The Barycentric Calculus of Möbius.

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August Ferdinand Möbius was born at Schulpforta, in Saxony, in the year 1790. He studied in the Universities of Leipsic and Göttingen, and, at the age of 26, was appointed extraordinary Professor of Astronomy and Superintendent of the Observatory at Leipsic. There he remained till his death, in 1868, being appointed ordinary professor in 1844. Between the years 1817 and 1868 Möbius wrote his Barycentric Calculus, a Treatise on Statics, another on the Mechanics of the Heavens, and a large number of papers on Mathematical, Dynamical, and Astronomical questions. Most of these papers were contributed to Crelle's Journal, which was founded in 1826. The works of Möbius have recently been collected under the direction of the Royal Scientific Society of Leipsic, and under the editorship of Klein, Scheibner, and Baltzer.

The Barycentric Calculus was published in 1827, and forms nearly two-thirds of the first volume of the collected works of Möbius. Though this Calculus is thus nearly two-thirds of a century old, its methods have never been introduced into ordinary mathematical courses, and I have, therefore, thought it worth while to give some account of it. The exposition of his Calculus, as given by Möbius, is exceedingly clear, and shows that, although the methods of the Calculus have not so wide an application as those of ordinary Co-ordinate Geometry, yet, within their own range, they are very powerful in the hands of one who has thoroughly mastered them. The work also possesses a large amount of historical interest. It was here that homogeneous co-ordinates were first used—Plücker's *Analytisch-geometrishe Entwickelungen*, in which the theory of trilinear co-ordinates, as they are now used, was first given, not appearing till 1828 (the second volume not till 1831.) Here also Möbius gave, for the first time, the method of determining the sign of a segment of a line in accordance with the, arbitrarily assumed, positive direction of the line (what may be called the quaternion method of determining the sign of a segment of a line); and discussed also, for the first time, the sign of an area and the sign of a volume. The distinction should be noted between the methods used for determining the sign of a line in quaternions and in co-ordinate geometry. Though one of these methods may be deduced from the other, they are really fundamentally different.

Some slight idea of the method of the Calculus may be given in a few words as follows :—

Let us confine our attention to a plane, and consider three points, A, B, C. If weights a, b, c, be imagined at the points A, B, C, then this system will have a definite centre of gravity, P say. In the determination of P only the ratios $a:b:c$ are involved ; and there is a one-one correspondence between the position of P and the values of these ratios. These quantities a, b, c, or their ratios, are therefore taken as the co-ordinates of P ; but, as will be seen later, the whole theory is established on a purely geometrical basis. It is obvious that this method may be extended to space of three dimensions, by taking four non-coplanar points. In the development of his theory, Möbius always represents a curve or surface, not by means of an equation connecting the co-ordinates, but by expressing the co-ordinates as functions of one or two independent variables.

Möbius begins by pointing out how signs are to be attached to segments of lines in accordance with the definition $\overline{AB} + \overline{BA} = 0$.

He then proceeds to give a purely geometrical theory of the centre of gravity, somewhat as follows :—

Through A and B (fig. 1) a pair of parallel straight lines are drawn ; to draw A′B′ so that $a.AA' + b.BB' = 0$.

Any straight line through a definite point P satisfies this condition.

P is the centre of gravity of the points A and B with weights a and b ; and is between a and b if a and b are both positive.

If A″P″B″ is any line, not through P, then

$$a.AA'' + b.BB'' = (a+b)PP''.$$

It is shown next that if A, B, C, (fig. 2) are three given points, then a point Q may always be found such that

$$a.AA' + b.BB' + c.CC' = 0$$
$$\text{and } a.AA'' + b.BB'' + c.CC'' = (a+b+c).QQ''.$$

Q is obviously the C.G. of the system A, B, C, with the weights a, b, c.

A similar theorem is next proved for four points, whether these be coplanar or not.

Any number of fundamental points A, B, C, ... N may be employed. A definite set of co-efficients (a, b, c...n) (or weights) determine one definite point (the centre of gravity).

Particular case.

If $a+b+c\ldots+n=0$, the point is at infinity.

Since $a \neq 0$, $\therefore b+c+\ldots n \neq 0$.

We may therefore determine T so that, if lines be drawn through B, C,...T parallel to a given line and be cut by any plane in B′,C′,...T′, then

$$b.BB'+c.CC'+\ldots+n.NN'=(b+c\ldots+n).TT'=-a.TT'$$
$$\therefore a.AA'+b.BB'+\ldots.+n.NN'=a.AA'-a.TT'.$$

Now this will be zero, provided the plane, B′C′...T′ be parallel to AT; *i.e.*, pass through the point at infinity in AT. Hence this point at infinity is the point required.

If then three fixed points A, B, C, be given in a plane, a point P is determined in accordance with the above when the coefficients a, b, c, are given.

We have then the relation

$$a.AA'+b.BB'+c.CC'=(a+b+c).PP'$$

This is written more shortly

$$aA+bB+cC=(a+b+c).P$$

And the notation $aA+bB+cC\equiv P$ is also employed to denote that P is the point whose co-ordinates are a, b, c.

Möbius next considers the application of algebraical signs to areas and to volumes; and then establishes the following results:—

If we put $c=-(a+b)$, we may denote the same fact in any of the three ways:—

$$aA+bB\equiv C,\ aA+bB=(a+b)C,\ aA+bB+cC=0.$$

In this case, $a:b:c=BC:CA:AB$; and A, B, C, are collinear.

If $aA+bB+cC\equiv D$, then A, B, C, D, are coplanar and $a:b:c=\triangle DBC:\triangle DCA:\triangle DAB$.

If A, B, C, D are not coplanar, and if $aA+bB+cC+dD\equiv E$,

$$\text{then } a:b:c:d=BCDE:CDEA:DEAB:EABC$$

where BCDE, etc., denote volumes of pyramids.

