [0,T].$$

The inequality (2.8) and the strong continuity of $L(t)$ and $U(t,s)$ imply that

$$\lim_{s\to t} E\left[\left|\frac{v\big(s,U(a,t)F\big) - v\big(t,U(a,t)F\big)}{s-t} - v\big(t,L(t)U(a,t)F\big)\right|^2\right] = 0. \tag{2.9}$$

By the strong continuity of $U(t,s)$, we get similarly

$$\lim_{s\to t} E\left[\left|\frac{v\big(s,[U(a,s)-U(a,t)]F\big) - v\big(t,[U(a,s)-U(a,t)]F\big)}{s-t}\right.\right.$$
$$\left.\left. -v\big(t,L(t)[U(a,s)-U(a,t)]F\big)\right|^2\right] = 0. \tag{2.10}$$

Since $L(t)$ generates the Kolmogorov evolution operator $U(t,s)$, we have

$$\lim_{s\to t} E\left[\left|v\big(t,L(t)U(a,s)F\big) - v\big(t,L(t)U(a,t)F\big)\right|^2\right] = 0$$

$$\lim_{s\to t} E\left[\left|v\big(t,L(t)U(a,t)F\big) + v\left(t,\frac{U(a,s)-U(a,t)}{s-t}F\right)\right|^2\right] = 0,$$

so that we get

$$\lim_{s\to t} E\left[\left|v\big(t, L(t)U(a,s)F\big) - \frac{v\big(t,U(a,s)F\big) - v\big(t,U(a,t)F\big)}{s-t}\right|^2\right] = 0. \tag{2.11}$$

From (2.9), (2.10) and (2.11), we get the desired equality claimed above. Hence $E\left[v\big(t,U(a,t)F\big)^2\right] =$ constant. Then letting $t \to 0$, by (2.8) and the definition of continuity of an $\mathcal{L}(\mathcal{D}_{E'})$-process in t, we have the constant $=$ 0. Taking the equalities $E\left[v\big(t,U(a,t)F\big)^2\right] = E\left[\big(v,(t,F) + v\big(t,[U(a,t) - U(a,a)]F\big)\big)^2\right]$ and $\lim_{t\to a} E\left[v\big(t,[U(a,t) - U(a,a)]F\big)^2\right] = 0$, into account, we have $E[v(a,F)^2] = 0$ for any $a > 0$, which implies $v(a,F) = 0$ almost surely. Thus the proof is complete.

3. **Proof of Lemma 2.** Assuming Lemma 1 which shall first be proved in the next Section, we will prove Lemma 2. To examine that $U(t,s)$ is the evolution operator stated in Lemma 2, we will check some regularities and integrabilities for $\eta_{s,t}(x)$. It is obvious that if $p \geq p_0$ and $x \in E'_p$, $\eta_{s,t}(x) \in E'_{n(p,T)}$ so that for $h \in E'_{p_4}$, $\eta_{s,t}(x+h) \in E'_{n(p_5,T)}$, where $p_5 = p \vee p_4$. Here $a \vee b = \max\{a,b\}$. Setting $n_1 = n(p_5,T)$ and following Kunita (p. 219 of [14]), we will show that $\xi_{s,t}(\tau) := \frac{1}{\tau}\{\eta_{s,t}(x+\tau h) - \eta_{s,t}(x)\}$ has a continuous extension at $\tau = 0$ for any s,t a.s. in E'_{n_1}. This can be shown by appealing to the Kolmogorov-Totoki criterion for continuity [27].

LEMMA 3. *For any $T > 0$ and any integer $j \geq 1$, we have*

$$E\{\|\xi_{s,t}(\tau) - \xi_{s',t'}(\tau')\|_{-n_1}^{2j}\} \leq C_{14}(T,h)\{|s-s'|^j + |t-t'|^j + |\tau-\tau'|^j\},$$

for $0 \leq s,s',t,t',\tau,\tau' \leq T$.

PROOF. Without loss of generality, we may assume $0 \leq s < s' < t < t' \leq T$. Then $\xi_{s,t}(\tau) - \xi_{s',t}(\tau')$ is a sum of the following terms:

$$\big(V(t,s) - V(t,s')\big)h + \int_s^{s'}\left(V(t,r)\int_0^1 DB\big(r,\zeta_{s,r}(\tau,y)\big)\big(\xi_{s,r}(\tau)\big)\,dy\right)dr, \tag{3.1}$$

where $\zeta_{s,r}(\tau,y) = \eta_{s,r}(x) + y\big(\eta_{s,r}(x+\tau h) - \eta_{s,r}(x)\big)$.

$$\int_{s'}^{t}\left(V(t,r)\int_0^1 \{DB\big(r,\zeta_{s,r}(\tau,y)\big)\big(\xi_{s,r}(\tau)\big) - DB\big(r,\zeta_{s',r}(\tau',y)\big)\big(\xi_{s',r}(\tau')\big)\}\,dy\right)dr. \tag{3.2}$$

By assumptions (V1) and (H2), the expectation of the $2j$-th power of the $\|\cdot\|_{-n_1}$-norm of (3.1) is dominated by

$$C_{15}\{|s-s'|^{2j} + E\left[\left(\int_s^{s'}\left\|V(t,r)\int_0^1 DB\big(r,\zeta_{s,r}(\tau,y)\big)\big(\xi_{s,r}(\tau)\big)\,dy\right\|_{-n_1}^2 dr\right)^j\right]$$
$$\leq C_{16}\left\{|s-s'|^{2j} + |s'-s|^{j-1}E\left[\int_s^{s'}\|\xi_{s,r}(\tau)\|_{-n_1}^{2j}\,dr\right]\right\}.$$

Again using the same assumptions and the Gronwall lemma, we have

$$E[\|\eta_{s,t}(x) - \eta_{s,t}(y)\|_{-n_1}^{2j}] \leq C_{17}\|x-y\|_{-p_5}^{2j}, \quad x,y \in E'_{p_5}, \tag{3.3}$$

which implies

$$E\Big[\int_s^{s'} \|\xi_{s,r}(\tau)\|_{-n_1}^{2j}\,dr\Big] \le C_{17}\|h\|_{-p_5}^{2j}|s'-s|. \tag{3.4}$$

Since the integrand in (3.2)

$$= V(t,r)\int_0^1 DB\big(r,\zeta_{s,r}(\tau,y)\big)\big(\xi_{s,r}(\tau)-\xi_{s',r}(\tau')\big)\,dy$$
$$+ V(t,r)\int_0^1\Big(\int_0^1 D^2B\big(r,\gamma_{s,s',r}(\tau,\tau',y_1)\big)\big(\zeta_{s,r}(\tau,y)-\zeta_{s',r}(\tau'y)\big)\,dy_1\Big)\big(\xi_{s',r}(\tau')\big)\,dy,$$

where $\gamma_{s,s',r}(\tau,\tau',y_1) = \zeta_{s',r}(\tau',y) + y_1\big(\zeta_{s,r}(\tau,y)-\zeta_{s',r}(\tau',y)\big)$, the $\|\cdot\|_{-n_1}$-norm of the integrand is dominated by

$$C_{18}\Big\{\|\xi_{s,r}(\tau)-\xi_{s',r}(\tau')\|_{-n_1} + \big(\|\eta_{s,r}(x)-\eta_{s',r}(x)\|_{-n_1} + \|\eta_{s,r}(x+\tau h)-\eta_{s',r}(x+\tau' h)\|_{-n_1}\big)\|\xi_{s',r}(\tau')\|_{-n_1}\Big\}. \tag{3.5}$$

The expectation of the $2j$-th power of $\|\cdot\|_{-n_1}$-norm of (3.2) is dominated by

$$C_{19}\Big\{\int_{s'}^t E\big[\|\xi_{s,r}(\tau)-\xi_{s',r}(\tau')\|_{-n_1}^{2j}\big]\,dr + \int_{s'}^t E\big[\|\eta_{s,r}(x)-\eta_{s',r}(x)\|_{-n_1}^{4j}\big]^{1/2}E\big[\|\xi_{s',r}(\tau')\|_{-n_1}^{4j}\big]^{1/2}\,dr + \int_{s'}^t E\big[\|\eta_{s,r}(x+\tau h)-\eta_{s',r}(x+\tau' h)\|_{-n_1}^{4j}\big]^{1/2}E\big[\|\xi_{s',r}(\tau')\|_{-n_1}^{4j}\big]^{1/2}\,dr\Big\}. \tag{3.6}$$

By the assumption (V1),

$$\|V(\tau,r)\|_{n_1}^2 = \sum_{j=1}^{\infty}\|V(\tau,r)h_j^{(0)}\|_{-n_1}^2 \le \sum_{j=1}^{\infty}\|h_j^{(0)}\|_{-p_5}^2 < \infty\ [15].$$

Then by the assumption that $\beta(t)$ is an E_1'-valued process and the Itô formula we have easily

LEMMA 4. *For any integer $j \ge 1$,*

$$E\Big[\Big\|\int_s^t V(\tau,r)\,d\beta(r)\Big\|_{-n_1}^{2j}\Big] \le C_{20}(j)E\Big[\Big(\int_s^t \|V(\tau,r)\|_{n_1}^2\,dr\Big)^j\Big] \le C_{21}|t-s|^j.$$

