REGULAR SURFACES OF GENUS TWO: PART I

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The present paper is a sequel to one I published lately (**3**) on regular surfaces of genus 3, and like it, is intended to fill up some of the gaps in our detailed knowledge of the regular surfaces of moderately low genus $p = p_g = p_a$ and linear genus $p^{(1)} = n + 1 > 1$. (The surfaces for which $p^{(1)} = 1$ form a rather separate field of study on which a good deal of work has been done, and I shall not consider them.) There is for $p = 2$ no canonical model, since the canonical system is only a pencil; but there is in general a unique bicanonical model, about which we shall find that something can be said. The problem, like most similar problems, increases sharply in difficulty with increase of the linear genus, and it is only for the first few values of $p^{(1)}$ that anything like a complete classification of the surfaces in question can be obtained. On account of the length of the work, I am publishing here some general results, and the detailed study of the cases in which $p^{(1)} \leqslant 4$, and shall hope to extend the classification to the cases $p^{(1)} = 5, 6$, with some examples of surfaces for $p^{(1)} = 7$, in a subsequent paper. The notation $[r]$ will be used throughout for the r-dimensional linear space.

1. Generalities. We may begin with one very general result:

THEOREM 1.1 *Every regular surface of genus* 2 *and linear genus* $n + 1$, *whose canonical system is irreducible, has as bicanonical model a surface* F^{4n} *of order* $4n$ *in* $[n + 2]$, *which lies on a quadric cone* $\Gamma_{n+1}{}^2$ *with* $[n - 1]$ *vertex* Ω_{n-1}, i.e., the cone generated by the $[n]$'s joining Ω_{n-1} to the points of a conic in a plane skew to Ω_{n-1}; *and the canonical pencil is traced on* F^{4n} *by the generating* $[n]$'s *of* $\Gamma_{n+1}{}^2$.

For the grade of the bicanonical system is four times that of the canonical, i.e., $4n$; and its freedom is $P - 1$, where by a known formula (**7,** p. 159)

$$P = p^{(1)} + p = n + 3.$$

This gives the order and ambient dimensions of the bicanonical model. Now the bicanonical system is adjoint to the canonical, i.e., on the bicanonical model each curve of the canonical pencil appears as its own canonical model, a curve C^{2n} in $[n]$. The canonical pencil $|C^{2n}|$ has n base points $A_1, \ldots, A_n$, which form a semicanonical set on the general curve of the pencil, i.e., a set in which the curve is touched by an $[n - 1]$ in its ambient $[n]$. Since, moreover, on a regular surface the characteristic series of a complete system, in particular of the canonical system, is complete, the set $A_1, \ldots, A_n$ forms a complete series on each C^{2n}, which means that their join is an $[n - 1]$ and not less, since every $[n - 1]$

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through them must trace on the curve an equivalent set. Their join is thus the unique $[n-1]$, Ω_{n-1}, which touches every C^{2n} in these points; Ω_{n-1} lies in the ambient $[n]$'s of all the curves C^n, which accordingly generate a cone of $n+1$ dimensions, and of order 2, since there are hyperplane sections of F^{4n} consisting of two curves C^{2n}, and hence hyperplane sections of the cone consisting of two of the $[n]$'s; i.e., the cone is the quadric cone Γ_{n+1}^2.

We have a more definite but restricted result in the following:

THEOREM 1.2 *Let G_3^{2n} be a 2n-ic threefold in $[n+2]$, whose general $[n]$ section is a canonical curve of genus $n+1$, so that its general $[n+1]$* (i.e., hyperplane) *section, is a surface G_2^{2n} of genus* 1, *whose unique canonical curve is the null curve* (**7**, p. 247); *and let G_3^{2n}, Γ_{n+1}^2 have simple contact along a normal rational n-ic curve K^n lying in a generating $[n]$ X_n of Γ_{n+1}^2; then their surface of intersection, F^{4n}, is the bicanonical model of a surface of genera $p=2$, $p^{(1)}=n+1$.*

(By simple contact is intended that the general point of K^n is double on F^{4n}, though simple on G_3^{2n} and Γ_{n+1}^2; this requires that the tangent [3] to G_3^{2n} at such a point is the join of the tangent planes to the two sheets of F^{4n}, and lies in Y_{n+1}, the tangent hyperplane to Γ_{n+1}^2 at all points of X_n.)

For the general hyperplane section f^{4n} of F^{4n} is a quadric section of a hyperplane section G_2^{2n} of G_3^{2n}; and it has n double points, at its intersections with K^n. On G_2^{2n} every complete linear system, in particular that of quadric sections, is adjoint to itself; the canonical series on f^{4n} is thus traced by quadrics through its n double points; and thus the system adjoint to the hyperplanes sections of F^{4n} is traced on it residually by quadrics through the double curve K^n. The difference between this adjoint system and the hyperplane sections themselves, i.e., the canonical system, is accordingly traced residually on F^{4n} by the pencil of hyperplanes through K^n, i.e., through X_n. But the generating $[n]$'s of Γ_{n+1}^2 trace on G_3^{2n} and hence on F^{4n}, canonical curves C^{2n} of genus $n+1$, any two of which are a hyperplane section of F^{4n}; they thus form a pencil on F^{4n}, traced residually by the pencil of hyperplanes through any one of them. But K^n, counted as double, is one of these curves, being the complete intersection of X_n with G_3^{2n}; the canonical system on F^{4n} is thus just the pencil $|C^{2n}|$, from which the theorem at once follows.

As a partial converse to this we have the result:

THEOREM 1.3 *If the bicanonical model F^{4n} of a regular surface of genera $p=2$, $p^{(1)}=n+1$, with irreducible canonical pencil, is the complete intersection of Γ_{n+1}^2 with some threefold, the latter can only be the threefold G_3^{2n} whose $[n]$ sections are canonical curves of genus $n+1$, touching Γ_{n+1}^2 along a normal rational curve K^n in a generating $[n]$, X_n, of Γ_{n+1}^2.*

For the threefold is of order $2n$, and has at least some $[n]$ sections which are canonical curves of genus $n+1$; thus its general $[n]$ section cannot be of lower genus than this, nor (since those which we know to be of genus $n+1$ are without singularities) can it be of higher genus; and since its ambient is $[n]$, it is the canonical curve of this genus. The threefold is thus G_3^{2n}. Now the adjoint system to the hyperplane sections $|2C^{2n}|$ of F^{4n} is $|3C^{2n}|$, and is traced residually on the surface by quadrics through any one curve C^{2n}, i.e., through any one generating $[n]$ of Γ_{n+1}^2; and by a known formula (**7**, p. 61) the genus of the general hyperplane section f^{4n} is $3n + 1$; but f^{4n} is a quadric section of a hyperplane section G_2^{2n} of G_3^{2n}, and on G_2^{2n} belongs to a complete linear system of genus $4n + 1$, adjoint to itself; to be of genus $3n + 1$, f^{4n} must have n double points, and its canonical series will be traced residually by quadrics through these, which must consequently be its complete intersection with a certain generating $[n]$, X_n, of Γ_{n+1}^2. By varying the secant hyperplane so as to keep one of these double points fixed, we see that X_n is the same generating $[n]$ of Γ_{n+1}^2 for all hyperplane sections of F^{4n}; F^{4n} has accordingly a double curve K^n which is its complete intersection with X_n, and is thus likewise the complete intersection of X_n with G_3^{2n}. This curve is rational, since its ambient X_n is of dimensions equal to the order of the curve.

We may devote a few lines to some properties of the surface constructed in these two theorems. The base points of the pencil $|C^{2n}|$ are of course the intersections of G_3^{2n} with Ω_{n-1}; and as such a point A_i is on the double curve K^n, the tangent [3] to G_3^{2n} at A_i lies in Y_{n+1}, and thus meets Ω_{n-1} in a line, the tangent line at A_i to the section of G_3^{2n} by a general $[n]$ through Ω_{n-1}, and thus to the general curve C^{2n}. Thus as we expect, Ω_{n-1} touches each C^{2n} in each of the points $A_1, \ldots, A_n$, i.e., these are a semicanonical set on C^{2n}. Since, moreover, K^n counted twice is a curve of the pencil $|C^{2n}|$, it is the image (on a model of the surface without singularities or exceptional curves) of a hyperelliptic curve of genus $n + 1$, the pairs of whose unique involution are neutral for the bicanonical system, and the united (or Jacobian) points of this involution, $2n + 4$ in number, correspond to pinch points of F^{4n}. Further, since $A_1, \ldots, A_n$ are a semicanonical set on this curve also, they are n of the pinch points. This explains how it is that the curves C^{2n} all have the same tangent in a point A_i, though on a non-singular model the corresponding point is only a simple base point of the pencil. For a curve on the non-singular model which passes simply through the coincident neutral pair corresponding to A_i, corresponds to a curve on F^{4n} with a cusp at A_i, whereas a curve passing through only the first point of the pair (not touching the hyperelliptic curve) corresponds to a curve on F^{4n} passing simply through A_i and touching a fixed line, the principal tangent; and all curves on a surface which pass simply through a pinch point must touch this line. Thus we may say that the curves C^{2n}, though they all touch each other *in space* at A_i, have only a simple intersection there *on the surface*. In this connexion we have the following simple remark:

THEOREM 1.4 *On the bicanonical model F^{4n} of any regular surface of genera $p = 2$, $p^{(1)} = n + 1$, whose canonical pencil is irreducible and non-hyperelliptic, every base point A_i of this pencil is either a simple point, in which case the tangent plane to F^{4n} in this point lies in Ω_{n-1}, or a pinch point, in which case the principal tangent lies in Ω_{n-1}, and is the tangent to all curves of the canonical pencil.*

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For since there are hyperplane sections of F^{4n} consisting of two curves C^{2n}, each of which passes simply through A_i, the multiplicity of A_i cannot exceed 2; and since on a model without singularities or exceptional curves A_i corresponds to a single point and not a curve, if it is a double point it can only be a pinch point; I have shown elsewhere that the only types of double point mapped by points only on a non-singular model are the pinch point and the intersection of two simple sheets. (Du Val (**2**). The result is not stated explicitly, but follows from the results of §11 as the multiplicity of the point is $\Sigma\sigma_i^2$, which can only equal 2 if $\sigma_1 = \sigma_2 = 1$.)