It is next shown that if A, B, C are any three non-collinear points in a plane, the ratios $a:b:c$ may be so determined that $aA+bB+cC\equiv D$, where D is any fourth point whatever; that is, any point in a plane may be regarded as the C.G. of any other three points.

Let AB and CD intersect in L; then we may write

$L\equiv\alpha A+\beta B\equiv\gamma C+\delta D$, where the ratios $\alpha:\beta$ and $\gamma:\delta$

are determined. Now determine the ratio $\beta : \gamma$ so that $\alpha + \beta = \gamma + \delta$, then $\alpha A + \beta B = \gamma C + \delta D$

$$\therefore \ D \equiv \alpha A + \beta B - \gamma C.$$

A similar proposition is proved for any four non-coplanar points in space and any fifth point taken arbitrarily.

Taking now $pA + qB + rC \equiv P$ as the expression for a point in the plane, we can determine the region in which P lies according to the relative signs and magnitudes of p, q, r. The result is as follows (fig. 3):—

P lies

I. in $\overline{ABC}$, if p, q, r have the same sign

$$\text{II.}\begin{cases}\text{in } \overline{BC}, \text{ if } q+r>p\\ \text{in } \overline{CA}, \text{ if } r+p>q\\ \text{in } \overline{AB}, \text{ if } p+q>r\end{cases}$$

$$\text{III.}\begin{cases}\text{in } \overline{A}, \text{ if } p>q+r\\ \text{in } \overline{B}, \text{ if } q>r+p\\ \text{in } \overline{C}, \text{ if } r>p+q\end{cases}$$

it being understood that letters on the same side of the symbol $>$ have the same sign, those on different sides opposite signs

A similar investigation is then given for space of three dimensions.

Change of Triangle of Reference.

The method of effecting a change of the triangle of reference will be most easily explained by means of the following simple example given by Möbius.

Let $A'B'C'$ be the new triangle of reference, where A' is the mid-point of BC, etc.

Then

$$2A' = B + C,\ 2B' = C + A,\ 2C' = A + B,\ \therefore\ A = B' + C' - A', \text{ etc.}$$

If $P \equiv pA + qB + rC$ is any point referred to ABC, its expression, when referred to $A'B'C'$, will be

$$\begin{aligned} P &\equiv p(B' + C' - A') + q(C' + A' - B') + r(A' + B' - C') \\ &\equiv (q + r - p)A' + (r + p - q)B' + (p + q - r)C'. \end{aligned}$$

Equations to Straight Lines and Planes.

If E and E' are any two points and w be taken as variable, $E + wE'$ may represent any point on the line joining E and E' and may therefore be taken as the expression for the straight line EE'.

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If A, B, C, are any three non-collinear points in the plane, we may put $e\mathrm{E} = a\mathrm{A} + b\mathrm{B} + c\mathrm{C}$, $e'\mathrm{E}' = a'\mathrm{A} + b'\mathrm{B} + c'\mathrm{C}$; $\therefore$ putting $w = ve'/e$,

$$\begin{aligned} \mathrm{E} + w\mathrm{E}' \equiv e\mathrm{E} + ve'\mathrm{E}' &= a\mathrm{A} + b\mathrm{B} + c\mathrm{C} + v(a'\mathrm{A} + b'\mathrm{B} + c'\mathrm{C}) \\ &= (a + va')\mathrm{A} + (b + vb')\mathrm{B} + (c + vc')\mathrm{C}, \end{aligned}$$

which is therefore the expression for the line joining the two points E and E′.

A simpler form may be got as follows :—

$\mathrm{P} \equiv \mathrm{A} + b\mathrm{C}$ and $\mathrm{Q} \equiv \mathrm{B} + c\mathrm{C}$ are any points in AC and BC respectively. The expression for PQ is $\mathrm{A} + b\mathrm{C} + x(\mathrm{B} + c\mathrm{C})$, that is, $\mathrm{A} + x\mathrm{B} + (b + cx)\mathrm{C}$.

Intersection of Two Straight Lines.

To determine the intersection of

$$\left.\begin{array}{r} \mathrm{A} + x\mathrm{B} + (a + bx)\mathrm{C} \\ \text{and } \mathrm{A} + y\mathrm{B} + (a' + b'y)\mathrm{C} \end{array}\right\} \text{ we have the relations}$$

$$1 : x : (a + bx) = 1 : y : (a' + b'y)$$

$$\therefore x = y = -(a - a')/(b - b') ;$$

and the point of intersection is $(b - b')\mathrm{A} - (a - a')\mathrm{B} + (a'b - ab')\mathrm{C}$.

Straight Line in Space.

It may be shown very easily that a straight line in space may be written in either of the forms

$$(a + a'v)\mathrm{A} + (b + b'v)\mathrm{B} + (c + c'v)\mathrm{C} + (d + d'v)\mathrm{D} \text{ or}$$

$$\mathrm{A} + x\mathrm{B} + (a + bx)\mathrm{C} + (c + dx)\mathrm{D}.$$

Plane.

If E, F, G, are three non-collinear points, any point in the plane EFG may be represented by $\mathrm{E} + v\mathrm{F} + w\mathrm{G}$, which is therefore the expression for the plane, when v and w are taken as variables.

Hence any plane may be written

$$(a + a'v + a''w)\mathrm{A} + (b + b'v + b''w)\mathrm{B} + \text{etc.} + \text{etc.}$$

This plane is determined by the points (a, b, c, d), (a', b', c', d') (a'', b'', c'', d''). [*N.B.*—Möbius does not use this notation.]

By taking for the points that determine the plane, points in three of the fundamental lines, we may reduce the expression to the form $\mathrm{A} + x\mathrm{B} + y\mathrm{C} + (a + bx + cy)\mathrm{D}$.