From the assumptions (H1) and (H2), we get

$$\|B\big(r,\eta_{s,r}(x)\big) - B\big(r,\eta_{s',r}(x')\big)\|_{-p_0} \le C_{22}\|\eta_{s,r}(x)-\eta_{s',r}(x')\|_{-n_1}$$

and taking the expectations of the $2n$-th power of both sides of the following inequality

$$\begin{aligned}\|\eta_{s,t}(x)-\eta_{s',t'}(x')\|_{-n_1} \le \|V(t,s)x-V(t',s')x'\|_{-n_1} &+ \Big\|\int_s^{s'} V(t,r)\,d\beta(r)\Big\|_{-n_1} \\ &+ \Big\|\int_s^{s'} V(t,r)B\big(r,\eta_{s,r}(x)\big)\,dr\Big\|_{-n_1} + \Big\|\int_t^{t'} V(t',r)\,d\beta(r)\Big\|_{-n_1} \\ &+ \Big\|\int_t^{t'} V(t',r)B\big(r,\eta_{s',r}(x')\big)\,dr\Big\|_{-n_1} \\ &+ \Big\|\int_{s'}^{t}\big(V(t,r)-V(t',r)\big)\,d\beta(r)\Big\|_{-n_1} \\ &+ \Big\|\int_{s'}^{t}\big(V(t,r)-V(t',r)\big)B\big(r,\eta_{s,r}(x)\big)\,dr\Big\|_{-n_1} \\ &+ \Big\|\int_{s'}^{t} V(t',r)\big\{B\big(r,\eta_{s,r}(x)\big)-B\big(r,\eta_{s',r}(x')\big)\big\}\,dr\Big\|_{-n_1},\end{aligned}$$

we have, by Lemma 4 and (V1),

$$\begin{aligned}&E\big[\|\eta_{s,t}(x)-\eta_{s',t'}(x)\|_{-n_1}^{2n}\|\big] \\ &\quad\le C_{23}(T)\Big\{|t-t'|^n+|s-s'|^n+\|x-x'\|_{-p_5}^{2n}+\int_s^t E\big[\|\eta_{s,r}(x)-\eta_{s',r}(x')\|_{-n_1}^{2n}\big]\,dr\Big\}.\end{aligned}$$

Hence we obtain

$$E\big[\|\eta_{s,t}(x)-\eta_{s',t'}(x')\|_{-n_1}^{2n}\big] \le C_{24}(T)\big\{|t-t'|^n+|s-s'|^n+\|x-x'\|_{-p_5}^{2n}\big\}. \tag{3.7}$$

Combining (3.1), (3.2), (3.3), (3.4), (3.6) and (3.7), we have

$$\begin{aligned}&E\big[\|\xi_{s,t}(\tau)-\xi_{s',t'}(\tau')\|_{-n_1}^{2j}\big] \\ &\qquad\le C_{25}(T)\|h\|_{-p_5}^{2j}\big\{|t-t'|^j+|s-s'|^j+|\tau-\tau'|^{2j}\|h\|_{-p_5}^{2j}\big\}.\end{aligned}$$

This completes the proof of Lemma 3.

Letting τ tend to 0 in $\xi_{s,t}(\tau)$, we have for each $x\in E_p'$,

$$D\eta_{s,t}(x)(h) = V(t,s)h+\int_s^t V(t,r)DB\big(r,\eta_{s,r}(x)\big)\big(D\eta_{s,r}(x)(h)\big)\,dr. \tag{3.8}$$

For the higher order differentiations, a formula similar to (3.8) can be proved inductively, together with the following lemma.

LEMMA 5. *Suppose that a natural number $q\ge p_0$ and any $T>0$. Then for $0\le s,t,$ $s',t'\le T$, a natural number j and $x,x',h_i\in E_1'$, $i=1,2,\dots,n$, we have for $n_2=n(q,T)$,*

$$E\big[\|D^n\eta_{s,t}(x)(h_1,h_2,\dots,h_n)\|_{-n_2}^{2j}\big] \le C_{26}(T)\|h_1\|_{-q}^{2j}\|h_2\|_{-q}^{2j}\cdots\|h_n\|_{-q}^{2j}. \tag{3.9}$$

$$\begin{aligned}&E\big[\|D^n\eta_{s,t}(x)(h_1,h_2,\dots,h_n)-D^n\eta_{s',t'}(x')(h_1,h_2,\dots,h_n)\|_{-n_2}^{2j}\big] \\ &\quad\le C_{27}(T)\big\{|t-t'|^j+|s-s'|^j+\|x-x'\|_{-q}^{2j}\big\}\|h_1\|_{-q}^{2j}\|h_2\|_{-q}^{2j}\cdots\|h_n\|_{-q}^{2j}.\end{aligned} \tag{3.10}$$

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PROOF. *First we will show (3.9) for the case $n = 1$. By assumptions (H1) and (H2), we get*

$$\left\|DB\big(r,\eta_{s,r}(x)\big)\big(D\eta_{s,r}(x)(h)\big)\right\|_{-q} \le C_{28}\|D\eta_{s,r}(x)(h)\|_{-n_2},$$

so that taking the expectations of $2j$-th powers of $\|\cdot\|_{-n_2}$ norms of both sides of (3.8), we get

$$E\Big[\|D\eta_{s,t}(x)(h)\|_{-n_2}^{2j}\Big] \le C_{29}(T)\Big\{\|h\|_{-q}^{2j} + \int_s^t E\Big[D\eta_{s,r}(x)(h)\|_{-n_2}^{2j}\Big]\,dr\Big\}$$

and the Gronwall inequality gives (3.9) for the case where $n = 1$. For $n \ge 2$, we will prove the inequality by mathematical induction. For $h_1, h_2, \ldots, h_n \in E_q'$,

$$D^n\eta_{s,t}(x)(h_1,h_2,\ldots,h_n) = \int_s^t \Big(D^nB\big(r,\eta_{s,r}(x)\big)\Big)(h_1,h_2,\ldots,h_n)\,dr.$$

Since

$$(3.11)\quad \Big(D^nB\big(r,\eta_{s,r}(x)\big)\Big)(h_1,h_2,\ldots,h_n) = DB\big(r,\eta_{s,r}(x)\big)\big(D^n\eta_{s,r}(x)(h_1,h_2,\ldots,h_n)\big)$$

+ a finite sum of terms of the type

$$D^mB\big(r,\eta_{s,r}(x)\big)\big(D^{k_1}\eta_{s,r}(x)(h_{j_1^{(1)}},h_{j_2^{(1)}},\ldots,h_{j_{k_1}^{(1)}}),$$
$$D^{k_2}\eta_{s,r}(x)(h_{j_1^{(2)}},h_{j_2^{(2)}},\ldots,h_{j_{k_2}^{(2)}}),\ldots,D^{k_m}\eta_{s,r}(x)(h_{j_1^{(m)}},h_{j_2^{(m)}},\ldots,h_{j_{k_m}^{(m)}})\big),$$

where $2 \le m \le n$, $k_1 + k_2 + \cdots + k_m = n$ and $0 \le k_i \le n-1$, so that using the inductive assumption, we get (3.9) by the same argument as before.

Before proceeding to the proof of (3.10), we note that for $h \in E_q'$, $\|D\eta_{s,t}(x)(h) - D\eta_{s',t'}(x')(h)\|_{-n_2}$ is dominated by

$$(3.12)\quad \begin{aligned}&\|V(t,s)h - V(t',s')h\|_{-n_2}\\ &+\Big\|\int_s^{s'} V(t,r)D\Big(B\big(r,\eta_{s,r}(x)\big)\Big)(h)\,dr\Big\|_{-n_2}\\ &+\Big\|\int_t^{t'} V(t',r)D\Big(B\big(r,\eta_{s',r}(x')\big)\Big)(h)\,dr\Big\|_{-n_2}\\ &+\Big\|\int_{s'}^{t}\Big\{V(t,r)D\Big(B\big(r,\eta_{s,r}(x)\big)\Big)(h) - V(t',r)D\Big(B\big(r,\eta_{s',r}(x')\big)\Big)(h)\Big\}\,dr\Big\|_{-n_2}.\end{aligned}$$

Now by the assumptions (H1) and (H2), we have

$$(3.13)\quad \begin{aligned}&\Big\|V(t,r)D\Big(B\big(r,\eta_{s,r}(x)\big)\Big)(h) - V(t',r)D\Big(B\big(r,\eta_{s',r}(x')\big)\Big)(h)\Big)\Big\|_{-n_2}\\ &\le |t-t'|\,\Big\|DB\big(r,\eta_{s,r}(x)\big)\big(D\eta_{s,r}(x)(h)\big)\Big\|_{-q}\\ &\quad+\Big\|\Big\{DB\big(r,\eta_{s,r}(x)\big) - DB\big(r,\eta_{s',r}(x')\big)\Big\}\big(D\eta_{s,r}(x)(h)\big)\Big\|_{-q}\\ &\quad+\Big\|DB(r,\eta_{s',r}(x')\big)\big(D\eta_{s,r}(x)(h) - D\eta_{s'r}(x')(h)\big)\Big\|_{-q}\\ &\le C_{30}(T)\Big\{\big(|t-t'| + \|\eta_{s,r}(x) - \eta_{s',r}(x')\|_{-n_2}\big)\|D\eta_{s,r}(x)(h)\|_{-n_2}\\ &\quad+\|D\eta_{s,r}(x)(h) - D\eta_{s',r}(x')(h)\|_{-n_2}\Big\}.\end{aligned}$$

Hence from (3.7), (3.12) and (3.13) we have

$$\begin{aligned} &E\Big[\|D\eta_{s,t}(x)(h) - D\eta_{s',t'}(x')(h)\|_{-n_2}^{2j}\Big] \\ &\quad \leq C_{31}(T)\Big\{\Big(|t-t'|^j + |s-s'|^j + \|x-x'\|_{-q}^{2j}\Big)\|h\|_{-q}^{2j} \\ &\qquad + \int_{s'}^{t} E\Big[\|D\eta_{s,t}(x)(h) - D\eta_{s',r}(x')(h)\|_{-n_2}^{2j}\Big]\,dr\Big\}, \end{aligned}$$

which gives (3.10) by the Gronwall lemma for the case $n = 1$. By (3.11) and the estimation of $\|D^n\eta_{s,t}(x)(h_1, h_2, \ldots, h_n) - D^n\eta_{s',t'}(x')(h_1, h_2, \ldots, h_n)\|_{-n_2}$ similar to that in (3.12), mathematical induction and Gronwall's lemma yield the proof of (3.10) for $n \geq 2$.

