On the relations between systems of curves which, together, cut their plane into squares.

By Professor TAIT.

If ρ be the vector of a corner of a square in one system, σ that in a system derived without inversion, we must obviously have

$$d\sigma = u\left(\cos\frac{\phi}{2} + k\sin\frac{\phi}{2}\right)d\rho\left(\cos\frac{\phi}{2} - k\sin\frac{\phi}{2}\right),$$
$$= u\{(i\cos\phi + j\sin\phi)dx - (i\sin\phi - j\cos\phi)dy\}, \quad \ldots \quad (1)$$

k being the unit-vector perpendicular to the common plane.

This requires that

$$\frac{d}{dy}\left\{u(i\cos\phi + j\sin\phi)\right\} = \frac{d}{dx}\left\{u(-i\sin\phi + j\cos\phi)\right\},$$

which gives the two equations

$$\frac{du}{dy}\cos\phi + \frac{du}{dx}\sin\phi = u\left(\sin\phi\frac{d\phi}{dy} - \cos\phi\frac{d\phi}{dx}\right),$$
$$\frac{du}{dy}\sin\phi - \frac{du}{dx}\cos\phi = u\left(-\cos\phi\frac{d\phi}{dy} - \sin\phi\frac{d\phi}{dx}\right),$$

or, in a simpler form,

$$\left.\begin{array}{l} \dfrac{1}{u}\dfrac{du}{dx} = \dfrac{d\phi}{dy}, \\ \dfrac{1}{u}\dfrac{du}{dy} = -\dfrac{d\phi}{dx}. \end{array}\right\} \quad \ldots \quad \ldots \quad (2).$$

Eliminating ϕ and u separately, we have

$$\frac{d^2\log u}{dx^2} + \frac{d^2\log u}{dy^2} = 0,$$
$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0.$$

Thus

$$\left.\begin{array}{r} \log u = C_1 \\ \phi = C_2 \end{array}\right\} \quad \ldots \quad \ldots \quad (3)$$

represent associated series of equipotential, and current, lines in two dimensions.

Assuming any lawful values for the members of (2) we obtain u and ϕ, and thence, by integration of (1), σ is given in terms of ρ.

Thus $\sigma = i\xi + j\eta$,

where ξ and η are known functions of x and y. From this x and y can be found in terms of ξ, η. Thus if

$$F_1(x, y) = A_1, \; F_2(x, y) = A_2 \quad \ldots \quad \ldots \quad (4)$$

be a pair of sets of curves possessing the required property, we

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obtain at once another pair by substituting for x and y their values in terms of ξ, η. These may now be written as x, y, and the process again applied, and so on.

Thus, let the values of the pairs of equal quantities in (2) be 1, 0, respectively (which is obviously lawful), we have

$$u=\epsilon^x,\ \phi=y;$$

so that (1) becomes

$$d\sigma=\epsilon^x\Big((i\cos y+j\sin y)dx-(i\sin y-j\cos y)dy\Big),$$

and $$\sigma=\epsilon^x(i\cos y+j\sin y)$$

or $$\xi=\epsilon^x\cos y,\ \eta=\epsilon^x\sin y.$$

From these we have

$$x=\log\sqrt{\xi^2+\eta^2},\ y=\tan^{-1}\frac{\eta}{\xi};$$

or, using polar cöordinates for the derived series,

$$x=\log r,\ y=\theta.$$

[This is easily seen to be only a special case of (3) above.] Hence, by (4), another pair of systems satisfying the condition is

$$F_1(\log r,\ \theta)=A_1,\ F_2(\log r,\ \theta)=A_2.$$

This, of course, is only one of the simplest of an infinite number of solutions of the equation (1), which may be obtained with the greatest ease from (2).

If there is inversion, all that is necessary is to substitute ρ^{-1} for ρ, or $-\rho^{-1}d\rho\rho^{-1}$ for $d\rho$. But the necessity for this may be avoided by substituting for any pair of systems which satisfy the condition their electric image, which also satisfies it, and which introduces the required inversion.

The solution of this problem without the help of quaternions is interesting. Keeping as far as possible to the notation above, it will be seen that the conditions of the problem require that

$$\left(\frac{d\xi}{dx}dx+\frac{d\xi}{dy}dy\right)^2+\left(\frac{d\eta}{dx}dx+\frac{d\eta}{dy}dy\right)^2=u^2(dx^2+dy^2)$$

whatever be the ratio $dx:dy$.

This gives at once

$$\left(\frac{d\xi}{dx}\right)^2+\left(\frac{d\eta}{dx}\right)^2=\left(\frac{d\xi}{dy}\right)^2+\left(\frac{d\eta}{dy}\right)^2=u^2,$$

$$\frac{d\xi}{dx}\frac{d\xi}{dy}+\frac{d\eta}{dx}\frac{d\eta}{dy}=0.$$

From these the equations (2) can be deduced by introducing ϕ an auxiliary angle.

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