Möbius works out the expression for the line of intersection of two planes, the condition that two planes be parallel, and a number of analagous problems. As a single example, I shall take the following :—

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Given the line $(au+a')A+(\beta u+\beta')B+uC+D$,
and the plane $A+vB+wC+(a+bv+cw)D$,
to find (1) their point of intersection, (2) the condition that the plane contain the line, (3) the condition that they be parallel.

For every point on their intersection we must have

$$au+a':\beta u+\beta':u:1=1:v:w:a+bv+cw \quad \ldots \quad \ldots \quad (1).$$

These three equations determine u, v and w, giving

$$u=(1-aa'-b\beta')/(aa+b\beta+c).$$

We get the point of intersection by substituting this value in the expression for the given straight line. Putting $m=1-aa'-b\beta'$, $n=aa+b\beta+c$, the point is $(am+a'n)A+(\beta m+\beta' n)B+mC+nD$.

If the plane contain the straight line, the equations (1) must hold good for all values of u; hence $m=n=0$.

If the plane and the straight line are parallel, then

$$(am+a'n)+(\beta m+\beta' n)+m+n=0,$$
$$\therefore\ (aa'+b\beta'-1)/(aa+b\beta+c)=(a+\beta+1)/(a'+\beta'+1).$$

Möbius next proceeds to the discussion of curved lines and surfaces. A curved line in a plane may be represented by the expression $pA+qB+rC$ where p, q, r are given functions of a single variable.

A curve of the second degree may be represented by

$$(a+a'v+a''v^2)A+(b+b'v+b''v^2)B+(c+c'v+c''v^2)C$$

provided the three coefficients do not contain a common factor and provided they cannot all by some transformation be reduced to linear form. Thus we must not have $a=b=c=0$, or $a'=b'=c'=0$. Each side of the triangle of reference is cut in two points, real or imaginary; and so is every other line, as it may be made a side of a new triangle of reference.

If we put

$$a_1A_1=aA+bB+cC$$
$$b_1B_1=a'A+b'B+c'C$$
$$c_1C_1=a''A+b''B+c''C$$

and take $A_1B_1C_1$ as the new triangle of reference, the expression becomes

$$a_1A_1+b_1vB+c_1v^2C_1$$

or when we put $c_1v/b_1=w$, and $a_1c_1/b^2=a$,

$$aA_1+wB_1+w^2C_1 \quad \ldots \quad \ldots \quad \ldots \quad (1).$$

To find the points at infinity on this curve we have to put $a+w+w^2=0$. The points will be real and different, imaginary, or equal, *i.e.*, the curve a hyperbola, ellipse or parabola, according as $a < > = 1/4$.

For these different cases the expression (1) may be reduced to one or other of the forms

$$A + xB + x^2C, \ -A + xB + x^2C, \ \tfrac{1}{4}A + xB + x^2C.$$

These curves pass through A and C and touch BA and BC at these points.

Other particular forms of the equation.

$$a(v-\beta)(v-\gamma)A + b(v-\gamma)(v-a)B + c(v-a)(v-\beta)C$$

represents a conic passing through A, B and C.

$$a(v-a)^2A + b(v-\beta)^2B + c(v-\gamma)^2C$$

represents a conic inscribed in ABC.

The next point discussed is the number of independent constants involved in the expression $pA + qB + rC$ where p, q, and r are integral functions of a single variable of the n^{th} degree. The number is $3n - 1$, which is less by $\frac{1}{2}(n-1)(n-2)$ than the number involved in the general equation of the n^{th} degree in Cartesian Co-ordinates As a matter of fact, when p, q, and r are rational functions, the above expression can only represent unicursal curves.

Möbius next considers the question of the contact of curves and of cusps and points of inflexion. The following will give some idea of his method.

Let $pA + qB + rC$ denote the curve, where p, q and r are functions of v. Consider a particular value v' of v and an adjacent value $v' + x$, then

$$p = p' + \frac{dp'}{dv'} \cdot x + \frac{1}{2!}\frac{d^2p'}{dv'^2} \cdot x^2 + \dots$$

$$q = q' + \frac{dq'}{dv'} \cdot x + \frac{1}{2!}\frac{d^2q'}{dv'^2} \cdot x^2 + \dots, \text{ etc.}$$

Now put $\quad p'A + q'B + r'C = a_1A_1,$

$$\frac{dp'}{dv'}A + \frac{dq'}{dv'}B + \frac{dr'}{dv'}C = b_1B_1, \text{ etc.}$$

Then the point on the curve corresponding to $v' + x$, is

$$a_1A_1 + b_1xB_1 + c_1x^2C_1 + \dots\dots$$

This, as a matter of fact, is another form of the expression for the curve, the new variable x occurring in the n^{th} degree. The first two terms $a_1A_1 + xb_1B_1$ give the tangent at the point (p', q', r'). Möbius examines thoroughly the nature of the contact of the curves of the 2^{nd}, 3^{rd}, etc., orders got by including x^2, x^3..... When one or more of the quantities b_1, c_1, etc., vanish, cases of contact of higher orders arises, and this leads to the discussion of points of inflexion and cusps.

I may pass over the discussion of asymptotes which is contained in the next section of the Calculus, with the remark that points at infinity are given by roots of the equation $p + q + r = 0$, and the asymptotes or tangents at these points are discussed by the method which has already been illustrated in the case of finite tangents.

Curves and Surfaces in Space.

A curve in space is represented by $pA + qB + rC + sD$ where p, q, r, s are functions of a single variable.

If p is a function (of v say) of the n^{th} degree, the n roots of $p = 0$, substituted in $qB + rC + sD$ give the n points where the curve meets the plane BCD.