For the proof of the generation problem of $L(t)$ we proceed as follows. By the assumptions (H1) and (H2), (3.7) and (3.9) of Lemma 5, we may exchange the order of differentiation and integration. They by assumption (H3) and Itô formula [16], we have the pointwise Kolmogorov forward and backward equations as in the finite dimensional case (Theorem 1 (page 73) of [8]):

$$\begin{aligned} \frac{d}{dt}\Big(U(t,s)F\Big)(x) &= \Big(U(t,s)L(t)F\Big)(x) \\ \frac{d}{ds}\Big(U(t,s)F\Big)(x) &= -\Big(L(s)U(t,s)F\Big)(x). \end{aligned}$$

Let $p \geq 0$, $q \geq 0$ and $n \geq 0$ be integers and $x \in E'_p$. Since

$$D^n\Big(F\big(\eta_{s,t}(x)\big)\Big)(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \ldots, h_{i_n}^{(q)})$$

is a finite sum of terms of the type

$$\begin{aligned} I = D^m F\big(\eta_{s,t}(x)\big)\Big(&D^{k_1}\eta_{s,t}(x)(h_{j_1^{(1)}}^{(q)}, h_{j_2^{(1)}}^{(q)}, \ldots, h_{j_{k_1}^{(1)}}^{(q)}), \\ D^{k_2}\eta_{s,t}(x)(h_{j_1^{(2)}}^{(q)}, h_{j_2^{(2)}}^{(q)}, \ldots, &h_{j_{k_2}^{(2)}}^{(q)}), \ldots, D^{k_m}\eta_{s,t}(x)(h_{j_1^{(m)}}^{(q)}, h_{j_2^{(m)}}^{(q)}, \ldots, h_{j_{k_m}^{(m)}}^{(q)})\Big), \\ &k_1 + k_2 + \cdots + k_m = n, \end{aligned}$$

so that from the nuclearity of E and (3.9), we have an integer $q' > n(p, p_0, q, T)$ such that

$$\sum_{j=1}^{\infty} \|h_j^{(q)}\|_{-q'}^2 < +\infty \tag{3.14}$$

and setting $n_3 = n(q', T)$, we have

$$\begin{aligned} E[|I|^2] \leq \|F\|_{n_3,n_3,n}^2 E\Big[&e^{2\|\eta_{s,t}(x)\|_{-n_3}}\|D^{k_1}\eta_{s,t}(x)(h_{j_1^{(1)}}^{(1)}, h_{j_2^{(1)}}^{(q)}, \ldots, h_{j_{k_1}^{(1)}}^{(q)})\|_{-n_3}^2 \\ \|D^{k_2}\eta_{s,t}(x)(h_{j_1^{(2)}}^{(q)}, h_{j_2^{(2)}}^{(q)}, \ldots, h_{j_{k_2}^{(2)}}^{(q)}\|_{-n_3}^2 &\cdots \|D^{k_m}\eta_{s,t}(x)(h_{j_1^{(m)}}^{(q)}, h_{j_2^{(m)}}^{(q)}, \ldots, h_{j_{k_m}^{(m)}}^{(q)}\|_{-n_3}^2\Big] \\ \leq C_{32}\|F\|_{n_3,n_3,n}^2 \|h_{i_1}^{(q)}\|_{-q'}^2 &\|h_{i_2}^{(q)}\|_{-q'}^2 \cdots \|h_{i_n}^{(q)}\|_{-q'}^2 E[e^{4\|\eta_{s,t}(x)\|_{-n_3}}]^{1/2}. \end{aligned} \tag{3.15}$$

Here we will prove

LEMMA 6. *For any* $\alpha > 0$ *and* $T > 0$, *there exists a constant* $C_{33} = C_{33}(\alpha, T)$ *such that*

$$\sup_{0 \le s,t \le T} E[e^{\alpha \|\eta_{s,t}(x)\|_{-n_3}}] \le C_{33}\, e^{\alpha \|x\|_{-q'}}.$$

PROOF. By (H1), $\|\eta_{s,t}(x)\|_{-n_3} \le \|x\|_{-q'} + C_{34} + \|\int_s^t V(t,r)\, d\beta(r)\|_{-n_3}$. Then it is enough to prove $E[\exp(\|\int_s^t \alpha V(t,r)\, d\beta(r)\|_{-n_3})] \le C_{35}$. Setting $y_{s,t} = \int_s^t \alpha V(t,r)\, d\beta(r)$ and following [9], by the Itô formula for $(1 + \|y_{s,t}\|_{-n_3}^2)^{m/2}$, we get the desired estimate.

Therefore (3.14), (3.15) and Lemma 6 yield

$$\|U(t,s)F\|_{p,q,n} \le C_{36}(T)\|F\|_{n_3,n_3,n}, \quad t, s \in [0,T],$$

which implies that $U(t,s)$ is a continuous linear operator from $\mathcal{D}_{E'}$ into itself.

In the same way as in [22], if we prove the strong continuity of $U(t,s)F$ in (t,s), the pointwise Kolmogorov forward and backward equations imply that $L(t)$ generates the evolution operator $U(t,s)$. Since $\|U(t,s)F - U(t',s')F\|_{p,q,n}^{2j}$ is dominated by a finite sum of terms of the type

$$\begin{aligned}\sup_{x \in E_p'} e^{-2j\|x\|_{-p}} \sum_{\substack{j_1^{(1)}, j_2^{(1)}, \ldots, j_{k_1}^{(1)} \\ \vdots \qquad \vdots \\ j_1^{(m)}, j_2^{(m)}, \ldots, j_{k_m}^{(m)}}} & E\Big[\Big|D^m F\big(\eta_{s,t}(x)\big) \\ & \big(D^{k_1}\eta_{s,t}(x)(h_{j_1^{(1)}}^{(q)}, h_{j_2^{(1)}}^{(q)}, \ldots, h_{j_{k_1}^{(1)}}^{(q)}), D^{k_2}\eta_{s,t}(x)(h_{j_1^{(2)}}^{(q)}, h_{j_2^{(2)}}^{(q)}, \ldots, h_{j_{k_2}^{(2)}}^{(q)}), \ldots \\ & \cdots D^{k_m}\eta_{s,t}(x)(h_{j_1^{(m)}}^{(q)}, h_{j_2^{(m)}}^{(q)}, \ldots, h_{j_{k_m}^{(m)}}^{(q)})\big) \\ & - D^m F\big(\eta_{s',t'}(x)\big)\big(D^{k_1}\eta_{s',t'}(x)(h_{j_1^{(1)}}^{(q)}, h_{j_2^{(1)}}^{(q)}, \ldots, h_{j_{k_1}^{(1)}}^{(q)}), \\ & D^{k_2}\eta_{s',t'}(x)(h_{j_1^{(2)}}^{(q)}, h_{j_2^{(2)}}^{(q)}, \ldots, h_{j_{k_2}^{(2)}}^{(q)}), \ldots, D^{k_m}\eta_{s',t'}(x)(h_{j_1^{(m)}}^{(q)}, h_{j_2^{(m)}}^{(q)}, \ldots, h_{j_{k_m}^{(m)}}^{(q)})\big)\Big|^{2j}\Big],\end{aligned}$$

so that by (3.7), Lemmas 5 and 6 and the nuclearity of E, we have

$$\|U(t,s)F - U(t',s')F\|_{p,q,n}^{2j} \le C_{37}\|F\|_{n_3,n_3,n+1}^{2j}\big\{|t - t'|^j + |s - s'|^j\big\}.$$

This completes the proof of Lemma 2.

4. **Proof of Lemma 1.** For any integers $p \ge 0$, $q \ge 0$ and $n \ge 0$, as before we choose an integer $q' > n\,(p, p_0, q, T)$ such that

$$\sum_{j=1}^{\infty} \|h_j^{(q)}\|_{-q'}^2 < +\infty \tag{4.1}$$

and set $n_3 = n(q', T)$. Then by the assumptions on $B(t,x)$ and for these q', $n+1$ and any $0 < \delta < 1$, there exists a bounded smooth functional

$$\tilde{B}(t,x) = \tilde{b}(t, \langle x, \zeta_1 \rangle, \langle x, \zeta_2 \rangle, \ldots, \langle x, \zeta_{m_k} \rangle)$$

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such that

$$\sum_{l=0}^{n+1} \sup_{\substack{x\in E'_{q'}\\ 0\le t\le T}} \|D^l B(t,x) - D^l\tilde{B}(t,x)\|^{(q')}_{\mathrm{H.S.}} < \delta. \tag{4.2}$$

Set $\beta_{s,t} = \int_s^t V(t,r)\,d\beta(r)$. For sufficiently large N, we put

$$\begin{aligned} z^N_{s,t}(x) = V(t,s)x + \beta_{s,t} + \int_s^t V(t,t_1)\tilde{B}\Big(t_1, V(t_1,s)x + \beta_{s,t_1} \\ + \int_s^{t_1} V(t_1,t_2)\tilde{B}\Big(t_2,\ldots, V(t_{N-1},s)x + \beta_{s,t_{N-1}} \\ + \int_s^{t_{N-1}} V(t_{N-1},t_N)\tilde{B}(t_N,x)\,dt_N\Big)\cdots\Big)\,dt_1. \end{aligned}$$

Setting

$$\begin{aligned} \hat{z}^{(n)}_{s,t}(x) = V(t,s)x + \beta_{s,t} + \int_s^t V(t,t_1)\tilde{B}\Big(t_1, V(t_1,s)x + \beta_{s,t_1} \\ + \int_s^{t_1} V(t_1,t_2)\tilde{B}\Big(t_2,\ldots, V(t_{n-1},s)x + \beta_{s,t_{n-1}} \\ + \int_s^{t_{n-1}} V(t_{n-1},t_n)\tilde{B}\big(t_n,\eta_{s,t_n}(x)\big)\,dt_n\Big)\cdots\Big)\,dt_1 \end{aligned}$$

$n = 1,2,\ldots,N$, where $t_0 = t$, we have for any $x \in E'_p$, $0 \le s,t \le T$ and any integer $j \ge 1$,

$$\begin{aligned} E\big[\|\eta_{s,t}(x) - z^N_{s,t}(x)\|^{2j}_{-n_3}\big] &\le cE\big[\|\eta_{s,t}(x) - \hat{z}^{(1)}_{s,t}(x)\|^{2j}_{-n_3}\big] \\ &\quad + \sum_{k=2}^{N} c^k E\big[\|\hat{z}^{(k-1)}_{s,t}(x) - \hat{z}^{(k)}_{s,t}(x)\|^{2j}_{-n_3}\big] \\ &\quad + c^N E\big[\|\hat{z}^{(N)}_{s,t}(x) - z^N_{s,t}(x)\|^{2j}_{-n_3}\big] \\ &\le c\delta^{2j}T + \sum_{k=2}^{N} c^k M^{2j(k-1)}\delta^{2j}T^k/k! \\ &\quad + c^N 2^{2j} M^{2jN} T^N/N! \\ &\le \delta^{2j}\exp\big(c(M\vee 1)^{2j}T\big) + c^N 2^{2j}M^{2jN}T^N/N!, \end{aligned} \tag{4.3}$$

where $c = 2^{2j-1}$ and $M = \max_{0\le l\le n+1} \sup_{\substack{x\in E'_{q'}\\ 0\le t\le T}} \|D^l\tilde{B}(t,x)\|^{(q')}_{\mathrm{H.S.}}$. Hence for any $\varepsilon > 0$, if we take sufficiently small δ and large N, we have

$$\sup_{x\in E'_p} E\big[\|\eta_{s,t}(x) - z^N_{s,t}(x)\|^{2j}_{-n_3}\big] < \varepsilon. \tag{4.4}$$