It must be understood for the purposes of this theorem that by saying that A_i is a pinch point we only mean that it is the image of a coincident neutral pair for the bicanonical system on a non-singular model, not that the pinch point necessarily occurs in the course of a double curve, i.e., that the coincident neutral pair is one of a singly infinite system of neutral pairs. On the surface we have just constructed this is so, and it is so on any surface in [3]; but in higher space, just as we can have an isolated intersection of two simple sheets which is not part of any multiple curve, so we can perfectly well have a pinch point which is not on any multiple curve of the surface.

We may next observe that

THEOREM 1.5 *Let F^{4n} be the bicanonical model of a surface of genera $p = 2$, $p^{(1)} = n + 1 \geqslant 3$, with irreducible and non-hyperelliptic canonical pencil $|C^{2n}|$; then if all the base points of this pencil are pinch points, the totality of quadrics on which F^{4n} lies is a linear system of freedom $\frac{1}{2}(n-1)(n-2)$, of which any sub-system of freedom $\frac{1}{2}n(n-3)$ (not containing Γ_{n+1}^2) traces on the general generating [n] of Γ_{n+1}^2 the complete system of quadrics through the corresponding curve C^{2n}.*

This depends on the lemma that the sections by any hyperplane of the quadrics through an irreducible normal manifold M are the complete system of quadrics in the hyperplane through the section of M. This must certainly be well known, but as I am not aware where it is to be found I give a brief proof here. Let H be a hyperplane whose intersection with M is M_H, and let Q_H be any quadric in H passing through M_H. Let Q be any quadric in the whole space whose intersection with H is Q_H, then its intersection with M consists of M_H together with something coresidual to a hyperplane section, and since M is normal this can only be the section M_K by some hyperplane K. The pencil of quadrics determined by Q and the hyperplane pair H, K all have the same complete intersection M_H, M_K with M, and all have the same intersection Q_H with H. But one quadric of this pencil can be found to pass through a further point of M (not on M_H or M_K), and hence, since M is irreducible, to contain the whole of M; thus Q_H is the section by H of a quadric through M.

We return to the proof of the theorem. The curves $|C^{2n}|$ all touch the principal tangent to the surface in each base point A_i, i.e., they all trace on Ω_{n-1} the same set of $2n$ points (coinciding by pairs). As C^{2n} is normal and irreducible, the sections by Ω_{n-1} of the quadrics Q_{n-1}^2 through C^{2n} in its ambient [n], a linear

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system of freedom $\frac{1}{2}n(n-3)$, are the complete system of quadrics Q_{n-2}^2 in Ω_{n-1} passing through these $2n$ points, i.e., touching the principal tangent to F^{4n} in each point A_i. Every quadric Q_{n-2}^2 of this system thus determines in each generating $[n]$ of Γ_{n+1}^2 a unique quadric Q_{n-1}^2 through C^{2n}, whose section it is; and the locus of the quadrics thus determined by any one Q_{n-2}^2 is an n-fold, S_n^4, which is clearly of order 4, and a quadric section of Γ_{n+1}^2, since it does not contain Ω_{n-1}, but meets the latter in Q_{n-2}^2. We thus obtain on Γ_{n+1}^2 a linear system $|S_n^4|$ of freedom $\frac{1}{2}n(n-3)$, the characteristic system of a linear system of freedom $\frac{1}{2}(n-1)(n-2)$, of which Γ_{n+1}^2 itself is one. It is clear that all these quadrics pass through F^{4n}, and that they are all the quadrics that do so.

An immediate corollary is:

THEOREM 1.6 *If all the hypothesis of* 1.5 *holds, and if in addition the general curve of the canonical pencil has neither a* g_3^1 *nor a* g_5^2, F^{4n} *is the complete intersection of the linear system of quadrics found in* 1.5.

(The notation g_n^r is used as usual for a linear series of order n and freedom r.)

For precisely in this case, the general C^{2n} is the complete intersection of the system of quadrics through it in its ambient.

It would be agreeable to be able to conclude at this point that under these circumstances the system of quadrics has a subsystem of freedom $\frac{1}{2}n(n-3)$ intersecting in a threefold, whose complete intersection with Γ_{n+1}^2 would then be F^{4n}, so that the threefold would be G_3^{2n} and the surface would be that referred to in Theorems 1.2 and 1.3. This however does not seem to be easy for $n > 4$; for $n = 4$ it is obvious, since in this case F^{16} is the complete intersection of a linear system of quadrics in [6], of freedom 3, whereas any system of quadrics in [6] of freedom 2 has an intersection which is at least threefold. For higher values of n it is not clear that $\frac{1}{2}(n-1)(n-2)$ quadrics need have an intersection which is as much as threefold. For $n = 5$ for instance, as we shall see later, if F^{20} is the complete intersection of a general G_3^{10} with Γ_6^2, G_3^{10} is in turn a complete quadric section of a certain quintic fourfold, U_4^5, which is the complete intersection of a linear system of quadrics, of freedom 4. Thus in the system of all quadrics through F^{20}, of freedom 6, no subsystem of freedom 5 can have a G_3^{10} as its intersection, nor consequently a threefold intersection at all, which does not contain this particular subsystem of freedom 4.

Some idea of the relation between the surfaces considered in Theorems 1.2 and 1.3 and a possible more general type of F^{4n} can be obtained from the following result:

THEOREM 1.7 *If as before* F^{4n} *is the bicanonical model of a surface of genera* $p = 2$, $p^{(1)} = n + 1 \geqslant 3$, *with irreducible canonical pencil, its projection* $'F^{4n}$ *into* [4] *from a general* $[n-3]$, O_{n-3}, *lying in* Ω_{n-1}, *is the complete section of* $'\Gamma_3^2$ (*the projection of* Γ_{n+1}^2) *by a hypersurface of order* $2n$. *The projections* $'A_1, \ldots,$ $'A_n$ *of* $A_1, \ldots, A_n$ *are pinch points of* $'F^{4n}$, *the principal tangent at each being the vertex line* $'\omega$ *of* $'\Gamma_3^2$; $'F^{4n}$ *has a double curve* $'d^{n(4n-7)}$, *of order* $n(4n-7)$, *which is the complete intersection of* $'F^{4n}$ *with a surface* $'\Sigma^{4n+7}$ *lying on* $'\Gamma_3^2$; $'\Sigma^{4n+7}$ *is in turn* *the residual section of* $'\Gamma_3^2$ *by a hypersurface of order* $2n - 3$ *through one of its generating planes. The necessary and sufficient condition for* F^{4n} *to have a double rational curve* K^n *lying in a generating* $[n]$ *of* Γ_{n+1}^2 *is that* $'\Sigma^{4n-7}$ *breaks up into a surface* $'\Sigma^{4n-8}$, *complete intersection of* $'\Gamma_3^2$ *with a hypersurface of order* $2n - 4$, *together with a generating plane of* $'\Gamma_3^2$, *the part of the double curve* $'d^{n(4n-7)}$ *which is the intersection of* $'F^{4n}$ *with this plane being the projection of the double curve* K^n *of* F^{4n}.

It is obvious that the projection of Γ_{n+1}^2 from O_{n-3} is $'\Gamma_3^2$; since Ω_{n-1} passes through O_{n-3} it is projected into a line $'\omega$, and every generating $[n]$ of Γ_{n+1}^2 is projected into a plane through $'\omega$, and of these planes two lie in a general [3] through $'\omega$. Each curve C^{2n} of the canonical pencil is projected into a plane curve $'C^{2n}$ in the corresponding generating plane of $'\Gamma_3^2$, which touches $'\omega$ in each of the points $'A_1, \ldots, 'A_n$; these are accordingly pinch points, $'\omega$ being the principal tangent at each. Since the curves $'C^{2n}$ have no variable intersection with $'\omega$, their locus $'F^{4n}$ is the complete intersection of $'\Gamma_3^2$ with a hypersurface of order $2n$. $'C^{2n}$ being of genus $n + 1$ has $2n(n - 2)$ double points, which (since the linear sections are part of the canonical series) are its complete intersection with a curve $'s^{2n-4}$ of order $2n - 4$. The locus of the ∞^1 curves $'s^{2n-4}$ is a surface $'\Sigma$ lying on $'\Gamma_3^2$, whose complete intersection with $'F^{4n}$ is the double curve of the latter, or part of it, any residual part of the double curve lying wholely in one or more generating planes of $'\Gamma_3^2$, and having no variable intersection with the general plane.

On the other hand the general hyperplane section of $'F^{4n}$ is a curve $'f^{4n}$, complete section by a surface of order $2n$, of the cone $'\Gamma_2^2$, hyperplane section of $'\Gamma_3^2$. Thus to be of genus $3n + 1$ (as it is), $'f^{4n}$ must have $n(4n - 7)$ double points, so that this is the order of the double curve on $'F^{4n}$. As $'f^{4n}$ is the complete intersection of two surfaces of orders 2, $2n$, its complete canonical series is traced residually by surfaces of order $2n - 2$ through its double points; on the other hand as the canonical series on the corresponding hyperplane section of F^{4n} is traced by quadrics through a generator of Γ_{n+1}^2, the residual intersection of $'f^{4n}$ with a quadric through a generator of $'\Gamma_3^2$, i.e., its complete intersection with a twisted cubic on $'\Gamma_2^2$, is a canonical set. This clearly means that the double points of $'f^{4n}$ are its complete intersection with a curve $'\sigma^{4n-7}$ on $'\Gamma_3^2$, obviously unique, which together with a general twisted cubic makes up the section of $'\Gamma_2^2$ by a surface of order $2n - 2$, and is thus itself the residual section by a surface of order $2n - 3$ through one generator.