The line joining A to the point $pA + qB + rC + sD$ meets the plane BCD in the point $qB + rC + sD$. Hence the curve may be formed as the intersection of any two of the four cones such as that which has A as vertex and $qB + rC + sD$ as generating curve.

The expression for the tangent and the osculating plane at any point are obtained by a method practically identical with that used in the case of plane curves.

A surface is represented by $pA + qB + rC + sD$ where p, q, r, s, are functions of two independent variables.

If p, q, r, s, are functions (say of v and w) of the n^{th} degree, the surface will in general be of a higher order than the n^{th}. For if we determine v and w by means of the equations $p = 0$, $q = 0$, and substitute the values obtained in $rC + sD$, we shall obtain the points in which CD meets the surface. Now $p = 0$, $q = 0$, will have in general more than n solutions.

The tangent-plane at a point of a surface is discussed as follows :—

Let $pA + qB + rC + sD$ be the surface, where p, q, r, s, are functions of v and w. If v' and w' be particular values and x and y infinitely small increments, p becomes

$$p' + \frac{dp'}{dv'}x + \frac{dp'}{dw'}y. \quad \text{Hence } \Sigma\left\{(p' + \frac{dp'}{dv'}x + \frac{dp'}{dw'}y)A\right\}$$

is an infinitely near point. If now x and y are taken to be any two independent variables, this will represent the tangent plane at $(p,' q', r', s'.)$

The conditions that ABC be a tangent plane are

$$s'=0,\ \frac{ds'}{dv'}=0,\ \frac{ds'}{dw'}=0.$$

Since s vanishes when $v=v'$, $w=w'$, assume $s=b(v-v')+c(w-w')$ where b and c are functions of v and w.

$$\therefore\ \frac{ds}{dv}=b+(v-v')\frac{db}{dv} \qquad +(w-w')\frac{dc}{dv}$$

$$\frac{ds}{dw}= \qquad (v-v')\frac{db}{dw}+c+(w-w')\frac{dc}{dw}$$

Hence b and c vanish when $v=v'$, $w=w'$.

Hence s must be of the form $f(v-v')^2+g(v-v')(w-w')+h(w-w')^2$.

If A is the point of contact, the equation to the surface must be of the form

$$\begin{aligned}&a\mathrm{A}+[b(v-v')+c(w-w')]\mathrm{B}+[d(v-v')+e(w-w')]\mathrm{C}\\&\quad+[f(v-v')^2+g(v-v')(w-w')+h(w-w')^2]\mathrm{D}.\end{aligned}$$

where a, b,...h are functions of v and w.

In considering the nature of the contact, we may assume a, b,...h to be constants and $v-v'$ and $w-w'$ to be infinitely small. Putting $t=[b(v-v')+c(w-w')]/a$; $u=[d(v-v')+e(w-w')]/a$ the expression becomes $\mathrm{A}+t\mathrm{B}+u\mathrm{C}+(it^2+ktu+lu^2)\mathrm{D}$.

Put $\mathrm{A}+t\mathrm{B}+u\mathrm{C}\equiv\mathrm{Q}$ and $\mathrm{Q}+(it^2+ktu+lu^2)\mathrm{D}\equiv\mathrm{P}$; then Q is a point in the tangent-plane ABC infinitely near the point of contact A ; while P is a point on the surface, lying in the line QD and distant from Q by a quantity of the second order. Further P lies towards D or away from D according as $it^2+ktu+lu^2$ is positive or negative. Hence if $k^2<4il$, the surface is altogether on one side of the tangent plane in the neighbourhood of A ; if $k^2>4il$ the surface crosses the tangent-plane. This leads to the distinction between synclastic and anticlastic surfaces. There is also a limiting case in which $k^2=4il$, when the surface touches the plane along a curve.

Möbius shows that, if the restriction that t and u be infinitely small be removed, then $\mathrm{A}+t\mathrm{B}+u\mathrm{C}^2+(it^2+ktu+lu^2)\mathrm{D}$ represents a quadric surface. To examine the different forms of quadric surface, put $s=lu^2+ktu+it^2+u+t+1$.

We may show that $s = \dfrac{1}{4l}\left[v^2 + \dfrac{w^2 + e}{4(4il - k^2)}\right]$

where v and w are linear functions of t and u and where $e = 16l(4il - k^2 - i + k + l)$.

The points at infinity are obtained by making s zero; hence there are no such points if $4il > k^2$ and e is positive. In this case the surface is an ellipsoid.

If $4il > k^2$, but e is negative, the surface does not cross the tangent plane at the point of contact, and there are real points at infinity (Hyperboloid of two sheets).

If $4il < k^2$, we get a hyperboloid of one sheet.

If $e = o$, we get an elliptic paraboloid if $4il > k^2$, a hyperbolic paraboloid if $4il < k^2$.

Möbius next proceeds to consider some simpler forms of the expressions for quadric surfaces and to deduce some of the properties of such surfaces; and then devotes a few pages to developable surfaces.

The next section of the Calculus is devoted to the problem of determining the equation to a curve when its barycentric expression is given; and to the converse problem.

The first of these problems is easily soluble; but, as has been already pointed out, the second is only soluble in particular cases. I shall only point out how the first of these problems is solved in the case of plane curves.

Let A, B, C, be the vertices of the triangle of reference, and let the co-ordinates of these points be (a, a'), (b, b'), (c, c') with references to a pair of Cartesian Axes.

Let $P \equiv pA + qB + rC$ and let the co-ordinates of P be (x, y). If P′, A′, B′, C′, are the feet of the ordinates from P, A, B, C, then, by the fundamental theorem of the Calculus,

$$p.AA' + q.BB' + r.CC' = (p + q + r).PP'$$

$$\text{i.e.,} \quad \left.\begin{aligned} pa' + qb' + rc' &= (p + q + r)y \\ \text{and similarly} \quad pa + qb + rc &= (p + q + r)x \end{aligned}\right\}$$

Thus we get x and y in terms of p, q and r. If p, q and r are given functions of a variable v, so that $pA + qB + rC$ represents a curve, we get the equation to the curve by eliminating v between the equations that give x and y.