Next we verify by mathematical induction that for any integer $1 \le k \le n$ and any $\varepsilon > 0$, there exists an integer $N(k,\varepsilon)$ such that if $N \ge N(k,\varepsilon)$,

$$E\big[\|D^k\eta_{s,t}(x)(h^{(q)}_{i_1}, h^{(q)}_{i_2},\ldots,h^{(q)}_{i_k}) - D^k z^N_{s,t}(x)(h^{(q)}_{i_1}, h^{(q)}_{i_2},\ldots,h^{(q)}_{i_k})\|^{2j}_{-n_3}\big] < \varepsilon. \tag{4.5}$$

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For any $\varepsilon' > 0$, (4.4) gives that for sufficiently small δ and large $n(\varepsilon')$, and for any $N \geq n(\varepsilon')$

$$\sup_{x\in E_p'} E\Big[\|\eta_{s,t}(x) - z_{s,t}^N(x)\|_{-n_3}^{4j}\Big]^{1/2} < \varepsilon'. \tag{4.6}$$

Here we need the following lemma for later use. In a manner similar to that in the proofs of (3.9) and Lemma 6, we get

LEMMA 7. *For any integers $q \geq p_0$, $j \geq 1$, $n \geq 1$ and any $T > 0$, we have for $n_2 = n(q, T)$,*

$$\begin{aligned}\sup_{0\leq s,t\leq T} & E\Big[\|D^n z_{s,t}^N(x)(h_1, h_2, \ldots, h_n)\|_{-n_2}^{2j}\Big] \\ &\leq C_{38}(T)\|h_1\|_{-q}^{2j}\|h_2\|_{-q}^{2j}\cdots\|h_n\|_{-q}^{2j}, \quad x, h_i, i = 1,2,\ldots, n \in E_q'.\end{aligned} \tag{4.7}$$

For any $\alpha > 0$ and $T > 0$,

$$\sup_{0\leq s,t\leq T} E[e^{\alpha\|z_{s,t}^N(x)\|_{-n_2}}] \leq C_{39}\, e^{\alpha\|x\|_{-q}}. \tag{4.8}$$

For any $\xi \in E$ and $\alpha > 0$ and $T > 0$, there exists $C_{40} = C_{40}(\xi, \alpha, T)$ such that

$$\begin{aligned}\sup_{0\leq s,t\leq T} \max\Big\{E\Big[\exp\Big(\alpha\sqrt{|\langle \eta_{s,t}(x), \xi\rangle|}\Big), E\Big[\exp\Big(\alpha\sqrt{|\langle z_{s,t}^N(x), \xi\rangle|}\Big)\Big]\Big\} \\ \leq C_{40}\exp\Big(\alpha\sqrt{|\langle x, \xi\rangle|}\Big).\end{aligned} \tag{4.9}$$

Setting

$$\begin{aligned} & y_{s,t}^{m,N}(x)(h_{i_1}^{(q)}) \\ &= V(t,s)h_{i_1}^{(q)} + \int_s^t V(t,t_1)D\tilde{B}\big(t_1, z_{s,t_1}^{N-1}(x)\big)\Big(V(t_1,s)h_{i_1}^{(q)} + \int_s^{t_1} V(t_1,t_2)D\tilde{B}\big(t_2, z_{s,t_2}^{N-2}(x)\big) \\ &\Big(V(t_2,s)h_{i_1}^{(q)} + \cdots + \int_s^{t_{m-1}} V(t_{m-1},t_m)D\tilde{B}(t_m, \eta_{s,t_m}(x))\big(D\eta_{s,t_m}(x)h_{i_1}^{(q)}\big)\, dt_m\Big)\cdots\Big)\, dt_1, \end{aligned}$$

$$\begin{aligned} & z_{s,t}^{m,N}(x)(h_{i_1}^{(q)}) \\ &= V(t,s)h_{i_1}^{(q)} + \int_s^t V(t,t_1)D\tilde{B}\big(t_1, z_{s,t_1}^{N-1}(x)\big)\Big(V(t_1,s)h_{i_1}^{(q)} + \int_s^{t_1} V(t_1,t_2)D\tilde{B}\big(t_2, z_{s,t_2}^{N-2}(x)\big) \\ &\Big(V(t_2,s)h_{i_1}^{(q)} + \cdots + \int_s^{t_{m-1}} V(t_{m-1},t_m)D\tilde{B}(t_m, z_{s,t_m}^{N-m}(x))\big(D\eta_{s,t_m}(x)h_{i_1}^{(q)}\big)\, dt_m\Big)\cdots\Big)\, dt_1, \end{aligned}$$

and taking $N \geq m + n(\varepsilon')$, we have by (3.9) and (4.7),

$$\begin{aligned}
E\Big[\|D\eta_{s,t}(x)(h_{i_1}^{(q)}) - Dz_{s,t}^N(x)(h_{i_1}^{(q)})\|_{-n_3}^{2j}\Big] \\
&\leq cE\Big[\|D\eta_{s,t}(x)(h_{i_1}^{(q)}) - y_{s,t}^{1,N}(x)(h_{i_1}^{(q)})\|_{-n_3}^{2j}\Big] \\
&\quad + c^2E\Big[\|y_{s,t}^{1,N}(x)(h_{i_1}^{(q)} - z_{s,t}^{1,N}(x)(h_{i_1}^{(q)})\|_{n_3}^{2j}\Big] \\
&\quad + \sum_{k=1}^{m-1}\Big\{c^{k+2}E\Big[\|z_{s,t}^{k,N}(x)(h_{i_1}^{(q)}) - y_{s,t}^{k+1,N}(x)(h_{i_1}^{(q)})\|_{-n_3}^{2j}\Big] \\
&\quad + c^{k+3}E\Big[\|y_{s,t}^{k+1,N}(x)(h_{i_1}^{(q)}) - z_{s,t}^{k+1,N}(x)(h_{i_1}^{(q)})\|_{-n_3}^{2j}\Big]\Big\} \\
&\quad + c^{m+2}E\Big[\|z_{s,t}^{m,N}(x)(h_{i_1}^{(q)}) - Dz_{s,t}^N(x)(h_{i_1}^{(q)})\|_{-n_3}^{2j}\Big] \\
&\leq C_{41}(h_{i_1}^{(q)})\{c\delta^{2j}T + \varepsilon' c^2 M^{2j}T \\
&\quad + \sum_{k=1}^{m-1}(c^{k+2}\delta^{2j}M^{2jk} + \varepsilon' c^{k+3}M^{2j(k+1)})T^{k+1}/(k+1)! \\
&\quad + c^{m+2}M^{2jm}T^m/m!\} \\
&\leq C_{42}(\delta^{2j} + \varepsilon' + c^{m+2}M^{2jm}T^m/m!),
\end{aligned}$$

which gives (4.5) for $k = 1$ if we take sufficiently small δ, ε' and large m. We assume (4.5) holds for integers $1 \leq k \leq l$, $l \geq 1$.

Since

$$\begin{aligned}
D^{l+1}\big(B(r,\eta_{s,r}(x))\big)(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \ldots, h_{i_{l+1}}^{(q)}) \\
&= DB\big(r,\eta_{s,r}(x)\big)\big(D^{l+1}\eta_{s,r}(x)(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \ldots, h_{i_{l+1}}^{(q)})\big) \\
&\quad + \text{a finite sum of terms of this type:} \\
&\quad D^u B\big(r,\eta_{s,r}(x)\big)\big(D^{k_1}\eta_{s,r}(x)(h_{j_1^{(1)}}^{(q)}, h_{j_2^{(1)}}^{(q)}, \ldots, h_{j_{k_1}^{(1)}}^{(q)}), \\
D^{k_2}\eta_{s,r}(x)(h_{j_1^{(2)}}^{(q)}, h_{j_2^{(2)}}^{(q)}, \ldots, h_{j_{k_2}^{(2)}}^{(q)}), \ldots, D^{k_u}\eta_{s,r}(x)(h_{j_1^{(u)}}^{(q)}, h_{j_2^{(u)}}^{(q)}, \ldots, h_{j_{k_u}^{(u)}}^{(q)})\big),
\end{aligned}$$

where

$$2 \leq u \leq l+1,\ k_1+k_2+\cdots+k_u = l+1, \{h_{j_{k_1}^{(i)}}^{(q)}, i = 1,2,\ldots,u\} = \{h_{i_j}^{(q)}, j = 1,2,\ldots,l+1\}$$

and

$$\begin{aligned}
D^{l+1}\eta_{s,r}(x)(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \ldots, h_{i_{l+1}}^{(q)}) \\
&= \int_s^t D^{l+1}\big(B(r,\eta_{s,r}(x))\big)(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \ldots, h_{i_{l+1}}^{(q)})\,dr,
\end{aligned}$$

so that (4.5) for $k = l + 1$ can be proved similarly by the assumption of the induction.