Now if k of the curves $'s^{2n-4}$ pass through a general point of $'\omega$, the surface $'\Sigma$ generated by them has $'\omega$ as k-ple line, and is of order $4n - 8 + k$; for even values of k it is the complete section of $'\Gamma_3^2$ by a hypersurface of order $2n - 4 + \frac{1}{2}k$, for odd values the residual section by one of order $2n - 4 + \frac{1}{2}(k + 1)$ through a generating plane. It is clear however that the hyperplane section of $'\Sigma$ is the curve $'\sigma^{4n-7}$, or part of it, any residual part consisting of the sections of any generating planes of $'\Gamma_3^2$ that contain double curves of $'F^{4n}$. Comparing the orders of $'\sigma$, $'\Sigma$ we see that $k \leqslant 1$. If $k = 1$, $'\Sigma$ is of order $4n - 7$ and is the

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residual section of $'\Gamma_3{}^2$ by a hypersurface of order $2n - 3$ through one generating plane; its complete intersection with $'F^{4n}$ is the whole double curve $'d^{n(4n-7)}$ of the latter, which passes simply through the pinch points $'A_1, \ldots, 'A_n$ and meets each generating plane of $'\Gamma_3{}^2$ residually in the $2n(n-2)$ double points of $'C^{2n}$. If on the other hand $k = 0$, $'\Sigma$ is of order $4n - 8$ and is the complete section of $'\Gamma_3{}^2$ by a hypersurface of order $2n - 4$; its intersection with $'F^{4n}$ is a double curve $'d^{4n(n-2)}$ on the latter, locus of the double points of $'C^{2n}$, which does not meet $'\omega$; there is thus a further double curve $'K^n$, which is the complete intersection of $'F^{4n}$ with a particular generating plane $'X_2$ of $'\Gamma_3{}^2$. In the latter case $'K^n$ doubled, and branching at those pinch points of $'F^{4n}$ which lie in it, is a curve of the pencil $|'C^{2n}|$, i.e., it is the projection of the canonical curve C^{2n} of genus $n + 1$ in the generator X_n of $\Gamma_{n+1}{}^2$ corresponding to $'X_2$. Since it is birationally equivalent to a double curve however, this C^{2n} is hyperelliptic, and (being the canonical model) is itself a double rational curve K^n, the complete intersection of X_n with F^{4n}. Conversely, of course, if F^{4n} has a double curve K^n in a generator X_n of $\Gamma_{n+1}{}^2$, this projects into a constituent $'K^n$ of the double curve of $'F^{4n}$, which is the complete intersection of $'F^{4n}$ with $'X_2$, the residual constituent being the complete intersection of $'F^{4n}$ with a hypersurface of order $2n - 4$.

We may plausibly conjecture that the existence of the double curve K^n on F^{4n} is a sufficient (as it is clearly a necessary) condition for F^{4n} to be the complete intersection of $\Gamma_{n+1}{}^2$, $G_3{}^{2n}$. To prove this it would be necessary to show that the hypersurface of order $2n - 4$ whose intersection with $'\Gamma_3{}^2$ is $'\Sigma^{4n-8}$ can be chosen so as to have on it a surface $\Delta^{2n(n-2)}$, whose complete intersection with $'\Gamma_3{}^2$ is $'d^{4n(n-2)}$, and further that the hypersurface $'G_3{}^{2n}$ whose intersection with $'\Gamma_3{}^2$ is $'F^{4n}$ can be so chosen as to have $\Delta^{2n(n-2)}$ as double locus. $'G_3{}^{2n}$ would then be the projection of a $G_3{}^{2n}$, projective model of the linear system traced residually on $'G_3{}^{2n}$ by hypersurfaces of order $2n - 3$ through its double surface $\Delta^{2n(n-2)}$. These two results however do not seem easy to prove, and I have not succeeded in establishing the sufficiency of the existence of the double curve for F^{4n} to be the complete intersection of $\Gamma_{n+1}{}^2$, $G_3{}^{2n}$, except for $n \leqslant 5$ (in which case it turns out that the property that $A_1, \ldots, A_n$ are pinch points is itself a sufficient condition.) The methods of proof however are different for the different values of n, and must be postponed until we come to consider the various values of n separately.

Meanwhile however we may remark that a number of the surfaces of genus 2 with canonical pencil of irreducible hyperelliptic curves, which I have studied in a recent paper (**4**), come under the specification of the special surfaces of 1.2, 1.3. For in the first place, a double rational ruled surface $R_2{}^n$, branching along a curve β^{2n+4} of order $2n + 4$ which meets each generator in four points, and which is consequently the residual section of $R_2{}^n$ by a quartic hypersurface through $2n - 4$ generators, is a surface $G_3{}^{2n}$, since the generators and hyperplane sections of $R_2{}^n$ form a base for curves on it, and each of these systems is clearly adjoint to itself on the double surface. Consequently a double rational threefold $R_3{}^n$, branching along a surface B^{2n+4} which is its residual section by a quartic hyper-

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surface through $2n - 4$ of its generating planes, is a G_3^{2n}. If now we consider a surface Φ^{2n}, the intersection of Γ_{n+1}^2 with R_3^n, and choose the branch surface B^{2n+4} so as to touch Φ^{2n} along the curve K^n traced on it by a particular generating $[n]$, X_n, of Γ_{n+1}^2 (e.g., but not necessarily, by letting it break up into the section of R_3^n by the tangent hyperplane Y_{n+1} to Γ_{n+1}^2 along X_n, together with the residual section of R_3^n by a cubic hypersurface through $2n - 4$ generating planes), then K^n is not a proper part of the branch curve of the double Φ^{2n}, since it counts twice in the intersection of B^{2n+4} with Φ^{2n}; and the branch curve of the double Φ^{2n} is its residual section by a cubic hypersurface through $2n - 4$ of the conics traced by the generating planes of R_3^n. Φ^{2n} has of course n double points $A_1, \ldots, A_n$, the intersections of R_3^n with the vertex Ω_{n-1} of Γ_{n+1}^2; and these are isolated branch points on the double Φ^{2n}, since a general curve on Φ^{2n} which passes simply through one of them does not touch B^{2n+4} there but intersects it simply, and therefore has a branch point there, regarded as a curve on the double surface. This double Φ^{2n} is thus precisely what I called the standard case of the bicanonical surface of genera $p = 2$, $p^{(1)} = n + 1$, with hyperelliptic canonical pencil, in the paper referred to.

Again, as the double plane with general sextic branch curve is the general G_2^2, the double Veronese surface whose branch curve is a general cubic section is one type of G_2^8, and one type of G_3^8 will be the double cone V_3^4 (projecting the Veronese surface from a point in [6]) whose branch surface is a general cubic section and having also an isolated branch point at its vertex (this last is necessary in order to ensure that every curve on V_3^4 shall have an even number of branch points, as it must for the double locus to exist at all). If now Φ^8 is the complete intersection of V_3^4, Γ_5^2, and the branch surface is again made to touch Φ^8 along K^4 (e.g., but not necessarily, by breaking up into the section of V_3^4 by Y_5 together with a general quadric section) the double Φ^8 has as branch curve a general complete quadric section, together with isolated branch points at its four double points, and is precisely what I called exceptional case no. xviii in that paper.

The seven other exceptional cases enumerated in the paper for $p = 2$ clearly do not give surfaces Φ^{2n} which are the complete intersection of Γ_{n+1}^2 with a threefold which, doubled and suitably branching, could be regarded as a G_3^{2n}; since any such threefold must have rational $[n]$ sections, and it is familiar that the only n-ic threefolds in $[n + 2]$ having this property are R_3^n and (for $n = 4$) V_3^4. It will remain to be considered, for any of these other exceptional cases, whether there can be a G_3^{2n} whose intersection with Γ_{n+1}^2 consists of the Φ^{2n} in question, counted twice, and if so what ought to be regarded as the branch curve when this situation is arrived at as the limit of a variable simple intersection of order $4n$.

2. The cases $n = 1$ ($p^{(1)} = 2$) and $n = 2$ ($p^{(1)} = 3$). These cases having been studied by Enriques (**7,** pp. 304, 312), comparatively little remains to be said of them; but a few remarks are worth making. There is one type of surface for $n = 1$, namely the double quadric cone Γ_2^2 in [3], branching along a general

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quintic section, and having an isolated branch point at the vertex (**4,** p. 208), and this can be thought of as the intersection of Γ_2^2 with the G_3^2 consisting of the ambient [3] doubled and having as branch surface a sextic surface which touches Γ_2^2 along a generator. For $n = 2$ there are two types of F^8; one is the double Φ^4, intersection of Γ_3^2 with another quadric in [4] (having two double points, and a pencil of conics passing through both of them, traced by the generating planes of Γ_3^2) whose branch curve is a general cubic section, and having also isolated branch points at the two double points; and this is precisely our standard case (**4,** p. 207) of the surface with hyperelliptic canonical pencil. The other is the section of Γ_3^2 by a quartic hypersurface which touches it along a conic in a generating plane, which is of course the surface F^8 given by 1.2. It is worth remarking that 1.7 shows clearly why, in this particular case, there can be no more general F^8, not a complete section of Γ_3^2; for on the one hand as the surface is already in [4] no projection is involved, and F^8, $'F^8$ of 1.7 are the same surface, so that as $'F^{4n}$ is in any case a complete section of $'\Gamma_3^2$, F^8 is a complete section of Γ_3^2; on the other hand the surface $'\Sigma$, of order $4n - 7 = 1$, lying on Γ_3^2, whose intersection with F^8 is the whole double curve of the latter, is just a generating plane of Γ_3^2, and there can be no question of its failing to break up into a plane and a residual surface, complete section of Γ_3^2 by a hypersurface of order $2n - 4 = 0$, since in this case this residual surface is null.