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This section concludes the first part of the Barycentric Calculus; the second part is devoted to the discussion of certain relations between figures and of some geometrical theorems connected with these relations. This part also contains a lengthy discussion of the theory of anharmonic ratio and what are called geometrical nets.

Möbius considers five different kinds of relations between figures. The first of these he describes by the phrase equality and similarity (Gleichheit and Aehnlichkeit); but as the second relation is similarity and the fourth is equality, it will be better to employ the single term congruence. The five relations then are (1) Congruence, (2) Similarity, (3) Affinity, (4) Equality, (5) Collineation. The most general of these relations is the last, all the others being particular cases of it; and towards the end of his discussion, Möbius shows that of any two plane figures between which the relation of collineation subsists, one may be exhibited as a conical projection of the other. The relation of collineation is thus the same as that of homology or homography, of which a discussion was given before the date of the Barycentric Calculus by Poncelet, and after that date by Chasles; and an account of the theory of homology is now contained in many geometrical text-books. But as Möbius's treatment is based on the methods of his Calculus, and for the sake of completeness, it may be well to give some account of his theory of these five geometrical relations.

The account of congruence and similarity I may pass over with the remark that he enumerates the number of conditions required to determine certain figures. The results of these enumerates I shall bring together in a table at the end of this account of his discussion.

The relation of affinity is defined as follows:—

Let ABC and A′B′C′ be any two triangles, and let D and D′ be two points determined with reference to these triangles by means of the relations $D \equiv aA + bB + cC$ and $D' \equiv aA' + bB' + cC'$, then D and D′ are corresponding points. From this it is obvious that if one system of points be given then the corresponding system of points will be determined when three of the points are given; but these three points may be given arbitrarily.

If P, Q, R, S, are any four points of the one system, P′, Q′, R′, S′, the corresponding points in the other system, we shall have relations of the form

$$\left.\begin{aligned} pP + qQ + rR + sS &= 0 \\ pP' + qQ' + rR' + sS' &= 0 \end{aligned}\right\}$$

and $\therefore \quad p : q : r : s = \triangle QRS : \triangle PSR : \triangle SPQ : \triangle RQP$

$$= \triangle Q'R'S' : \triangle P'S'R' : \triangle S'P'Q' : \triangle R'Q'P'.$$

Hence the ratio PQR : P′Q′R′ is constant.

It is next shown that the above definition is equivalent to that given in terms of Cartesian co-ordinates by Euler in his *Introductio in Analysin Infinitorum*; and then the theory is extended to space of three dimensions by means of the definition that E and E′ are corresponding points if $E \equiv aA + bB + cC + dD$ and $E' \equiv aA' + bB' + cC' + dD'$.

It is easily shown, by a method perfectly analogous to that used above for plane figures, that corresponding solids are to one another in a constant ratio.

Further if in one figure, three points are collinear or four points coplanar, then the same will be true of the corresponding points in the other figure; and if A, B, C, are collinear points and A′, B′, C′, the corresponding points, then $AB : BC = A'B' : B'C'$.

It has been pointed out that in two plane figures that bear to one another the relation of affinity, if P, Q R, and P′, Q′, R′, are corresponding points, then $\triangle PQR : \triangle P'Q'R' = \text{const}$ (m, say), and in space of three dimensions if P, Q, R, S, and P′, Q′, R′, S′, are corresponding points, then vol.PQRS : vol.P′Q′R′S′ = const (n, say). If m is unity, we get the relation of equality for plane figures; and if n is unity, for solid figures. To obtain two such figures, it is only necessary to make the triangles of reference equal in area in the one case, and the tetrahedrons of reference equal in the other case.

Möbius works out a number of cases of the calculation of parts of figures in plane and solid geometry, in connection with these relations, and applies this theory to prove some properties of conic sections and of quadric surfaces. I select the following as an example.

Given five points ABCDE (fig. 4) in a plane, and given the four ratios

$$\left.\begin{aligned} ABC : CDA &= p \\ BCD : DAB &= q \\ ABE : CDE &= r \\ DAE : BCE &= s \end{aligned}\right\} \quad \begin{gathered}\text{to calculate the ratio} \\ DBE \div ACE = t.\end{gathered}$$

Assume $d''D = dA + d'B + C,\ e''E = eA + e'B + C \quad \ldots \quad (1), (2),$
so that $d'' = 1 + d + d'$ and $e'' = 1 + e + e'$.

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Eliminating first C and then A, we get

$$(d-e)\mathrm{A}+(d'-e')\mathrm{B}-d''\mathrm{D}+e''\mathrm{E}=0, \quad \ldots \quad \ldots \quad (3)$$
$$(de'-ed')\mathrm{B}+(d-e)\mathrm{C}+ed''\mathrm{D}-de''\mathrm{E}=0. \quad \ldots \quad \ldots \quad (4)$$

From (1) we get the relations

$$\mathrm{BCD} : \mathrm{DCA} : \mathrm{DAB} : \mathrm{ABC} = d : d' : 1 : d''$$

with similar results from (2), (3), and (4).

These enable us to express p, q, r, s and t in terms of d, d', e, e'; viz., we get

$$p = -d''/d', \; q=d, \; r=d''/(de'-d'e), \; s=(e'-d')/ed'' \text{ and } t=(e-d)/e'd''.$$

On eliminating d, d', e, e' from these four equations, we get

$$t=(1+p)^2(1-q^2rs)/(1+q)^2(p^2s-r).$$

The next chapter is devoted to the discussion of anharmonic ratio (Doppelschnittsverhältniss) (ratio bissectionalis). It is here that this theory was first fully discussed; but as the subject is now well known, I may pass it over altogether.