Since $F \in \mathcal{D}_{E'}$, for any $0 < \varepsilon' < 1$, we have a weighted Schwartz functional $\tilde{F}(x) = f(\langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \ldots, \langle x,\xi_m\rangle)$ such that

$$(4.10) \qquad \sum_{k=0}^{n+1} \sup_{x\in E'_{n_3}} e^{-\|x\|_{-n_3}} D^k\big(F(x) - \tilde{F}(x)\big)\|_{\text{H.S.}}^{(n_3)} < \varepsilon'.$$

Then to prove Lemma 1, it is enough to show $\big(U(t,s)F\big)(x)$ is approximated by weighted Schwartz functionals in $\|\cdot\|_{p,k}^{(q)}$, $0 \le k \le n$. Since $D^k\big(F(\eta_{s,t}(x))\big)\,(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \ldots, h_{i_k}^{(q)})$ is a finite sum of terms of the type

(4.11)
$$I_{h_{i_1}^{(q)},h_{i_2}^{(q)},\ldots,h_{i_k}^{(q)}}\big(\eta_{s,t}(x)\big) = D^uF\big(\eta_{s,t}(x)\big)\big(D^{k_1}\eta_{s,t}(x)(h_{j_1^{(1)}}^{(q)}, h_{j_2^{(1)}}^{(q)}, \ldots, h_{j_{k_1}^{(1)}}^{(q)}),$$
$$D^{k_2}\eta_{s,t}(x)(h_{j_1^{(2)}}^{(q)}, h_{j_2^{(2)}}^{(q)}, \ldots, h_{j_{k_2}^{(2)}}^{(q)}), \ldots, D^{k_u}\eta_{s,t}(x)(h_{j_1^{(u)}}^{(q)}, h_{j_2^{(u)}}^{(q)}, \ldots, h_{j_{k_u}^{(u)}}^{(q)})\big)$$

where $0 \le u \le k$ and $k_1 + k_2 + \cdots + k_u = k$, so that setting

(4.12)
$$J_{h_{i_1}^{(q)},h_{i_2}^{(q)},\ldots,h_{i_k}^{(q)}}\big(z_{s,t}^N(x)\big)$$
$$= D^u\tilde{F}\big(z_{s,t}^N(x)\big)(D^{k_1}z_{s,t}^N(x)(h_{j_1^{(1)}}^{(q)}, h_{j_2^{(1)}}^{(q)}, \ldots, h_{j_{k_1}^{(1)}}^{(q)}),$$
$$D^{k_2}z_{s,t}^N(x)(h_{j_1^{(2)}}^{(q)}, h_{j_2^{(2)}}^{(q)}, \ldots, h_{j_{k_2}^{(2)}}^{(q)}), \ldots, D^{k_u}z_{s,t}^N(x)(h_{j_1^{(u)}}^{(q)}, h_{j_2^{(u)}}^{(q)}, \ldots, h_{j_{k_u}^{(u)}}^{(q)})),$$

we see that $\Big(\big\|U(t,s)F - E\big[\tilde{F}\big(z_{s,t}^N(\cdot)\big)\big]\big\|_{p,k}^{(q)}\Big)^2$ is dominated by a finite sum of terms of the type

$$(4.13)\quad C_{43}\sup_{x\in E_p'} e^{-2\|x\|_{-p}} \sum_{i_1,i_2,\ldots,i_k=1}^{\infty} E\Big[\Big|I_{h_{i_1}^{(q)},h_{i_2}^{(q)},\ldots,h_{i_k}^{(q)}}\big(\eta_{s,t}(x)\big)$$
$$- J_{h_{i_1}^{(q)},h_{i_2}^{(q)},\ldots,h_{i_k}^{(q)}}\big(z_{s,t}^N(x)\big)\Big|^2\Big]$$
$$\le C_{44}\Big\{\sup_{x\in E_p'} e^{-2\|x\|_{-p}} \sum_{i_1,i_2,\ldots,i_k=1}^{\infty} E\Big[e^{2\|\eta_{s,t}(x)\|_{-n_3}}(\varepsilon')^2$$
$$\|D^{k_1}\eta_{s,t}(x)(h_{j_1^{(1)}}^{(q)}, h_{j_2^{(1)}}^{(q)}, \ldots, h_{j_{k_1}^{(1)}}^{(q)})\|_{-n_3}^2\, \|D^{k_2}\eta_{s,t}(x)(h_{j_1^{(2)}}^{(q)}, h_{j_2^{(2)}}^{(q)}, \ldots, h_{j_{k_2}^{(2)}}^{(q)})\|_{-n_3}^2$$
$$\cdots\|(D^{k_u}\eta_{s,t}(x)(h_{j_1^{(u)}}^{(q)}, h_{j_2^{(u)}}^{(q)}, \ldots, h_{j_{k_2}^{(u)}}^{(q)})\|_{-n_3}^2\Big]$$
$$+ \sup_{s\in E_p'} e^{-2\|x\|_{-p}} \sum_{i_1,i_2,\ldots,i_k=1}^{\infty} E\Big[\Big|J_{h_{i_1}^{(q)},h_{i_2}^{(q)},\ldots,h_{i_k}^{(q)}}\big(\eta_{s,t}(x)\big) - J_{h_{i_1}^{(q)},h_{i_2}^{(q)},\ldots,h_{i_k}^{(q)}}\big(z_{s,t}^N(x)\big)\Big|^2\Big]\Big\}.$$

Lemmas 6 and 7 and (4.10) show that

$$(4.14)\quad \sup_{x\in E_p'} e^{-\|x\|_{-p}} \max\Big\{E\Big[\Big(\big\|D^u\tilde{F}\big(z_{s,t}^N(x)\big)\big\|_{\mathrm{H.S.}}^{(n_3)}\Big)^2\Big]^{1/2},$$
$$E\Big[\Big(\big\|D^{u+1}\tilde{F}\Big(z_{s,t}^N(x) + \tau\big(\eta_{s,t}(x) - z_{s,t}^N(x)\big)\Big)\big\|_{\mathrm{H.S.}}^{(n_3)}\Big)^2\Big]^{1/2}\Big\}$$
$$\le C_{45}(T), \quad 0 \le \tau \le 1,\ 0 \le s,\ t \le T.$$

Hence from (3.9), (4.1), (4.14) and Lemma 6, we have constants C_{46} and C_{47} independent

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of ε', and for any $\varepsilon > 0$, a natural number N_0 such that (4.13) is dominated by

$$(4.15)\quad \varepsilon/3 + C_{46}\,\varepsilon' + C_{47} \sum_{i_1,i_2,\ldots,i_k=1}^{N_0} E\Big[\|\eta_{s,t}(x) - z_{s,t}^N(x)\|_{-n_3}^2$$
$$\|D^{k_1}\eta_{s,t}(x)(h_{j_1^{(1)}}^{(q)}, h_{j_2^{(1)}}^{(q)}, \ldots, h_{j_{k_1}^{(1)}}^{(q)})\|_{-n_3}^2 \cdots \|(D^{k_u}\eta_{s,t}(x)(h_{j_1^{(u)}}^{(q)}, h_{j_2^{(u)}}^{(q)}, \ldots, h_{j_{k_u}^{(u)}}^{(q)})\|_{-n_3}^2$$
$$+ \sum_{r=1}^{u} \|D^{k_1} z_{s,t}^N(x)(h_{j_1^{(1)}}^{(q)}, h_{j_2^{(1)}}^{(q)}, \ldots, h_{j_{k_1}^{(1)}}^{(q)})\|_{-n_3}^2 \cdots$$
$$\|(D^{k_{r-1}} z_{s,t}^N(x)(h_{j_1^{(r-1)}}^{(q)}, h_{j_2^{(r-1)}}^{(q)}, \ldots, h_{j_{k_{r-1}}^{(r-1)}}^{(q)})\|_{-n_3}^2$$
$$\|D^{k_r}\eta_{s,t}^N(x)(h_{j_1^{(r)}}^{(q)}, h_{j_2^{(r)}}^{(q)}, \ldots, h_{j_{k_r}^{(r)}}^{(q)}) - D^{k_r} z_{s,t}^N(x)(h_{j_1^{(r)}}^{(q)}, h_{j_2^{(r)}}^{(q)}, \ldots, h_{j_{k_r}^{(r)}}^{(q)})\|_{-n_3}^2$$
$$\|D^{k_{r+1}}\eta_{s,t}(x)(h_{j_1^{(r+1)}}^{(q)}, h_{j_2^{(r+1)}}^{(q)}, \ldots, h_{j_{k_{r+1}}^{(r+1)}}^{(q)})\|_{-n_3}^2 \cdots$$
$$\|D^{k_u}\eta_{s,t}(x)(h_{j_1^{(u)}}^{(q)}, h_{j_2^{(u)}}^{(q)}, \ldots, h_{j_{k_u}^{(u)}}^{(q)})\|_{-n_3}^2\Big].$$

Therefore noting (3.9), (4.4), (4.5), (4.7), (4.13), and (4.15) and taking sufficiently small ε', δ and large N, we obtain

$$\sup_{x\in E_p'} e^{-\|x\|_{-p}} \Big\|D^k\Big(\big(U(t,s)F(x)\big) - D^k\Big(E\big[\tilde{F}\big(z_{s,t}^N(x)\big)\big]\Big)\Big\|_{H.S}^{(q)} < \varepsilon.$$

It remains to prove that $E\big[\tilde{F}\big(z_{s,t}^N(x)\big)\big]$ is a weighted Schwartz functional. Of course $E\big[\tilde{F}\big(z_{s,t}^N(x)\big)\big] = \phi_{s,t}(\langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \ldots, \langle x,\xi_m\rangle, \langle x,\zeta_1\rangle, \langle x,\zeta_2\rangle, \ldots, \langle x,\zeta_l\rangle)$ is a smooth functional. To prove $g(\mathbf{x})\phi_{s,t}(\mathbf{x}) \in \mathcal{S}(\mathbf{R}^{l+m})$, by the Leibniz formula, it is sufficient to examine the finiteness of

$$\sup_{x\in R^{l+m}} (1 + |\mathbf{x}|^2)^n |g^{(r)}(\mathbf{x})\phi_{s,t}^{(k)}(\mathbf{x})|, \qquad \text{for any integers } 0 \le r,\ k \le n.$$

For any differentiable function $c(\mathbf{x})$, we denote $(\frac{d}{d\mathbf{x}})^n c(\mathbf{x})$ by $c^{(n)}(\mathbf{x})$.