It may also be pointed out here, with all diffidence, that Enriques appears to be wrong when he says (**7,** p. 315) that these two types of surface form distinct families with the same number of moduli; in fact,

THEOREM 2.1 *The regular surfaces of genera $p = 2$, $p^{(1)} = 3$, whose canonical pencil is irreducible and hyperelliptic, are a subfamily of those whose canonical pencil consists of general irreducible curves of genus* 3.

In other words, the general double Φ^4, with branch points at its two double points and branch curve which is a general cubic section, is contained in the family of sections of Γ_3^2 by a quartic touching it along a conic in a generating plane, and can be obtained as the limit of a variable surface of this latter type.

For let $(x_0, \ldots, x_4)$ be a homogeneous coordinate system in [4], so chosen that the equation of Γ_3^2 is

$$x_0 x_2 = x_1^2,$$

and let $\phi_2 = 0$ be any quadric whose intersection with Γ_3^2 is Φ^4, and $f_3 = 0$ any cubic whose intersection with Φ^4 is the branch curve of the double surface. This double Φ^4 can be taken as the section by Γ_3^2 of the double quadric $\phi_2 = 0$, with branch surface consisting of its sections by the hyperplane $x_0 = 0$ and the cubic $f_3 = 0$; since as the former partial branch surface touches Φ^4 along the conic K^2 traced by the plane $x_0 = x_1 = 0$, this conic will contribute nothing to the branching of the double Φ^4 except isolated branch points at the two nodes (as in the example considered at the end of §1.) But the double quadric so branching is the limit for $\lambda \to 0$ of the variable quartic

$$\phi_2^2 + \lambda x_0 f_3 = 0,$$

which for a general value of λ is irreducible and simple, and touches $x_0 = 0$, and hence Γ_3^2, along K^2. Thus this pencil of quartics traces on Γ_3^2 a pencil of surfaces F^8 of genera $p = 2$, $p^{(1)} = 3$, whose general member is non-singular except for the double conic K^2, whereas one surface of the pencil is the double Φ^4 from which we started, which was the most general of its kind. It is clear therefore that the whole family of double Φ^4's is contained as a subfamily in that of the F^8's.

Enriques' arguments to the contrary are twofold. He projects F^8 into an octavic surface $'F^8$ in [3], with two coincident and coplanar fourfold lines, and a double conic, and then says that the projection of Φ^8 into [3] is a quartic surface with a double conic, which cannot (counted twice) be the limit of a variable $'F^8$ unless the double conic reduces to a tacnodal line; but he seems to have forgotten that the projection of F^8 into $'F^8$ was made, not from a general point of [4], but from a general point of Γ_3^2, and that when Φ^4 is similarly projected from a point of Γ_3^2, the double conic of the projected surface does in fact reduce to a tacnodal line. Secondly, Enriques says that each of these families of surfaces has 24 moduli, without stating in either case how this figure is arrived at. For the double Φ^4, I think it is correct, as there are ∞^1 projectively distinct Φ^4's, as can be seen from the plane mapping by cubics with five base points $X_1, \ldots, X_5$, of which X_2 is in the neighbourhood of X_1, and the other three are in a line; the whole figure is projectively determined by the cross ratio of the lines joining X_1 to the other four points; Φ^4 has ∞^{24} cubic sections, of which however only ∞^{23} are projectively distinct, as the surface has ∞^1 projective transformations into itself (in the plane mapping, the pencil of homologies with centre X_1 and axis $X_3X_4X_5$); there are thus ∞^{24} projectively distinct figures consisting of Φ^4 together with a cubic section of itself. On the other hand it seems to me that F^8 has 26 moduli. In the first place there are ∞^6 lines Ω_1, each of which is the vertex of ∞^5 cones Γ_3^2, each of which has ∞^1 generating planes X_2, in each of which are ∞^5 conics K^2. Thus the cone Γ_3^2 and the conic K^2 can be chosen in ∞^{17} ways. The quartic hypersurfaces touching Γ_3^2 along K^2 are ∞^{48}, since every such quartic must trace on Y_3 (the tangent hyperplane to Γ_3^2 over X_2) a quartic surface with K^2 as double conic, of which there are ∞^{13}, as they are birationally equivalent to the quadric sections of a quadric in [4]; while there are ∞^{35} quartics in [4] tracing any given quartic surface on Y_3. On the other hand the quartic hypersurfaces tracing any given F^8 on Γ_3^2 are ∞^{15}; there are thus $\infty^{17+48-15} = \infty^{50}$, or ∞^{26} projectively distinct, surfaces F^8, since there are ∞^{24} projective transformations in [4].

3. The case $n = 3$ $(p^{(1)} = 4)$. For $n > 2$ we have as yet no absolute guarantee that there are any surfaces of the required genera, other than those whose canonical curves are hyperelliptic, though of course the presumption is that there are; the construction of 1.2 requires contact of G_3^{2n} with Γ_{n+1}^2 of a kind whose possibility is not obvious, and we have still no information as to

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whether the more general type of surface envisaged in 1.7 can exist at all. We shall therefore begin by actually constructing some surfaces, of both kinds, for $n = 3$.

THEOREM 3.1 *A G_3^6 can be constructed in* [5] *to touch a* Γ_4^2 *along a rational cubic curve* K^3 *lying in a generating* [3] X_3 *of* Γ_4^2, *so that their intersection* F^{12} *is the bicanonical model of a regular surface of genera* $p = 2$, $p^{(1)} = 4$.

Since every non-hyperelliptic canonical curve of genus 4 is the complete intersection of a quadric and a cubic surface in [3], the general G_2^6 and G_3^6 are the complete intersections of a quadric and a cubic hypersurface in [4], [5] respectively. Since Γ_4^2 has the same tangent hyperplane Y_4 at all points of X_3, for G_3^6 to touch Γ_4^2 along K^3 is the same thing as for it to touch Y_4 along K^3, i.e., for its section by Y_4 to be a surface, virtually G_2^6, that is the intersection of a quadric and a cubic, but having K^3 as double curve, so that its hyperplane sections are not of genus 4 but elliptic, and the surface must be the projection of the sextic del Pezzo surface from some line. (We recall for comparison that in the case $n = 2$ the corresponding surface, section of G_3^4 by Y_3, being a quartic with the double conic K^2, was a projection of the quartic del Pezzo surface.)

We first show therefore that the sextic del Pezzo surface U_2^6 can be projected from a suitably chosen line l to give a surface $'U_2^6$ in [4] which has a double rational cubic curve K^3, forming a complete hyperplane section, and that $'U_3^6$ is the complete intersection of a quadric with a cubic hypersurface.

In the first place, just as the normal elliptic quartic E^4 has four points (the vertices of the four cones in the pencil of quadrics whose intersection it is) from each of which it projects into a double conic, the normal elliptic $2n$-ic curve E^{2n} has n^2 $[n-2]$'s from each of which it projects into a double K^n, image of one of the n^2 quadratic involutions on E^{2n} which have the property that any n pairs of the involution are together a hyperplane section of the curve.

Now let E^6 be a general hyperplane section of U_2^6, and let l be any one of the nine lines from which it projects into a double cubic K^3, image of an involution I^2; the projection of U_2^6 from l is a sextic surface $'U_2^6$ in [4], with the double curve K^3 constituting its whole section by a hyperplane X_3. We shall show that $'U_2^6$ is in fact the intersection of a quadric and a cubic. For U_2^6 has on it two homaloidal nets of rational cubics (represented, when U_2^6 is mapped on a plane by cubics with three base points, by the lines of the plane and the conics through the base points); the cubics of either net that join pairs of I^2 are a pencil with a base point on E^6, say P′, P″ for the two nets; and since the two nets are residual with respect to hyperplane sections of U_2^6, P′ P″ is also a pair of I^2. These two pencils of cubics appear on $'U_2^6$ as pencils of *plane* cubics, with double points on K^3, and base points at a point P of K^3, projection of P′ and P″. Every plane containing a cubic of one system meets every plane containing a cubic of the other in a line (which of course passes through P), since two cubics on U_2^6 one of each net, have two intersections. Thus the two systems of planes containing the plane cubics on $'U_2^6$ are the two systems of generating planes of a

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quadric cone R_3^2, with point vertex P. $'U_2^6$ lies on this quadric, and since it traces a cubic curve on every generating plane, is its complete section by a cubic hypersurface Θ_3^3. (This is of course not determinate, but can be taken to be a general member of the linear system of freedom 5 which all trace the same surface on R_3^2.)

We can now take X_3 to be a generating [3] of Γ_4^2 in [5], and the ambient [4] of $'U_2^6$ to be the tangent hyperplane Y_4 to Γ_4^2 at all points of X_3. If now Q_4^2 is a general quadric whose section by Y_4 is R_3^2, and Θ_4^3 a general cubic whose section by Y_4 is Θ_3^3, the intersection of Q_4^2, Θ_4^3 is a G_3^6, whose section by Y_4 is $'U_2^6$; thus G_3^6 touches Y_4, and hence Γ_4^2, along K^3, so that the surface of intersection F^{12} of G_3^6, Γ_4^2 is precisely as specified in 1.2. Theorem 3.1 is thus proved.