The next subject considered is that of geometrical nets in a plane and in space of three dimensions. If we start with four points in a plane, and join them in every possible way, we get three new points. If we draw the three straight lines joining these three points in pairs, we obtain three new points as the intersections of these lines with lines already drawn in the figure. Proceeding in this way we may get an infinite number of points and straight lines in the plane. The figure formed by all these straight lines is called a plane geometrical net; and the analogous figure in space of three dimensions may be formed by starting with five points in space.

It is easily shown by mathematical induction that if A, B, C, D are the four principal points of a plane net, and if $\mathrm{D} \equiv a\mathrm{A}+b\mathrm{B}+c\mathrm{C}$, then every other point of the net may be expressed in the form $\lambda a\mathrm{A}+\mu b\mathrm{B}+\nu c\mathrm{C}$ where λ, μ, ν are rational numbers, and are independent of a, b, and c. Further, by calculating the anharmonic ratio of four collinear points, we may show that its value is independent of a, b, c; and therefore this anharmonic ratio does not depend on the position of A, B, C, D, but only on the steps by which the four points considered have been derived from A, B, C, D. The converse to the theorem quoted above is then proved, namely, that any point $\mathrm{P} \equiv \lambda a\mathrm{A}+\mu b\mathrm{B}+\nu c\mathrm{C}$ where λ, μ, ν are rational numbers is a point in the net. In proving this we may assume λ, μ, ν, to be integers. Constructions are given for the points $2aA+bB+cC$, $3aA+bB+cC$, etc., in succession, and this shows that we may construct the point $\lambda aA+bB+cC$. Then constructions for the points $\lambda aA+2bB+cC$, $\lambda aA+3bB+cC$ are given, showing that we may construct $\lambda aA+\mu bB+cC$, and then similarly that we may construct $\lambda aA+\mu bB+\nu cC$. In order to apply this theory to the last of the geometrical relations, that of collineation, Möbius proves (what is really obvious) that every point of the plane is either a point of the net, or infinitely near to a point of the net. He proves a number of theorems in plane and solid geometry by means of the theory of nets, and then proceeds to consider what he calls a "Vieleckschnittsverhältniss." This word may be translated by the phrase "polygonal cross-ratio." There is a triangular cross-ratio, tetragonal cross-ratio, and so on. If P, Q, R, are points in the sides BC, CA, AB of a triangle, then

$$\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}\quad\text{is a triangular cross-ratio.}$$

It is shown that, just as in the case of an ordinary cross-ratio, if A, B, C, P, Q, R, are points of a net, then the value of the triangular cross-ratio is independent of the positions of the four principal points of the net.

The defining property of the last kind of geometrical relation considered, that of collineation, is this, that to three collinear points in the one figure shall correspond three collinear points in the other figure. Two geometrical nets (both plane or both solid) satisfy this definition, and are the most general figures that can satisfy it. For if we start with two tetrads of corresponding points A, B, C, D, and A′, B′, C′, D′, we may obtain from these by the first step, three new points in each figure, E, F, G, and E′, F′, G′, and by the definition of collineation E′ must correspond to E, F′ to F, and G′ to G; and so on for all the other points of the two nets. Now we may put $D\equiv aA+bB+cC$ and $D'\equiv a'A'+b'B'+c'C'$, where a, b, c, a', b', c', are arbitrary, and any point of the first net may be written $P\equiv\lambda aA+\mu bB+\nu cC$, and the corresponding point in the second net will be $\lambda a'A'+\mu b'B'+\nu c'C'$. Hence this is the analytical definition of the relation of collineation. If we suppose λ, μ, ν to be functions of a single variable, we shall get two curves which are subject to this relation; and it is obvious that any two corresponding curves are of the same order. From what has been proved above it follows that a cross-ratio or a polygonal cross-ratio in one figure is equal to the cross-ratio or the polygonal cross-ratio formed by the corresponding points in the other figure.

As has been mentioned above, Möbius shows that of any two figures between which this relation of collineation subsists either may be regarded as the conical projection of the other. Hence the relation of collineation is identical with that of homography, or if the relative position of the two figures be restricted in a certain way, that of homology. The above definition is easily extended to space of three dimensions; and, among his illustrations of the theory, Möbius shows that by the method of collineation, any one conic section may be transformed into any other, but that a similar theorem does not hold good of quadric surfaces. These divide themselves into two groups, the first of which includes the ellipsoid, the hyperboloid of two sheets and the elliptic paraboloid, and the second the hyperboloid of one sheet and the hyperbolic paraboloid; in other words, the first class includes the non-ruled surfaces, and the second class the ruled surfaces. Any one surface of one of these classes may be transformed into any other of that class; but one from one class cannot be transformed into one from the other class.

Möbius next explains what he calls the contracted (abgekürzte) Barycentric Calculus. This contracted form is applicable to problems which are true of all figures collinearly related. The contraction consists in omitting the a, b, c, from the equations, and taking for the four principal points A, B, C and $D \equiv A + B + C$. This contracted method is applied to certain problems connected with the triangle and the tetrahedron.

Möbius next proceeds to classify certain geometrical problems according to the geometrical relation (congruence, similarity, etc.) under which they come. If there are n points in a straight, in a plane or in space, we may want to calculate the length of a line, the ratio of two lines, an area, etc. The following table gives the number of conditions necessary in each case:—

	I.	II.	III.	
Congruence	$n-1$	$2n-3$	$3n-6$	
Similarity .	$n-2$	$2n-4$	$3n-7$	Ratios.
Equality .	$n-1$	$2n-5$	$3n-11$	Portions of space (areas, etc.)
Affinity .	$n-2$	$2n-6$	$3n-12$	Ratios between portions of space.
Collineation	$n-3$	$2n-8$	$3n-15$	Polygonal cross-ratios.