By the expression (4.12) of $D^k\big(\tilde{F}\big(z_{s,t}^N(x)\big)\big)(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \ldots, h_{i_k}^{(q)})$, (4.7) and the fact that $f(\mathbf{x}) = h(\mathbf{x})\varphi(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^m$ and $|g^{(r)}(\mathbf{x})| \le C_{48}\exp(-\sum_{i=1}^{l+m}\sqrt{|x_i|})$, it is enough to show the finiteness of

$$(4.16)\quad \sup_{Q}\Big(1 + \sum_{i=1}^{m}\langle x,\xi_i\rangle^2 + \sum_{j=1}^{l}\langle x,\zeta_j\rangle^2\Big)^n \times \exp\Big(-\sum_{i=1}^{l}\sqrt{|\langle x,\xi_i\rangle|} - \sum_{j=1}^{m}\sqrt{|\langle x,\zeta_j\rangle|}\Big) \times E\Big[\bar{h}^{(\mu(Gk))}\big(z_{s,t}^N(x)\big)\bar{\varphi}^{(\nu)}\big(z_{s,t}^N(x)\big)\Big)^2\Big]^{1/2}$$

where

$$Q = \big\{x\,;\,(\langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \ldots, \langle x,\xi_m\rangle, \langle x,\zeta_1\rangle, \langle x,\zeta_2\rangle, \ldots, \langle x,\zeta_l\rangle) \in \mathbf{R}^{l+m}\big\},$$
$$\bar{h}^{(\mu)}\big(z_{s,t}^N(x)\big) = h^{(\mu)}(\langle z_{s,t}^N(x),\xi_1\rangle, \langle z_{s,t}^N(x),\xi_2\rangle, \ldots, \langle z_{s,t}^N(x),\xi_m\rangle)$$

and

$$\bar{\varphi}^{(\nu)}\big(z_{s,t}^N(x)\big) = \varphi^{(\nu)}(\langle z_{s,t}^N(x),\xi_1\rangle, \langle z_{s,t}^N(x),\xi_2\rangle, \ldots, \langle z_{s,t}^N(x),\xi_m\rangle).$$

Since $|h^{(\mu)}(x)| \leq C_{49} \exp(\sum_{i=1}^m \sqrt{|x_i|})$, (4.9) of Lemma 6 shows that (4.16) is dominated by

(4.17)

$$\begin{aligned}
&\sup_Q \Big(1 + \sum_{i=1}^m \langle x, \xi_i \rangle^2 + \sum_{j=1}^l \langle x, \zeta_j \rangle^2\Big)^n \times \exp\Big(-\sum_{j=1}^l \sqrt{|\langle x, \zeta_j \rangle|}\Big) E\Big[\big(\bar{\varphi}^{(\nu)}(z_{s,t}^N(x))\big)^4\Big]^{1/4} \\
&\leq C_{50} \sup_Q \Big(1 + \sum_{i=1}^m \langle x, \xi_i \rangle^2 + \sum_{j=1}^l \langle x, \zeta_j \rangle^2\Big)^n \exp\Big(-\sum_{j=1}^l \sqrt{|\langle x, \zeta_j \rangle|}\Big) \\
&\quad \times E\left[\frac{\big(1 + \sum_{i=1}^m \langle z_{s,t}^N(x), \xi_i \rangle^2\big)^{4n}}{\big(1 + \sum_{i=1}^m \langle z_{s,t}^N(x), \xi_i \rangle^2\big)^{4n}} \big|\bar{\varphi}^{(\nu)}(z_{s,t}^N(x))\big|^4\right]^{1/4} \\
&\leq C_{51} \|\varphi\|_n \sup_Q \Big(1 + \sum_{i=1}^m \langle x, \xi_i \rangle^2 + \sum_{j=1}^l \langle x, \zeta_j \rangle^2\Big)^n \exp\Big(-\sum_{j=1}^l \sqrt{|\langle x, \zeta_j \rangle|}\Big) \\
&\quad \times E\left[\frac{1}{\big(1 + \sum_{i=1}^m \langle z_{s,t}^N(x), \xi_i \rangle^2\big)^{4n}}\right]^{1/4}
\end{aligned}$$

where $\|\varphi\|_n = \sup_{\substack{x \in R^m \\ 0 \leq r \leq n}} (1 + |\mathbf{x}|^2)^n |\varphi^{(r)}(\mathbf{x})|$.

On the other hand, we can verify the following lemma.

LEMMA 8. *For any $\xi_1, \xi_2, \ldots, \xi_m \in E$ and any integer $p \geq 1$, we have*

$$E\left[\frac{1}{\big(1 + \sum_{i=1}^m \langle z_{s,t}^N(x), \xi_i \rangle^2\big)^p}\right] \leq C_{52}(T) \frac{1}{\big(1 + \sum_{i=1}^m \langle x, \xi_i \rangle^2\big)^p}, \quad 0 \leq s,\ t \leq T.$$

PROOF. Setting $\theta(x) = \frac{1}{\left(1 + \sum_{i=1}^m \langle x, \xi_i \rangle^2\right)^p}$ and applying the Itô formula for $\theta\big(z_{s,t}^n(x)\big)$, we get

(4.18)

$$\begin{aligned}
E\left[\frac{1}{\big(1 + \sum_{i=1}^m \langle z_{s,t}^N(x), \xi_i \rangle^2\big)^p}\right] &= \frac{1}{\big(1 + \sum_{i=1}^m \langle x, \xi_i \rangle^2\big)^p} \\
&\quad + E\Big[\int_s^t -2p\Big(1 + \sum_{i=1}^m \langle z_{s,r}^N(x), \xi_i \rangle^2\Big)^{-(p+1)} \\
&\quad \times \Big(\sum_{i=1}^m \langle z_{s,r}^N(x), \xi_i \rangle \langle V(t,r)\tilde{B}\big(r, z_{s,r}^{N-1}(x)\big), \xi_i \rangle\Big)\, dr\Big] \\
&\quad + \Big[\int_s^t \sum_{j=1}^\infty \Big\{2p(p+1)\Big(1 + \sum_{i=1}^m \langle z_{s,r}^N(x), \xi_i \rangle^2\Big)^{-(p+2)} \\
&\quad \times \Big(\sum_{i=1}^m \langle z_{s,r}^N(x), \xi_i \rangle \langle V(t,r) h_j^{(0)}, \xi_i \rangle\Big)^2 \\
&\quad - p\Big(1 + \sum_{i=1}^m \langle z_{s,r}^N(x), \xi_i \rangle^2\Big)^{-(p+1)} \Big(\sum_{i=1}^m \langle V(t,r) h_j^{(0)}, \xi_i \rangle^2\Big)\Big\}\, dr\Big].
\end{aligned}$$

By the boundedness of $\tilde{B}(t,x)$, (4.18) is dominated by

$$\frac{1}{\left(1+\sum_{i=1}^{m}\langle x,\xi_i\rangle^2\right)^p}+C_{53}\int_s^t E\left[\frac{1}{\left(1+\sum_{i=1}^{m}\langle z_{s,r}^N(x),\xi_i\rangle^2\right)^p}\right]dr,$$

which yields the proof of the lemma, together with the Gronwall lemma.

Using this lemma, we see that the right hand side of (4.16) is dominated by

$$C_{54}\|\varphi\|_n \sup_Q\left(1+\sum_{i=1}^{m}\langle x,\xi_i\rangle^2+\sum_{j=1}^{l}\langle x,\zeta_j\rangle^2\right)^n \exp\left(-\sum_{j=1}^{l}\sqrt{|\langle x,\zeta_j\rangle|}\right)$$
$$\times\frac{1}{\left(1+\sum_{i=1}^{m}\langle x,\xi_i\rangle^2\right)^n}<\infty.$$

Hence $E\left[\tilde{F}\left(z_{s,t}^N(x)\right)\right]$ is a weighted Schwartz functional and the rest of Lemma 1 follows immediately.

5. **A fluctuation theorem for a system of interacting, spatially distributed neurons.** A problem in neurophysiology that has received considerable attention in recent years, is the stochastic behavior of the voltage potential of a spatially distributed neuron [12, 28]. When the spatial dimension of the neuronal membrane is greater than one, the voltage potential is modeled as a stochastic process taking values in the dual of some nuclear space such as the space of Schwartz distributions $\mathcal{S}'(\mathbf{R^d})$. The SDE satisfied by the voltage potential is best introduced via the following general model: Let H be a real separable Hilbert space, in applications, usually $H = L^2(\mathcal{X}, d\mu)$ where $\mathcal{X}$ is the membrane of the spatially extended neuron (e.g. $\mathcal{X} = [0,b]$, a d-dimensional rectangle or a compact Riemannian manifold and μ is the appropriate natural measure on $\mathcal{X}$). Let T_t be a strongly continuous semigroup on H generated by a closed, densely defined operator $\mathcal{K}$ such that $(\mathcal{K}\xi,\xi)_H \leq 0$ for $\xi \in \mathrm{Dom}(\mathcal{K})$ where $(\cdot,\cdot)_H$ denotes the inner product of H. Assume that some power of the resolvent of $\mathcal{K}$ is a Hilbert-Schmidt operator *i.e.*

(5.1) $$(\lambda I-\mathcal{K})^{-r_1} \text{ is Hilbert-Schmidt for some } r_1>0.$$

For example, (parallel fiber neurons), $\mathcal{K}$ is usually the Laplacian on a bounded region with nice properties; then the above condition (5.1) is satisfied [3].

Then there is a CONS $\{\varphi_j\}_{j\geq 1}$ in H such that $-\mathcal{K}\varphi_j = \lambda_j\varphi_j$ for any $j \geq 1$ and $0 \leq \lambda_1 < \lambda_2 < \cdots$. Set

$$E=\left\{\xi\in H\,;\,\sum_{j=1}^{\infty}(1+\lambda_j)^{2r}(\xi,\varphi_j)_H^2<\infty \text{ for any } r\geq 0\right\}.$$

Define the inner product on E,

$$(\xi,\zeta)_r=\sum_{j=1}^{\infty}(1+\lambda_j)^{2r}(\xi,\varphi_j)_H(\zeta,\varphi_j)_H$$

and E_r as the $\|\cdot\|_r$-completion of E, $\left(\|\xi\|_r^2 = (\xi, \xi)_r\right)$ and E_r' as the dual of the Hilbert space E_r. For $r < s$, $E_s \subseteq E_r$ and $E_0 = H$. Condition (5.1) implies that the canonical injection $E_p \longrightarrow E_r$ is Hilbert-Schmidt if $p > r + r_1$. Hence E is nuclear.