Before investigating what other surfaces may exist for $n = 3$ it is convenient to recall briefly some properties of the threefold loci ${}^{\pi}H_3^m$, of order m, having hyperelliptic curve sections of genus π, and not generated by a hyperelliptic pencil of planes. These were considered in rather general terms by Enriques (**6**) long ago; proofs of any statement made here which may not appear obvious will be found in a recent paper of my own (**5**).

The surface ${}^{\pi}H_2^m$ of order m in $[m - \pi + 1]$ ($\pi \geqslant 2$, $\pi + 2 \leqslant m \leqslant 4\pi + 4$), with hyperelliptic hyperplane sections, and not ruled, was studied by Castelnuovo (**1**); it is rational, being mapped on a plane (in general) by $(\pi + 2)$-ic curves with one π-ple and $4\pi + 4 - m$ simple base points, and has a pencil of conics, represented by the lines through the π-ple base point, which trace the unique g_2^1 on each hyperplane section. Enriques showed that any threefold ${}^{\pi}H_3^m$ whose general hyperplane section is ${}^{\pi}H_2^m$, is rational and has on it a pencil of quadrics $|Q_2^2|$ which trace the unique pencil of conics on each hyperplane section. The ambient [3]'s of these generate a normal rational fourfold $R_4^{m-\pi-1}$, on which ${}^{\pi}H_3^m$ is coresidual to a quadric section, together with $2\pi + 2 - m$ of its generating [3]'s. The projection of ${}^{\pi}H_3^m$ from a general point of itself is a ${}^{\pi}H_3^{m-1}$; not, however, the general ${}^{\pi}H_3^{m-1}$, since one quadric surface of its pencil $|Q_2^2|$ breaks up into a pair of planes, arising respectively from the neighbourhood of the centre of projection and from the Q_2^2 through this point, whereas the general ${}^{\pi}H_3^{m-1}$ has no such reducible Q_2^2. Any base point of the pencil $|Q_2^2|$ is a $(\pi + 1)$-ple point on ${}^{\pi}H_3^m$.

We will now prove:

THEOREM 3.2 *In* [5], *let* Θ_4^3 *be a cubic hypersurface containing a plane* Ω_2, *and having three non-collinear double points* A_1, A_2, A_3 *in this plane. There are three* [3]'*s*, $X_3^{(i)}$ $(i = 1, 2, 3)$ *through* Ω_2 *whose residual intersections with* Θ_4^3 *are quadric surfaces* $Q_2^{2(i)}$, *tracing on* Ω_2 *the three pairs of sides of the triangle* $A_1 A_2 A_3$. *If* Γ_4^2 *is a quadric cone with vertex* Ω_2, *and having* $X_3^{(1)}, X_3^{(2)}, X_3^{(3)}$ *as generators, then the residual intersection* F^{12} *of* Θ_4^3, Γ_4^2, *and another cubic hypersurface through the three quadrics* $Q_2^{2(i)}$, *is the bicanonical model of a regular surface of genera* $p = 2$, $p^{(1)} = 4$; *the base points of the canonical pencil are* A_1, A_2, A_3, *which are simple points on the surface*, Ω_2 *being the tangent plane at each of them.*

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In the first place the quadric surfaces traced residually by [3]'s through a fixed plane on a cubic hypersurface through the plane trace on the plane a net of conics, in projective correspondence with the net of [3]'s through the plane. The necessary and sufficient condition for this net to have a base point at a point A of the plane is that A is a double point of the cubic; thus in the case of the cubic Θ_4^3 the net is that of all conics through A_1, A_2, A_3, three members of which are the pairs of sides of the triangle $A_1A_2A_3$, so that there are, as stated, [3]'s $X_3^{(1)}$, $X_3^{(2)}$, $X_3^{(3)}$, whose quadric residual sections $Q_2^{2(1)}$, $Q_2^{2(2)}$, $Q_2^{2(3)}$, trace these three degenerate concics on Ω_2, and thus touch Ω_2 in A_1, A_2, A_3 respectively. (Any [3] through Ω_2 cuts the quadric cone tangent to Θ_4^3 at A_i in a pair of planes, namely Ω_2 and the tangent plane to the quadric residual section; thus $X_3^{(i)}$ is the unique [3] whose section with the tangent cone at A_i consists of Ω_2 counted twice, namely the base of the pencil of hyperplanes which touch this cone along its pencil of generating lines in Ω_2.) The quadric residual sections $|Q_2^2|$ by the generating [3]'s of Γ_4^2 thus trace on Ω_2 a quadratic family of conics of which the three pairs of sides of the triangle are members, and whose envelope is accordingly a quartic with cusps at A_1, A_2, A_3. The intersection of Θ_4^3, Γ_4^2 is a special type of ${}^3H_3^6$, on which Ω_2 is double (generated by the quadratic family of conics, and hence having the three cusped quartic as locus of pinch points); for whereas the general hyperelliptic sextic curve of genus 3 in [3] is the residual section of a quadric surface by a quartic through two generators of one system, when the quadric is a cone this curve reduces to the complete section by a cubic through the vertex: so that though the general ${}^3H_3^6$ is the residual section of the general R_4^2, a cone with line vertex and two systems of generating [3]'s, by a quartic through two [3]'s of the same system, when R_4^2 becomes Γ_4^2, ${}^3H_3^6$ becomes its section by a cubic through Ω_2. The points A_i are quadruple points of ${}^3H_3^6$, being double on each of the intersecting hypersurfaces; the tangent cone to ${}^3H_3^6$ at A_i is in fact a (non-normal) R_3^4, generated by the tangent planes to the pencil of quadrics $|Q_2^2|$ at A_i; each of these planes meets Ω_2 in a line (the tangent to the conic traced by Q_2^2 on Ω_2), and Ω_2 is itself one of the family, being the tangent plane at A_i to $Q_2^{(i)}$; thus the cone is that projecting a normal rational R_2^4 in [5], with directrix line, from a point coplanar with the directrix line and a generator.

Now consider the surface F^{12}, residual section of ${}^3H_3^6$ by a general cubic hypersurface through $Q_2^{2(1)}$, $Q_2^{2(2)}$, $Q_2^{2(3)}$. F^{12} has no curve of intersection with Ω_2, since the three quadric surfaces meet this plane altogether in the lines A_2A_3, A_3A_1, A_1A_2, each twice, which accounts for the whole intersection of the secant cubic with the double Ω_2. Moreover, since the secant cubic meets Ω_2 in these three lines, it touches the plane in A_1, A_2, A_3; it is clear in fact that F^{12} touches Ω_2 in these three points; for the tangent planes to the three quadric surfaces at A_i are Ω_2 (tangent to $Q_2^{2(i)}$) and two other generating planes of the tangent cone R_3^4; these are joined by a hyperplane, which is necessarily the tangent hyperplane to the secant cubic, and whose residual intersection with R_3^4 is Ω_2 counted a second time; Ω_2 is thus the tangent plane at A_i to the residual intersection F^{12}. On each quadric of the pencil $|Q_2^2|$ on ${}^3H_3^6$, F^{12} traces a sextic curve C^6 of genus 4, complete section of Q_2^2 by the secant cubic, since $Q_2^{2(i)}$ has no curve of intersection with the general Q_2^2; and this C^6 clearly touches Ω_2 in A_1, A_2, A_3, both because the secant cubic does so, and because F^{12} does so. The hyperplane sections $|f^{12}|$ of F^{12} are the double of the pencil $|C^6|$, i.e., any two curves of $|C^6|$ are a hyperplane section of F^{12}, and every hyperplane through Ω_2 meets F^{12} in two curves of $|C^6|$, since it meets Γ_4^2 in two generating [3]'s, and ${}^3H_3^6$ in two surfaces of the pencil $|Q_2^2|$.

We shall now show that the canonical series on the general f^{12} is traced on it residually by quadrics in its ambient [4], through its intersections with any one curve of the pencil $|C^6|$, i.e., with any generating [3] of Γ_4^2. For this purpose we consider the corresponding hyperplane section ${}^3H_2^6$ of ${}^3H_3^6$. The general ${}^3H_2^6$ is mapped on a plane by quintics with a triple base point X and ten simple base points $Y_1, \ldots, Y_{10}$, the lines through X representing the conics on the surface; since in the present case ${}^3H_2^6$ has a double line (the section of Ω_2) which with any two conics forms a hyperplane section, this is represented by a cubic curve on which all the base points lie. f^{12}, being the residual section of the surface by a cubic through three particular conics of the pencil, is mapped by a curve of order 12 with a sextuple point at X and triple points at $Y_1, \ldots, Y_{10}$; and on this the canonical series is traced by curves of order 9 with a quintuple base point at X and double base points at $Y_1, \ldots, Y_{10}$, which are just what represent the residual scetions of ${}^3H_2^6$ by quadrics through any one of its conics.

Thus the adjoint system to the hyperplane sections $|f^{12}|$ of F^{12} is traced on F^{12} residually by quadrics through any one Q_2^2 of the pencil on ${}^3H_3^6$, i.e., through any one C^6 of the pencil on F^{12}, and is accordingly the system $|2f^{12} - C^6| = |f^{12} + C^6|$, which means that $|C^6|$ is the canonical system on F^{12}. Theorem 3.2 is thus established.