The meaning of this table may be illustrated by taking (say) the second column under the head of "affinity." If there are n points in a plane, and if $2n-6$ independent ratios of areas of triangles or polygons formed by certain of these points are given, then all the figures that can be constructed to satisfy these conditions will have the relation of affinity to one another; and from the $2n-6$ given ratios any other ratio may be calculated. Many illustrations are given by Möbius, and one of these has been quoted above.

Möbius points out that problems connected with congruence and similarity, *e.g.*, such as involve angles, cannot be solved by means of the Barycentric Calculus at all; those that are true of all figures which satisfy the relation of affinity can be solved by the ordinary form of the Calculus; while those that are true of all figures collinearly related may be solved by the contracted form. This forms a three-fold classification of geometrical theorems.

The third and last part of the work is concerned with the application of the Calculus to the investigation of certain properties of the Conic Sections and with a discussion of the principle of duality. From this part I shall only select a few examples by way of illustration. The first question considered is the determination of a conic section by means of five given points or five given tangents.

As we have seen already, the general expression for a conic circumscribing the triangle of reference is

$$a(v-\beta)(v-\gamma)\mathrm{A}+b(v-\gamma)(v-\alpha)\mathrm{B}+c(v-\alpha)(v-\beta)\mathrm{C}.$$

This expression may be reduced to the form

$$fx\mathrm{A}-gx(1-x)\mathrm{B}+h(1-x)\mathrm{C} \quad \ldots \quad \ldots \quad (1)$$

where

$$x = -\frac{\alpha-\beta}{\gamma-\alpha}\cdot\frac{v-\gamma}{v-\beta};\ f=a(\beta-\gamma),\ g=b(\gamma-\alpha),\ h=c(\alpha-\beta).$$

Let $D \equiv aA+bB+cC$, then the conditions that the conic pass through this point are

$$\frac{f}{1-x} : -g : \frac{h}{x} = a:b:c,$$

whence $$f/a+g/b+h/c=0.$$

If $E \equiv a'A+b'B+c'C$ is the fifth point, we must also have $f/a'+g/b'+h/c'=0$; and from these two equations we get

$$f:g:h = \left(\frac{1}{bc'}-\frac{1}{b'c}\right) : \left(\frac{1}{ca'}-\frac{1}{c'a}\right) : \left(\frac{1}{ab'}-\frac{1}{a'b}\right)$$

$$= a(\beta-\gamma) \quad : \quad b(\gamma-\alpha) \quad : \quad c(\alpha-\beta),$$

where $$a'=a/\alpha,\ b'=b/\beta,\ c'=c/\gamma.$$

Hence the expression for the conic through the five points A, B, C, D, E, is

$$a(\beta-\gamma)xA-b(\gamma-\alpha)x(1-x)B+c(\alpha-\beta)(1-x)C;$$

and the values of x that give these points are

$$1,\ \beta,\ 0,\ -(\alpha-\beta)/(\gamma-\alpha) \text{ and } -\gamma(\alpha-\beta)/\beta(\gamma-\alpha).$$

Putting $(\alpha-\beta)/(\gamma-\alpha)=m$, and $\therefore\ (\beta-\gamma)/(\gamma-\alpha)=-1+m$, we get the expression $a(1-m)xA+bx(1-x)B+cm(1-x)C$, which, if we suppose m to be arbitrary, will represent any conic passing through A, B, C, D.

Taking the form (1) we may determine the condition that it represent an ellipse, a hyperbola or a parabola, by considering the sum of the co-efficients. This sum is

$$fx-gx(1-x)+h(1-x)=gx^2-(g+h-f)x+h.$$

The discriminant of this is

$$f^2+g^2+h^2-2(gh+hf+fg).$$

The condition for a parabola is

$$g+h-f=2\sqrt{gh} \text{ or } f=(\sqrt{g}-\sqrt{h})^2.$$

Putting $e=\sqrt{(h/g)}$, the expression for a parabola circumscribing ABC becomes

$$(1-e)^2xA-x(1-x)B+e^2(1-x)C.$$

Now take a fourth point $D \equiv aA + bB + cC$, and put $a + b + c + d = 0$, then the equation to determine e, so that the parabola shall pass through D reduces to

$$(1 - e)^2/a + 1/b + e^2/c = 0$$

that is,
$$(a + c)be^2 - 2bce + (a + b)c = 0.$$

The discriminant of this reduces to the form $abcd$.

Now if this expression is positive, two of the factors must be positive and the other two negative, and therefore each of the points ABCD will be outside the triangle formed by the other three. On the other hand, if the expression is negative, three of the factors must have the same sign, and therefore one of the points will be inside the triangle formed by the other three. In the first case it will be possible to construct two parabolas to pass through ABC and D, and in the other case no such parabola can be described.

Using the form $a(1 - m)xA + bx(1 - x)B + cm(1 - x)C$ . . (2), we find that the discriminant of the sum of the co-efficients is

$$M \equiv (b + c)^2 m - (c + a)^2 m(1 - m) + (a + b)^2(1 - m).$$

The discriminant of M, considered as a function of m, is $abcd$; and hence the expression (2) may represent a parabola if $abcd$ is positive, as we have already shown. If, on the other hand, $abcd$ is negative M will always have the same sign, and (2) can represent only hyperbolas or only ellipses. When $m = 0$, M becomes $(a + b)^2$, and hence if $abcd$ is negative (2) can represent only hyperbolas. This case, it is to be remembered, occurs when one of the four points A, B, C, D, lies within the triangle formed by the other three.