Assumptions 1 and 2 are satisfied for the nuclear spaces E' which are duals of the spaces E defined above. To see this, note that, in view of (5.1) there exists an integer $i_0 \geq 1$ such that $\lambda_{i_0} > 0$. For simplicity, take $i_0 = 1$, then

$$\|\xi\|_{p+r}^2 = \sum_{j=1}^{\infty}(1+\lambda_j)^{2(p+r)}(\xi, \varphi_j)_H^2 \geq (1+\lambda_1)^{2r}\|\xi\|_p^2.$$

Take sufficiently large r_0 such that $(1+\lambda_1)^{r_0} \geq 2$, then we have

$$2\|\xi\|_p \leq \|\xi\|_{p+r_0}.$$

Further, E_p' has a C.O.N.S. $\{(1+\lambda_j)^p h_j\}$ where $\langle h_j, \varphi_i\rangle = \delta_{ij}$, $h_j \in H' \subset E'$. Hence we can carry out in a manner similar to the proof in Φ that Assumptions 1 and 2 are satisfied for this case.

Since $\mathcal{K}$ generates T_t on H, we have for $\xi \in E$ and $t > 0$,

$$T_t\xi = \sum_{j=1}^{\infty} e^{-t\lambda_j}(\xi, \varphi_j)_0\varphi_j.$$

The following properties of T_t can be easily verified:

(a) $T_tE \subset E$,

(b) The restriction of T_t to E is an E-continuous semigroup,

(c) $t \longrightarrow T_t\xi$ is continuous for every $\xi \in E$,

(d) The restriction of $\mathcal{K}$ on E maps E into E and is the generator of the semigroup T_t on E,

(e) For any $\xi \in E$ and $t, s > 0$,

$$\|T_t\xi\|_r \leq \|\xi\|_r \text{ and } \|(T_t - T_s)\xi\|_r \leq |t-s|\ \|\xi\|_r.$$

The voltage potential is then derived as the solution of an E'-valued SDE

$$dX(t) = d\beta(t) + \mathcal{K}'X(t)\,dt, \tag{5.2}$$

where $\mathcal{K}'$ is the adjoint of $\mathcal{K}$ on E and $\beta(t)$ is the standard E'-Wiener process.

Let us now define

$$\langle V(t)x, \xi\rangle = \langle x, T_t\xi\rangle \qquad \text{for any } x \in E',\ \xi \in E.$$

Then, using property (e) above we have

$$\|V(t)x\|_{-r} = \sup_{\|\xi\|_r\leq 1} |\langle x, T_t\xi\rangle| \leq \|x\|_{-r} \sup_{\|\xi\|_r\leq 1} \|T_t\xi\|_r \leq \|x\|_{-r}$$

and so

$$\sup_{0\leq t\leq T} \|V(t)x\|_{-r} \leq \|x\|_{-r}. \tag{5.3}$$

Thus the condition (V1) stated before is satisfied for the class of spatially extended neurons whose voltage potentials are modeled by (5.2). For specific examples of $L^2(\mathcal{X}, d\mu)$ and the semigroup T_t which describes the deterministic part of the behavior of the neuron, see [12].

We now come to the question of interacting assemblies of a very large number of neurons. This appears to be an important problem of physiological interest since such large systems are involved in the functioning of the central nervous system. The difficulty consists in discovering the precise nature of the interaction in a mathematical form. In this section we consider an interaction similar to the mean-field interaction in particle diffusions. Another, possibly more realistic interaction known in the physiological literature as "parallel fiber interaction" will be investigated in our future work.

We now consider an infinite dimensional version of the fluctuation result for the McKean model in the following setting. Let $b(x, y)$ be a mapping from $E' \times E'$ to some E'_{p_0} such that $b(\cdot, \cdot)$ is infinitely many times E'_p-Fréchet differentiable for every integer $p \geq 0$ and with all derivatives bounded;

$$(V2) \qquad \sup_{x,y \in E'} \|D_x^k D_y^m b(x, y)\|_{\text{H.S.}}^{(p)} < \infty$$

for any integers k, m and $p \geq 0$. Here D_x and D_y denote the Fréchet derivatives with respect to variables x and y. The i-th component $X_i^{(n)}(t)$ of the n-system of diffusions obeys the following stochastic differential equation:

$$(5.4) \quad dX_i^{(n)}(t) = d\beta_i(t) + \Big\{ \mathcal{K}(t)X_i^{(n)}(t) + \frac{1}{n}\sum_{j=1}^{n} b\big(X_i^{(n)}(t), X_j^{(n)}(t)\big)\Big\}\, dt, \quad i = 1, 2, \ldots, n,$$

where $\{\beta_i(t)\}$ are independent copies of $\beta(t)$ and $\mathcal{K}(t)$ is a continuous linear operator stated in the Introduction. Then (5.4) is equivalent to

$$(5.5) \quad X_i^{(n)}(t) = V(t, 0)\sigma_i + \int_0^t V(t, s)\, d\beta_i(s) + \int_0^t V(t, s)\Big(\frac{1}{n}\sum_{j=1}^{n} b\big(X_i^{(n)}(s), X_j^{(n)}(s)\big)\Big)\, ds.$$

For simplicity we assume the initial values σ_i to be independent copies of σ such that $E[\exp(\varepsilon\|\sigma\|_{-p_0})] < \infty$ for every $\varepsilon > 0$.

The solution of (5.5) until time T is easily obtained by the usual method of successive approximations in $E'_{n(p_0,T)}$.

For the finite measure $\nu(dx)$ on E', set $b[x, \nu] = \int_{E'} b(x, y)\nu(dy)$, where the integral is the Bochner integral on E' and consider

$$(5.6) \qquad \begin{aligned} dX_i(t) &= d\beta_i(t) + \{\mathcal{K}(t)X_i(t) + b[X_i(t), u]\}\, dt, \\ u(t, dx) &= \text{ the distribution of } X_i(t). \end{aligned}$$

Then according to the following lemma the empirical distribution $\frac{1}{n}\sum_{j=1}^{n}\delta_{X_j^{(n)}(t)}$ converges to $u(t, dx)$ in probability in the usual weak convergence of measures, where δ_x is the Dirac measure at x in E'.

LEMMA 9. *For any $T > 0$ and integer $j \geq 1$,*

$$E\Big[\|X_i^{(m)}(t) - X_i(t)\|_{-n(p_0,T)}^{2j}\Big] \leq C_{55}(T)/m^j, \quad 0 \leq t \leq T.$$

PROOF. Put $n_0 = n(p_0, T)$. Then the condition (V2) yields

$$\begin{aligned}\Big\|b\big(X_i^{(m)}(t), X_j^{(m)}(t)\big) - b\big(X_i(t), X_j^{(m)}(t)\big)\Big\|_{-p_0} &\\ \leq \sup_{x,y\in E'} \|D_x b(x,y)\|_{\text{H.S.}}^{(n_0)} \|X_i^{(m)}(t) - X_i(t)\|_{-n_0} &\\ \leq C_{56}\|X_i^{(m)}(t) - X_i(t)\|_{-n_0} &\end{aligned}$$

and

$$\Big\|b\big(X_i(t), X_j^{(m)}(t)\big) - b\big(X_i(t), X_j(t)\big)\Big\|_{-p_0} \leq C_{57}\,\|X_j^{(m)}(t) - X_j(t)\|_{-n_0},$$

so that we have

$$\begin{aligned}&E\Big[\|X_i^{(m)}(t) - X_i(t)\|_{-n_0}^{2j}\Big]\\ &\leq C_{58}(T)\int_0^t E\Big[\Big\|V(t,s)\Big\{\frac{1}{m}\sum_{j=1}^m b\big(X_i^{(m)}(s), X_j^{(m)}(s)\big) - b[X_i(s),u]\Big\}\Big\|_{-n_0}^{2j}\Big]\,ds\\ &\leq C_{59}(T)\int_0^t \Big\{E\Big[\|X_i^{(m)}(s) - X_i(s)\|_{-n_0}^{2j}\Big] + \frac{1}{m}\sum_{j=1}^m E\Big[\|X_j^{(m)} - X_j(s)\|_{-n_0}^{2j}\Big]\\ &\quad + E\Big[\Big\|\frac{1}{m}\sum_{j=1}^m\big\{b\big(X_i(s), X_j(s)\big) - b[X_i(s),u]\big\}\Big\|_{-p_0}^{2j}\Big]\Big\}\,ds.\end{aligned} \tag{5.7}$$

From the independence of $X_i(t)$, $i = 1, 2, \ldots, m$ and condition (V2), we have

$$E\Big[\Big\|\frac{1}{m}\sum_{j=1}^m\big\{b\big(X_i(s), X_j(s)\big) - b[X_i(s),u]\big\}\Big\|_{-p_0}^{2j}\Big] \leq C_{60}(T)/m^j. \tag{5.8}$$

Therefore Gronwall's inequality, together with (5.7) and (5.8), implies the assertion of Lemma 9.

Now we are able to proceed to the fluctuation problem. Suppose that $u(dx)$ is a probability measure on E' and $\int_{E'}\exp\big(\alpha\sqrt{|\langle x,\xi\rangle|}\big)u(dx) < \infty$ for any $\alpha > 0$. Then for $F \in C_{0,n}^\infty(E')$, $|\langle u, F\rangle| \leq C_{61}\|F\|_0$, so that $u(dx) \in C_{0,n}^\infty(E')'$. Hence, together with the following Lemma 10, the Dirac measure $\delta_x, x \in E'$ and $u(t, dx)$ are considered as elements of $C_0^\infty(E')'$. We are able to consider $\mathcal{N}_n(t) = \sqrt{n}\big(\frac{1}{n}\sum_{j=1}^n \delta_{X_j^{(n)}(t)} - u(t,dx)\big)$ as a $C_0^\infty(E')'$-valued continuous stochastic process [19], [23]. To check the tightness of $\mathcal{N}_n(t)$ in $C([0,\infty)\,;\,C_0^\infty(E')')$, the space of all continuous mappings from $[0,\infty)$ into $C_0^\infty(E')'$, it is enough to verify the Kolmogorov tightness criterion for $\langle\mathcal{N}_n(t), F\rangle$, $F \in C_0^\infty(E')$, where $\langle\ ,\ \rangle$ denotes the canonical bilinear form on $C_0^\infty(E')' \times C_0^\infty(E')$ [6], [20].

We have the following exponential integrability.