THEOREM 3.3 *There is a type of* ${}^2H_3^5$ *in* [5], *residual intersection of* Γ_4^2 *with a cubic hypersurface through one of its generating* [3]'*s, with the following features*: *the pencil of quadrics* $|Q_2^2|$ *on* ${}^2H_3^5$ *trace on* Ω_2 *a pencil of conics with three base points* A_1, A_2, A_3, *and with a fixed tangent* k *in* A_3; *and one quadric of the pencil* $|Q_2^2|$ *breaks up into a pair of planes* κ, λ, *meeting* Ω_2 *in the lines* k, A_1A_2 *respectively. There are also cubic hypersurfaces* Θ_4^3, *containing the plane* λ, *containing also that quadric* Q_2^2 *of the pencil which traces on* Ω_2 *the pair of lines* A_2A_3, A_3A_1, *and further touching* ${}^2H_3^5$ *along a line* s *lying in the plane* κ *and passing through* A_3. *If* ${}^2H_3^5$, Θ_4^3 *satisfy these conditions, their residual intersection* F^{12} *is the bicanonical model of a surface of genera* $p = 2$, $p^{(1)} = 4$, *on which the canonical pencil is traced by the pencil* $|Q_2^2|$, *and has the base points* A_1, A_2, A_3, *of which the two former are simple points on the surface whereas* A_3 *is a pinch point.*

In the first place we will satisfy ourselves that a ${}^2H_3^5$ exists with the desired peculiarities. The residual section of Γ_4^2 by a general cubic through one generat-

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ing [3] is a ${}^2H_3{}^5$, its pencil of quadric surfaces $|Q_2{}^2|$ being the sections of the cubic by the generating [3]'s of $\Gamma_4{}^2$, residual to Ω_2, and these trace on Ω_2 a pencil of conics, since Ω_2 is simple on ${}^2H_3{}^5$, and hence one $Q_2{}^2$ passes through a general point of it. The base points of this pencil of conics are double on the secant cubic and triple on ${}^2H_3{}^5$, the tangent cone at each being an $R_3{}^3$ (cone projecting a ruled cubic surface from a point), intersection of $\Gamma_4{}^2$ with the tangent cone to the secant cubic, residual to the common [3]; it is generated by the tangent planes to the quadrics $|Q_2{}^2|$, and has Ω_2 as directrix plane, on which the generating planes trace a pencil of lines.

Now let $R_4{}^3$ in [6] be a cone with line vertex l, and a directrix [3] on which its generating [3]'s trace the pencil of planes through l; the section of this by a general quadric is a ${}^2H_3{}^6$, with two triple points A_1, A_2, the intersections of l with the secant quadric; these are base points of the pencil $|Q_2{}^2|$ on ${}^2H_3{}^6$; there is also a quadric surface $\bar{Q}_2{}^2$ traced by the secant quadric on the directrix [3], and on this the pencil $|Q_2{}^2|$ trace the pencil of plane sections with base points A_1, A_2. If we now project ${}^2H_3{}^6$ into [5] from a point K of $\bar{Q}_2{}^2$, $\bar{Q}_2{}^2$ is projected into a plane Ω_2, $R_4{}^3$ into $\Gamma_4{}^2$ with Ω_2 as vertex, and of course ${}^2H_3{}^6$ into a ${}^2H_3{}^5$ lying on $\Gamma_4{}^2$. The conics traced by the projected pencil $|Q_2{}^2|$ on Ω_2 are the projections of those traced by the original pencil $|Q_2{}^2|$ on $\bar{Q}_2{}^2$, so that they form a pencil on conics in Ω_2, whose base points are the projections of A_1, A_2, together with the points A_3, A_4 arising from the two generators of $\bar{Q}_2{}^2$ through K. The neighbourhood of K on ${}^2H_3{}^6$ gives rise to a plane κ on ${}^2H_3{}^5$, passing through A_3, A_4, and the quadric $Q_2{}^2$ through K to a plane λ passing through A_1, A_2. We can thus get a ${}^2H_3{}^5$ answering our requirements by letting the secant quadric in [6] either touch the directrix [3] of $R_4{}^3$, so that $\bar{Q}_2{}^2$ is a cone and A_3, A_4 coincide, or touch l so that A_1, A_2 coincide. The notation we have used supposes the former, but the ${}^2H_3{}^5$'s obtained by these two specializations of the construction are in fact identical.

We remark also that at the point A_3, where the conics in Ω_2 have the common tangent k, the (cubic) tangent cone to ${}^2H_3{}^5$ has not a point vertex but the line vertex k, and any three of its generating planes form a hyperplane section. Ω_2 is one of these, and κ is another. We may call this cone $\Gamma_3{}^3$.

Consider now an arbitrary line s in κ, passing through A_3. Let X_3 be the generating [3] of $\Gamma_4{}^2$ which contains κ, λ, and Y_4 the tangent hyperplane to $\Gamma_4{}^2$ at all points of X_3. The tangent [3]'s to ${}^2H_3{}^5$ at points of s all contain κ, and all lie in Y_4, and thus form a pencil, each member of which touches ${}^2H_3{}^5$ in just one point, since the secant cubic has a double point in s, namely A_3, and thus no plane or [3] through s is bitangent to it. In this one-one (and hence projective) correspondence between the pencil of [3]'s through κ in Y_4 and their points of contact on s, the tangent [3] to $\Gamma_3{}^3$ at all points of κ corresponds to A_3, and X_3 to the point of intersection of s with λ, since X_3 clearly touches ${}^2H_3{}^5$ at all points of the line of intersection of κ with λ.

Now if a cubic hypersurface in [5] contains a line, its tangent hyperplanes at points of the line form a cone $\Gamma'_4{}^2$, since a general plane through the line touches the cubic in two points; thus there is a unique plane through the line, the vertex of $\Gamma'_4{}^2$, which touches the cubic at all points of the line, i.e., whose intersection with the cubic consists of this line counted twice and some other line; and the [3]'s traced by the tangent hyperplanes in question on any one of them are just the pencil of [3]'s in this [4], passing through the vertex plane of $\Gamma'_4{}^2$, and in projective correspondence with their points of contact on the line. If therefore we make a cubic hypersurface $\Theta_4{}^3$ contain the plane λ and that particular quadric $Q'_2{}^2$ of the pencil $|Q_2{}^2|$ which traces the line pair A_3A_1, A_2A_3 on Ω_2, and make it also touch Y_4 in A_3 and κ at all points of s, so that its intersection with κ consists of s counted twice together with the line of intersection of κ with λ, the tangent hyperplanes to $\Theta_4{}^3$ at points of s will trace on Y_4 the pencil of [3]'s through κ, in projective correspondence with their points of contact on s; that at the point of intersection of s with λ traces X_3; so that it is only necessary to make $\Theta_4{}^3$ touch in two further points of s the tangent [3]'s to ${}^2H_3{}^5$, to ensure that it shall do so in all points of s, so that s is a double line on the intersection of $\Theta_4{}^3$ with ${}^2H_3{}^5$. All this imposes only 31 linear conditions, while the freedom of cubics in [5] is 55, so that there is an ample supply of cubics satisfying all the conditions.

The intersection of $\Theta_4{}^3$, ${}^2H_3{}^5$, residual to λ and $Q'_2{}^2$, is a surface F^{12} of order 12. At A_1, A_2 it has simple points with Ω_2 as tangent plane, since the tangent hyperplane to $\Theta_4{}^3$ at either of these contains two generating planes of the tangent cone to ${}^2H_3{}^5$ (namely λ and the tangent plane to $Q'_2{}^2$) and thus meets this cone further in its directrix plane Ω_2, which is accordingly the tangent plane to the residual intersection F^{12}. At A_3 on the other hand the tangent hyperplane to $\Theta_4{}^3$ is Y_4, which meets the tangent cone $\Gamma_3{}^3$ to ${}^2H_3{}^5$ in the planes κ (doubly) and Ω_2 (simply), the latter being the tangent plane to $Q'_2{}^2$. F^{12} has thus a double point at A_3 (as of course it must, since s is a double line) with κ counted twice as tangent cone; the natural assumption from this is that A_3 is a pinch point, which will become certain when it appears in the sequel that A_3 is an improper singularity. $\Theta_4{}^3$ traces on Ω_2 the three lines A_2A_3, A_3A_1, A_1A_2 (the first two being on $Q'_2{}^2$, the third on λ), and thus touches Ω_2 in A_1, A_2, A_3; it thus traces on the general surface $Q_2{}^2$ a sextic curve C^6 of genus 4, touching Ω_2 in A_1, A_2, A_3, and this curve belongs wholly to F^{12}, since neither λ nor $Q'_2{}^2$ has any curve of intersection with the general $Q_2{}^2$. On the other hand, λ and $Q'_2{}^2$ account for the whole intersection of $\Theta_4{}^3$ with Ω_2, so that F^{12} meets Ω_2 only in the three points A_1, A_2, A_3.