Of five points in a plane there will always be four such that each of them lies outside the triangle formed by the other three. Let these be A, B, C, D (fig. 5).

If E lies in one of the spaces H the curve must be a hyperbola; if in one of the spaces G it may be either a hyperbola or an ellipse. Construct the two parabolas through ABCD (fig. 6). These parabolas must lie altogether in the spaces G. If we letter the spaces as in the figure, we see that, if E lies in one of the spaces L, the curve is an ellipse; if E lies in one of the other spaces, the curve is a hyperbola. This follows from the continuity of M and from the fact that M is zero when E is on one of the parabolas.

As a last illustration of the application of the Calculus we may take the following:—

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In the triangle ABC (fig. 7) is inscribed the triangle A′B′C′, in this the triangle A″B″C″, and so on, so that the three sets of corresponding vertices lie on straight lines which pass through a fixed point D. In the triangle ABC, a conic is inscribed touching the sides at A′, B′, C′, in A′B′C′ a conic is inscribed touching at A″, B″, C″. To show that the centres of all these conics lie in a straight line passing through D.

Let $$d\mathrm{D} = i\mathrm{A} + k\mathrm{B} + l\mathrm{C},$$

and put $$k + l = i', \; l + i = k', \; i + k = l'$$
$$k' + l' = i'', \; l' + i' = k'', \; i' + k' = l'', \text{ etc.}$$

Then $$i'\mathrm{A}' = k\mathrm{B} + l\mathrm{C}, \; k'\mathrm{B}' = l\mathrm{C} + i\mathrm{A}, \; l'\mathrm{C}' = i\mathrm{A} + k\mathrm{B}.$$

Similarly $$i''\mathrm{A}'' = k'\mathrm{B}' + l'\mathrm{C}', \text{ etc., etc.}$$

Now the centre of the conic which is inscribed in ABC and touches the sides at the points $k\mathrm{B} + l\mathrm{C}$, $l\mathrm{C} + i\mathrm{A}$, $i\mathrm{A} + k\mathrm{B}$, is

$$\begin{aligned} m\mathrm{M} &= i(k+l)\mathrm{A} + k(l+i)\mathrm{B} + l(i+k)\mathrm{C} \\ &= ii'\mathrm{A} + kk'\mathrm{B} + ll'\mathrm{C}, \end{aligned}$$

where $$m = ii' + kk' + ll'.$$

Similarly the centre of the second conic is

$$\begin{aligned} m'\mathrm{M}' &= i'(k'+l')\mathrm{A}' + k'(l'+i)\mathrm{B}' + l'(i'+k')\mathrm{C} \\ &= i'i''\mathrm{A}' + k'k''\mathrm{B}' + l'l''\mathrm{C}' \\ &= i''(k\mathrm{B} + l\mathrm{C}) + k''(l\mathrm{C} + i\mathrm{A}) + l''(i\mathrm{A} + k\mathrm{B}) \\ &= i(k''+l'')\mathrm{A} + k(l''+i'')\mathrm{B} + l(i''+k'')\mathrm{C} \\ &= ii'''\mathrm{A} + kk'''\mathrm{B} + ll'''\mathrm{C}, \end{aligned}$$

where $$m' = i'i'' + k'k'' + l'l'' = ii''' + kk''' + ll'''.$$

From these we get

$$m'\mathrm{M}' - m\mathrm{M} = i(i''' - i')\mathrm{A} + k(k''' - k')\mathrm{B} + l(l''' - l')\mathrm{C}.$$

Now

$$i''' - i' = k'' + l'' - i' = l' + i' + i' + k' - i' = i' + k' + l' = 2d;$$

and hence

$$m'\mathrm{M}' - m\mathrm{M} = 2d(i\mathrm{A} + k\mathrm{B} + l\mathrm{C}) = 2d^2\mathrm{D}.$$

Hence M, M′ and D are collinear and hence all the centres lie on a straight line passing through D.

Möbius calculates the values of the successive ranges formed by these centres, and finds that

$$(\text{DMM}'\text{M}'') = 5, \ (\text{MM}'\text{M}''\text{M}''') = (\text{M}'\text{M}''\text{M}'''\text{M}'''') = \text{etc.} = \frac{25}{21}.$$

The last two chapters of the Calculus are devoted to a discussion of the principle of duality. This principle, which is now so well known, was only being investigated when Möbius wrote in 1827. As a consequence of the theory of reciprocal polars, it was first stated by Poncelet in the year 1818, and as an independent principle by Gergonne in 1826, and the Barycentric Calculus was published in 1827. In his preface Möbius states that before the part of his work dealing with duality was sent to press he had heard of the papers of Poncelet and Gergonne, but had not seen them. It is unnecessary to give an account of his treatment of the subject; but it may be well to state that he enunciates the principle quite generally as a characteristic property of space and treats the subject independently of the properties of conic sections.

Second Meeting, December 11, 1891.

Professor J. E. A. STEGGALL, M.A., President, in the Chair.

Note on an approximate fractional expression for the expansion of $(1+x)^n$ to any odd number of terms.

By Professor J. E. A. STEGGALL.

In Bonnycastle's *Arithmetic* a rule is given for finding any root of a number by approximation which in substance reduces to this statement:—If a and b are nearly equal, then the n^{th} root of a/b is nearly equal to

$$\frac{(n+1)a+(n-1)b}{(n+1)b+(n-1)a};$$

or, if we call a/b, $1+x$, we have $(1+x)^{\frac{1}{n}} = \dfrac{2n+(n+1)x}{2n+(n-1)x} \quad \ldots \quad (1)$

For example, let $a=2$, $b=1$, $n=3$, then an approximation to the cube root of 2 is

$$\frac{4.2+2.1}{4.1+2.2} = \frac{5}{4},$$