LEMMA 10. *For any $\alpha > 0$, $T > 0$ and any integer $p \geq n(p_0, T)$, there exists a constant $C_{62} = C_{62}(\alpha, T, p)$ such that*

$$\sup_{0\leq t\leq T} E[e^{\alpha\|X_i^{(n)}(t)\|_{-p}}] \vee E[e^{\alpha\|X_i(t)\|_{-p}}] \leq C_{62}.$$

PROOF. Set $n_0 = n(p_0, T)$. Assumptions (V1) and (V2) give

$$\max\left\{\|X_i^{(n)}(t)\|_{-n_0}, \|X_i(t)\|_{-n_0}\right\} \leq \|\sigma_i\|_{-p_0} + C_{63} + \left\|\int_0^t V(t,s)\, d\beta_i(s)\right\|_{-n_0}$$

and hence the lemma can be proved in the same way as Lemma 6.

Once we know Lemmas 9 and 10, we can check the moment condition;

$$E\left[|\langle \mathcal{N}_n(t) - \mathcal{N}_n(s), F\rangle|^4\right] \leq C_{64}(F)|t-s|^2, \tag{5.9}$$

(see [9]). Similarly we have

$$\sup_{0\leq t\leq T} E[\langle \mathcal{N}_n(t), F\rangle^2] \leq C_{65}(T)\|F\|^2_{n_0,n_0,1}. \tag{5.10}$$

Then a subsequence of $\mathcal{N}_n(t)$ converges to $\mathcal{N}(t)$ in $C([0,\infty)\,;\, C_0^\infty(E')')$.

By the Itô formula, for $F \in C_0^\infty(E')$, we have

$$\begin{aligned}\langle \mathcal{N}_n(t), F\rangle - \langle \mathcal{N}_0(t), F\rangle &= \int_0^t \left[\int_{E'} \big(\mathcal{M}(s)F\big)(x)\, d\mathcal{N}_n(s)\right] ds \\ &\quad + \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^t DF\big(X_j^{(n)}(s)\big)\big(d\beta_j(s)\big) = R_F^{(n)}(t),\end{aligned}$$

where $R_F^n(t)$ is the negligible term and

$$\begin{aligned}\big(\mathcal{M}(t)F\big)(x) &= \frac{1}{2}\operatorname{trace}_{E_0} D^2F(x) + DF(x)\big(b[x,u] + \mathcal{K}(t)x\big) \\ &\quad + \int_{E'} DF(y)\big(b(y,x)\big)u(t,dy).\end{aligned}$$

Since $\mathcal{M}(t)$ does not leave $C_0^\infty(E')$ invariant, to derive the SDE of type (1.1) satisfied by $\mathcal{N}(t)$, we extend $\mathcal{N}_n(t)$ and $\mathcal{N}(t)$ to continuous $\mathcal{L}(\mathcal{D}_{E'})$-processes by using (5.9) and (5.10) and so we denote the extensions by $(\mathcal{N}_n)_F(t)$ and $\mathcal{N}_F(t)$.

Now we impose a rather technical condition on $b(x,y)$.

For any $\varepsilon > 0$ and any integers $p, q, n \geq 0$, there exists a C_b^∞-function $\bar{b}(\mathbf{x},\mathbf{y})$ of $\mathbf{R}^m \times \mathbf{R}^{m'}$ to E'_{p_0} such that

$$\begin{aligned}\sup_{x\in E'_p}\Big\|D_x^\mu D_y^\nu[b(x,y) - \bar{b}(\langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \dots, \langle x,\xi_m\rangle, \\ \langle y,\zeta_1\rangle, \langle y,\zeta_2\rangle, \dots, \langle y,\zeta_{m'}\rangle)]\Big\|^{(q)}_{\text{H.S.}} < \varepsilon,\end{aligned} \tag{V3}$$

$$0 \leq \mu + \nu \leq n,\ \xi_i, \zeta_j \in E, \quad i = 1,2,\dots,m \text{ and } j = 1,2,\dots,m'.$$

Here C_b^∞-function means $\bar{b}(\mathbf{x}, \mathbf{y})$ itself and all the derivatives are bounded.

We set

$$\big(J(t)F\big)(x) = \int_{E'} DF(y)\big(b(y,x)\big)u(t,dy).$$

By Assumptions 1 and 2, (V1), (V2), (V3) and a part of the proof of Theorem 1, we can show that

$$L(t)\mathcal{D}_{E'} \subset \mathcal{D}_{E'} \text{ and } J(t)\mathcal{D}_{E'} \subset \mathcal{D}_{E'}.$$

Define

$$W_F(t) = \mathcal{N}_F(t) - \mathcal{N}_F(0) - \int_0^t \mathcal{N}_{M(s)F}(s)\,ds.$$

Then noticing that the characteristic function of $(\mathcal{N}_n)_F(t)$ converges to the characteristic function of $\mathcal{N}_F(t)$ and following the argument of [9] word by word, we have the proof that $W_F(t)$ is a continuous $\mathcal{L}(\mathcal{D}_{E'})$-Wiener process. Thus any limit process of convergent subsequences of $\mathcal{N}_n(t)$ satisfies the SDE of type (1.1).

By Theorem 1, $L(t)$ generates the Kolmogorov evolution operator from $\mathcal{D}_{E'}$ into itself. Further since $J(t)$ satisfies the condition of Proposition 2 in [22] the proof of Proposition 2 in [22] is valid for any Fréchet space. $\mathcal{M}(t) = L(t) + J(t)$ generates the Kolmogorov evolution operator like $U(t,s)$ in Theorem 1. Since Theorem 1 gives the identification of the distributions of the limit processes $\mathcal{N}(t)$, we obtain the desired conclusion.

THEOREM 2. *Under assumptions (V1)–(V3) and the exponential integrability of σ, $\mathcal{N}_n(t)$ converges to a Gaussian field governed by the weak* SDE *of type (1.1) in $C([0,\infty)\,;\,C_0^\infty(E')')$, namely,*

$$d\mathcal{N}_F(t) = dW_F(t) + \mathcal{N}_{M(t)F}(t)\,dt.$$

Unfortunately we have no criterion under which an $\mathcal{L}(\mathcal{D}_{E'})$-process is a $\mathcal{D}'_{E'}$-valued process.

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REFERENCES

1. P. Billingsley, *Convergence of Probability Measures*, Wiley, New York-London-Sydney-Toronto, 1968.

2. T. Bojdecki and L. G. Gorostiza, *Langevin equation for $\mathcal{S}'$-valued Gaussian processes and fluctuation limits of infinite particle systems*, Probab. Th. Rel. Fields **73**(1986), 227–244.

3. R. Courant and D. Hilbert, *Methods of Mathematical Physics 1*, Interscience Publishers, Inc., New York, 1966.

4. D. A. Dawson, *Critical dynamics and fluctuations for a mean-field model of cooperative behavior*, J. Statist. Phys. **31**(1983), 29–85.

5. J. D. Deuschel, *Central limit theorem for an infinite lattice system of interacting diffusion processes*, Ann. Probab. **16**(1988), 700–716.

6. J. P. Fouque, *La convergence en loi pour les processus a valeurs dans un espace nucleaire*, Ann. IHP **20**(1984), 225–245.

7. I. M. Gelfand and G. E. Shilov, *Generalized functions 2*, Academic press, New York and London, 1964.

8. J. I. Gikhman and A. V. Skorokhod, *Stochastic Differential Equations*, Springer, Berlin, 1972.

9. M. Hitsuda and I. Mitoma, *Tightness problem and stochastic evolution equation arising from fluctuation phenomena for interacting diffusions*, J. Multivariate Anal. (1986), 311–328.

10. R. Holley and D. W. Stroock, *Central limit phenomena of various interacting systems*, Ann. Math. **110** (1979), 333–393.

11. K. Itô, *Infinite dimensional Ornstein-Uhlenbeck processes*, Taniguchi Symp. SA, Katata, Kinokuniya, Tokyo, (1984), 197–224.

12. G. Kallianpur and R. Wolpert, *Infinite dimensional stochastic models for spatially distributed neurons*, Appl. Math. Optim **12**(1984), 125–172.

13. H. Komatsu, *Semi-groups of operators in locally convex spaces*, J. Math. Soc. Japan, **16**(1964), 232–262.

14. H. Kunita, *Stochastic differential equations and stochastic flows of diffeomorphisms*, Lect. Notes in Math. **1097**, Springer 1984.

15. H. H. Kuo, *Gaussian measures on Banach spaces*, Lect. Notes in Math. **463**, Springer, Berlin, 1975.

16. ———, *Stochastic integrals in abstract Wiener space II, regularity properties*, Nagoya Math. J. **50**(1973), 89–116.

17. H. P. McKean, *Propagation of chaos for a class of non-linear parabolic equations*, Lecture series in Differential Equations **7**, Catholic Univ. (1967), 41–57.

18. R. A. Minlos, *Generalized random processes and their extension to a measure*, Selected Transl. Math. Statist. Probab. **3**(1962), 291–313.

19. I. Mitoma, *On the sample continuity of $\mathcal{S}'$-processes*, J. Math. Soc. Japan, (1983), 629–636.

20. ———, *Tightness of probabilities on* $C([0,1]\,;\mathcal{S}')$ *and* $D([0,1]\,;\mathcal{S}')$, Ann. Probab. **11**(1983), 989–999.

21. ———, *An ∞-dimensional inhomogeneous Langevin's equation*, J. Functional Analysis **61**(1985), 342–359.

22. ———, *Generalized Ornstein-Uhlenbeck process having a characteristic operator with polynomial coefficients*, Probab. Th. Rel. Fields **76**(1987), 533–555.

23. C. Martias, *Sur les support des processus a valeurs dans des espaces nucleairs*, Ann. IHP **24**(1988), 345–365.

24. I. Segal, *Non-linear functions of weak processes*, J. Funct. Anal. **4**(1969), 404–457.

25. T. Shiga and H. Tanaka, *Central limit theorem for a system of Markovian particles with mean-field interactions*, Z. Wahrsch. verw. Gebiete **69**(1985), 439–459.

26. H. H. Schaefer, *Topological vector spaces*, Springer, Berlin, 1972.

27. H. Totoki, *A method of construction for measures on function spaces and its applications to stochastic processes*, Mem. Fac. Sci. Kyushu Univ. Ser. A, Math. **15**(1962), 178–190.

28. J. Walsh, *An introduction to stochastic partial differential equations*, École d'eté de Probabilités de Saint-Flour XIV, Lect. Notes in Math. **1180**, Springer, Berlin, 1984.

29. S. Watanabe, *Lectures on stochastic differential equations and Malliavin's calculus*, Springer, Berlin, 1984.

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