The general hyperplane section f^{12} of F^{12} can best be studied by means of the plane mapping of the corresponding hyperplane section ${}^2H_2{}^5$ of ${}^2H_3{}^5$. This is by quartics with a double base point X (lines through which represent the conics on the surface, sections of the pencil $|Q_2{}^2|$) and seven simple base points $Y_1, \ldots, Y_7$. The fact that the $R_3{}^2$ generated by the planes of the conics is not the general $R_3{}^2$ (cone with point vertex) but a $\Gamma_3{}^2$ with line vertex ω, section of Ω_2, means that there is a line ω on the surface, bisecant to the conics, which together with any two conics makes up a hyperplane section; and this in turn requires that

$Y_1, \ldots, Y_7$ all lie on a conic, the image of ω. Seven of the conics break up into line pairs, represented by XY_i and the neighbourhood of Y_i; we can suppose that the neighbourhood of Y_7 represents the section of the plane κ, and the line XY_7 that of λ. f^{12}, being the residual section of ${}^2H_2{}^5$ by a cubic through this last line and also through the conic, section of $Q'_2{}^2$, is mapped by a curve of order 10, with a quadruple point at X, triple points at $Y_1, \ldots, Y_6$, and a double point at Y_7; it has also a double point Z in the neighbourhood of Y_7, corresponding to the actual double point of f^{12} at its intersection with s. The canonical series is traced on this curve by septimics with a triple base point at X, double base points at $Y_1, \ldots, Y_6$, and simple base points at Y_7, Z, amongst which are those which have a double point at Y_7 and do not pass through Z; these last clearly represent residual sections of ${}^2H_2{}^5$ by quadrics through one of its pencil of conics; thus the canonical series on f^{12} is traced residually by quadrics through its intersections with any one surface $Q_2{}^2$, and the adjoint system to the hyperplane sections $|f^{12}|$ is traced residually on F^{12} by quadrics through any one curve C^6, i.e., it is the system $|2f^{12} - C^6| = |f^{12} + C^6|$ which means that the pencil $|C^6|$ is the canonical system. Theorem 3.3 is thus established; it remains only to note that the general f^{12} is of genus 10, which is what the genus of $|2C^6|$ would be if the three base points of $|C^6|$ were all simple points of the surface; A_3 is thus an improper singularity, and as it cannot be the intersection of two simple sheets (since the pencil $|C^6|$ which passes simply through A_3, and of which any two curves are a hyperplane section, is irreducible) it can only be a pinch point.

Before showing that the surfaces constructed in these three theorems include the bicanonical models of all surfaces of genera $p = 2$, $p^{(1)} = 4$, with irreducible and non-hyperelliptic canonical pencil, it is convenient to digress, and devote some study to a surface which exhibits many of the features which we should expect of our bicanonical surface F^{12}, but which proves nevertheless to be of genus 1, and not 2.

THEOREM 3.4 *There exist surfaces* $\bar{F}^{12}$ *in* [5], *lying on* $\Gamma_4{}^2$, *and having a pencil* $|C^6|$ *of canonical curves of genus* 4 *traced by the generating* [3]'*s of* $\Gamma_4{}^2$, *with three base points* A_1, A_2, A_3, *simple points of the surface, in each of which the tangent plane to the surface is the vertex* Ω_2 *of* $\Gamma_4{}^2$, *so that the base points are a semicanonical set on each curve of the pencil; but on which nevertheless the canonical system is not the pencil* $|C^6|$, *but contains only one curve, which breaks up into three conics, each forming part of a different curve of the pencil* $|C^6|$ *and each passing through one of the base points* A_1, A_2, A_3; *these three conics are exceptional curves, and the surface belongs tothe familiar series of surfaces with all genera equal to unity, the unique curve of whose reduced canonical system is the null curve, so that every linear system on the surface is adjoint to itself* (**7**, p. 247).

First we must show that there exists a type of ${}^2H_3{}^5$, special in two respects: the $R_2{}^4$ generated by the ambient [3]'s of its $Q_2{}^2$'s is $\Gamma_4{}^2$ (with plane instead of line vertex), and three of these $Q_2{}^2$'s break up into pairs of planes. The latter peculiarity is evidently ensured by obtaining ${}^2H_3{}^5$ as the projection of ${}^2H_3{}^8$ in

[8] (residual section of R_4^5 by a quadric through two of its generating [3]'s from three points K_1, K_2, K_3 of itself); the former can be ensured by choosing K_1, K_2, K_3 suitably. For the general R_4^5 is generated by the [3]'s joining corresponding points of three lines and a conic, projectively related; it thus has on it an R_3^3, generated by the planes joining corresponding points of the three lines, which is its residual section by a hyperplane through two generating [3]'s, and which of course meets each generating [3] in a plane. The quadric which cuts R_4^5 in two [3]'s and ${}^2H_3^8$, cuts R_3^3 in two planes and a ruled quartic surface R_2^4, which is accordingly the intersection of ${}^2H_3^8$ and R_3^3 on R_4^5. The conics on R_2^4 are traced by the planes of R_3^3, and are the sections by these planes of the Q_2^2's in the corresponding [3]'s of R_4^5. If now K_1, K_2, K_3 are not general points of ${}^2H_3^8$, but are on R_2^4, it is clear that the directrix lines a_1, a_2, a_3 of R_3^3 through K_1, K_2, K_3 respectively are projected into points A_1, A_2, A_3, that R_2^4, as well as every generating plane of R_3^3, is projected into the plane $\Omega_2 = A_1 A_2 A_3$, and that the generating [3]'s of R_4^5 are projected into [3]'s which all pass through Ω_2, so that the projection of R_4^5 is not the general R_4^2 with line vertex, but a Γ_4^2 with vertex Ω_2. The conics on R_2^4 are projected into a pencil of conics in Ω_2, which are the traces on it of the pencil of quadrics $|Q_2^2|$ on ${}^2H_3^5$, the projections of those on ${}^2H_3^8$; this pencil of conics has as base points A_1, A_2, A_3, and a fourth point B, the projection of the unique rational cubic curve b on R_2^4 through A_1, A_2, A_3. These four points are triple points of ${}^2H_3^5$, the tangent cone at each being generated by the tangent planes to the quadric surfaces $|Q_2^2|$, and having Ω_2 as directrix plane. The three plane pairs in the pencil $|Q_2^2|$ evidently trace on Ω_2 the three line pairs in the pencil of conics; if we denote by κ_i, λ_i respectively the planes arising from the neighbourhood of K_i and from the Q_2^2 through K_i on ${}^2H_3^8$, we see that as a_i and b both pass through K_i, A_i and B both lie in κ_i, while as the Q_2^2 through K_i meets a_j, a_k, in general points, λ_i passes through A_j, A_k. Thus κ_i, λ_i trace on Ω_2 the lines $A_i B$, $A_j A_k$ respectively.

Now consider the surface $\bar{F}^{12}$, residual section of this ${}^2H_3^5$ by a general cubic hypersurface through the planes $\lambda_1, \lambda_2, \lambda_3$. It is of order 12, and meets Ω_2 in no curve, since the whole intersection of the secant cubic with Ω_2 consists of the three lines $A_2 A_3$, $A_3 A_1$, $A_1 A_2$, traced by $\lambda_1, \lambda_2, \lambda_3$; $\bar{F}^{12}$ passes simply through A_1, A_2, A_3, its tangent plane in each of these being Ω_2, since the tangent hyperplane to the secant cubic at A_i contains the planes λ_j, λ_k, which are generating planes of the tangent cone to ${}^2H_3^5$, and thus cuts this cone residually in Ω_2, which is accordingly the tangent plane there to the residual intersection $\bar{F}^{12}$. Since $\lambda_1, \lambda_2, \lambda_3$ trace no curve on the general Q_2^2, $\bar{F}^{12}$ traces on each Q_2^2 its complete section by the secant cubic, a canonical curve C^6 of genus 4, touching Ω_2 in A_1, A_2, A_3. $\bar{F}^{12}$ traces on each plane κ_i a conic s_i, since κ_i meets the secant cubic in a cubic curve of which the intersection of κ_i, λ_i is part. The curve t_i traced by $\bar{F}^{12}$ on λ_i is accordingly a quartic, since s_i, t_i together form a curve of the pencil $|C^6|$ on $\bar{F}^{12}$; and it is clear that s_i touches Ω_2 in A_i, and t_i touches it in A_j, A_k.

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All the properties of $\bar{F}^{12}$ so far·found are consistent with (and indeed strongly suggest) the idea that it is the bicanonical model of a surface of genera $p = 2$, $p^{(1)} = 4$, $|C^6|$ being its canonical pencil. We shall show however that $\bar{F}^{12}$ is on the contrary of genus 1, its unique canonical curve consisting of the three conics s_1, s_2, s_3.

To prove this we shall consider as before the general hyperplane section $\bar{f}^{12}$ of $\bar{F}^{12}$ as a curve on the corresponding section ${}^2H_2{}^5$ of ${}^2H_3{}^5$. ${}^2H_2{}^5$ as we have seen is rational, being mapped on a plane by quartics with a double base point X and seven simple base points $Y_1, \ldots, Y_7$, the latter all lying on a conic. Seven conics of the pencil break up into pairs of lines; we may suppose that amongst these, the lines represented by the neighbourhoods of Y_5, Y_6. Y_7 are the sections of $\kappa_1, \kappa_2, \kappa_3$, and those represented by XY_5, XY_6, XY_7 are the sections of $\lambda_1, \lambda_2, \lambda_3$. Thus $\bar{f}^{12}$, being the residual section of ${}^2H_2{}^5$ by a cubic through these last three lines, is represented by a curve of order 9 with triple points at $X, Y_1, \ldots, Y_4$, and double at Y_5, Y_6, Y_7; on this curve the canonical series is traced residually by sextics with double base points at $X, Y_1, \ldots, Y_4$, and simple at Y_5, Y_6, Y_7; amongst which are those that break up into the conic through $Y_1, \ldots, Y_7$ (which has no residual intersection with the curve) and quartics with a double base point at X and simple base points at $Y_1, \ldots, Y_4$; and in this latter system, those which pass also through Y_5, Y_6, Y_7 trace on the curve sets corresponding to hyperplane sections of $\bar{f}^{12}$, together of course with the pairs of points coinciding in Y_5, Y_6, Y_7, which represent the pairs of points traced on $\bar{f}^{12}$ by s_1, s_2, s_3. Thus canonical sets are traced on $\bar{f}^{12}$ by all the reducible curves on $\bar{F}^{12}$ consisting of another hyperplane section together with the three conics s_1, s_2, s_3; which means that these three conics are together a curve of the canonical system. That there is no other curve of this system is fairly obvious, and becomes certain when we remark that the three conics are of negative grade, and are in fact exceptional curves. That the grade of s_i is -1 follows from the fact that as s_i, t_i are together a curve of the pencil $|C^6|$, the virtual intersections of s_i with itself are equivalent to its intersections with a general C^6, minus its intersections with t_i, which are two in number, namely its intersections with the line traced by λ_i on its ambient plane κ_i. Theorem 3.4 is thus established.

It is now comparatively easy to prove:

THEOREM 3.5 *Every regular surface on which is a pencil $|C|$ of irreducible non-hyperelliptic curves of genus* 4, *with three base points which are a semicanonical set on the general curve of the pencil, has as projective model of the system $|2C|$ one of the four surfaces constructed in Theorems* 3.1, 3.2, 3.3, 3.4.

In the first place, as in Theorem 1.1, the projective model of $|2C|$ is a surface F^{12} lying on $\Gamma_4{}^2$ in [5]; the curves $|C|$ are canonical sextics C^6 on this model, and are traced by the generating [3]'s of $\Gamma_4{}^2$, whose vertex Ω_2 they all touch in the base points A_1, A_2, A_3 of the pencil. Also, as in Theorem 1.4, these base points are simple points or pinch points of F^{12} according as the variable curve

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of the pencil has variable or fixed tangent there. The general curve of $|2C|$ is of genus 10.

The general C^6 lies on a unique quadric surface Q_2^2 in its ambient [3], which traces on Ω_2 a conic passing through A_1, A_2, A_3 and touching there the tangents to C^6. If m of these conics pass through a general point of Ω_2, the threefold locus generated by the surfaces Q_2^2 is of order $4 + m$, having Ω_2 as m-ple locus, and is (for even values of m) the section of Γ_4^2 by a hypersurface of order $2 + \frac{1}{2}m$, passing $\frac{1}{2}m$-ply through Ω_2, or (for odd values) the residual section by a hypersurface of order $2 + \frac{1}{2}(m + 1)$ passing simply through a generating [3] and $\frac{1}{2}(m + 1)$-ply through Ω_2. We shall show that $m \leqslant 2$. For if the tangents at at least two of the base points to the curves $|C^6|$ and the conics in Ω_2 are fixed, the conics all coincide, and $m = 0$; if the tangents are variable at at least two base points, they correspond projectively, so that the conics are transformed by a quadratic transformation with base points at $A_1 A_2 A_3$ into a family of lines which trace projective ranges of points on at least two sides of the triangle; such a family of lines is either a pencil, or the tangents to a conic which touches the sides of the triangle, and the family of conics is thus either a pencil (giving $m = 1$) or a quadratic family enveloping a quartic with cusps at A_1, A_2, A_3 ($m = 2$).

If $m = 0$, the locus of the quadrics Q_2^2 is a quadric section S_3^4 of Γ_4^2, and since F^{12} traces a C^6 of genus 4 on each Q_2^2, the virtual difference on S_3^4 between F^{12} and a cubic section must be of order zero, and have no intersection with the general Q_2^2, i.e., must be null; thus F^{12} is a cubic section of S_3^4, i.e., it is the complete intersection of Γ_4^2, another quadric, and a cubic, i.e., of Γ_4^2 with the G_3^6 which is the intersection of the second quadric with the cubic. Thus the surface is that constructed in Theorem 3.1.

In the case $m = 2$ on the other hand, when the conics in Ω_2 envelope a tricuspidal quartic, the locus of the quadrics Q_2^2 is the section of Γ_4^2 by a cubic through Ω_2, i.e., it is a ${}^3H_3^6$, and is precisely the ${}^3H_3^6$ considered in Theorem 3.2. On this the virtual difference between F^{12} and a cubic section must be of order 6, and must as before have no intersection with the general Q_2^2, i.e., F^{12} is the residual section of ${}^3H_3^6$ by a cubic through some surface of order 6, which consists of quadrics Q_2^2 or planes forming parts of these. Moreover, since each C^6 touches Ω_2 in A_1, A_2, A_3, the secant cubic must touch Ω_2 in these points, i.e., must cut it in the lines $A_2 A_3$, $A_3 A_1$, $A_1 A_2$, and its total intersection with the double Ω_2 on ${}^3H_3^6$ consists of these three lines each counted twice; and this must be accounted for entirely by the trace on Ω_2 of the sextic surface of intersection residual to F^{12}, since F^{12} has no curve of intersection with Ω_2. But the three pairs of these three lines are the traces on Ω_2 of particular quadrics of the pencil $|Q_2^2|$; these three quadrics thus form the residual intersection, i.e., F^{12} is the residual section of ${}^3H_3^6$ by a cubic through these three quadrics, and is thus the surface constructed in Theorem 3.2.

If on the other hand the conics traced by $|Q_2^2|$ on Ω_2 are a pencil, they have either a fourth base point B, distinct from A_1, A_2, A_3, or a fixed tangent k in one of these, say A_3. In either case $m = 1$, and the locus of the quadrics $|Q_2^2|$ is a ${}^2H_3^5$, residual section of Γ_4^2 by a cubic through one generating [3]. F^{12} must differ from a cubic section by some residual surface of total order 3, which as before must consist entirely of quadrics Q_2^2 or planes forming part of these; as before, the secant cubic must cut Ω_2 in the three lines $A_2 A_3$, $A_3 A_1$, $A_1 A_2$, and these must be entirely accounted for as the trace on Ω_2 of the residual intersection of order 3.

If the conics in Ω_2 have a fixed tangent k in A_3, two of them are line pairs, namely $A_2 A_3$, $A_3 A_1$ and $A_1 A_2$, k; thus the Q_2^2 tracing the former line pair must be part of the residual surface, and that tracing the latter must break up into two planes κ, λ, tracing k, $A_1 A_2$ respectively, of which λ must be the other part of the residual surface. A general F^{12} however, residual to this quadric and plane, would trace a conic on κ, which (by an analysis of the properties of the hyperplane section of the surface, on that of ${}^2H_3^5$, precisely similar to that in Theorem 3.3, merely omitting the double point Z in the neighbourhood of Y_7) can be seen to be a fixed part of the canonical system, the variable part being the pencil $|C^6|$; thus though the surface is of genus 2, it is not the bicanonical model, nor is the canonical pencil irreducible; moreover, the general curve of $|f^{12}| = |2C^6|$ is of genus 11, instead of 10, showing that A_3 is a proper singularity. The only way to make this singularity improper, and the general hyperplane section of genus 10, is to give the surface a double line passing through A_3; and this double line must lie in κ, replacing the conic which would otherwise be traced on the surface by κ, since it is precisely the pair of points traced by this conic on the general hyperplane section which we need to subtract from the canonical series on the latter. Thus we have the surface constructed in Theorem 3.3.

Finally, if the pencil of conics in Ω_2 has a fourth base point distinct from A_1, A_2, A_3, the lines $A_2 A_3$, $A_3 A_1$, $A_1 A_2$ form parts of three distinct conics of the pencil, so that the residual surface must consist of three planes, meeting Ω_2 in these three lines respectively, and forming parts of three distinct quadrics of the pencil $|Q_2^2|$. Thus we have the surface $\bar{F}^{12}$ of Theorem 3.4.

From this result an obvious corollary is

THEOREM 3.6 *Every regular surface of genera $p = 2$, $p^{(1)} = 4$, has as its bicanonical model one of the following four surfaces:*

I. *That constructed in Theorem* 3.2, *without singularities*;

II. *That constructed in Theorem* 3.3, *with a double line, passing through one of the base points of the canonical pencil, which is a pinch point, the other two being simple*;

III. *That constructed in Theorem* 3.1, *with a double rational cubic curve passing through all three base points of the canonical pencil, all of which are pinch points*;

IV. *That given in my paper referred to above* (**4**), *on which the general curve of the canonical pencil is hyperelliptic*; *the surface is the double ${}^2H_2^6$, complete intersection of Γ_4^2 with a general R_3^3, with branch curve of order* 14, *residual section of*

${}^2H_2{}^6$ by a cubic hypersurface through two of its pencil of conics, and having also isolated branch points at its three nodes, the intersections of Ω_2 with $R_3{}^3$, which are the base points of the canonical pencil.

The point of view adopted in Theorem 1.8 makes it natural to regard type I as the general case, of which all the others are specializations. Whether, however, every surface of types II, III, IV can be obtained as the limit of a variable surface of type I seems to be quite a difficult problem, to which we shall not attempt a definite answer. In this connexion, however, two simple and obvious remarks present themselves:

(i) Though type III seems to be in some sense more special than type II, the general surface of type III cannot be the limit of a variable surface of type II, as if this assumes such a limiting form the double cubic curve of the latter will break up into a line (limit of the double line on the variable surface) and a conic.

(ii) Similarly, though type IV is in fact, as was remarked at the close of §1, the complete intersection of $\Gamma_4{}^2$ with a $G_3{}^6$, in this case the double $R_3{}^3$, with branch surface of order 10 which is its residual section by a quartic hypersurface through two generating planes, the general surface of type IV is not the limit of a variable surface of type III, since the general $R_3{}^3$ is not, counted twice, the complete intersection of a quadric and a cubic. Every quadric hypersurface containing the general $R_3{}^3$ is in fact an $R_4{}^2$ with line vertex (which is any one of the ∞^2 directrix lines of $R_3{}^3$); and this has on it two distinct systems of $R_3{}^3$'s, whose generating planes lie respectively in the two systems of generating [3]'s of $R_4{}^2$; and a cubic hypersurface through an $R_3{}^3$ of one system cuts $R_4{}^2$ residually in one of the other system. $R_3{}^3$ can only be the complete intersection of a quadric and a cubic if it is a cone with (in general) a point vertex and a directrix plane; the quadric is then a cone with plane vertex, which is the directrix plane of $R_3{}^3$.

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