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Uniform Distribution in Model Sets

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Abstract. We give a new measure-theoretical proof of the uniform distribution property of points in model sets (cut and project sets). Each model set comes as a member of a family of related model sets, obtained by joint translation in its ambient (the 'physical') space and its internal space. We prove, assuming only that the window defining the model set is measurable with compact closure, that almost surely the distribution of points in any model set from such a family is uniform in the sense of Weyl, and almost surely the model set is pure point diffractive.

When thoughts spring up,
The wind freshens, and like waves
A thousand worlds arise.

1 Introduction

Model sets are discrete point sets formed by projections into 'physical space' of a lattice in some super-space, the projection being controlled by a relatively compact set, called the window, in the internal space. Initially introduced by Y. Meyer in the context of Diophantine approximation and harmonic analysis in his extraordinary book [4], model sets have become an important tool in the mathematical study of aperiodic order and quasicrystals.

Most often in applications, the setting for model sets has been real Euclidean spaces, but, as is already evident in [4], the extension to σ-compact, locally compact Abelian groups is natural, and that is the setting that we adopt here.

Unfortunately two of the best results in the subject, the property of uniform distribution of the points of a model set and the property of their being pure point diffractive have depended on the requirement of the window having boundary of (Haar) measure 0. The restriction would not be so bad, except that it is not a property that we know how to identify on the physical side of the picture, and it can be hard to verify in practice.

Here we derive a result proving that the properties of uniform distribution and pure point diffractivity hold for arbitrary relatively compact and measurable windows almost surely, in a measure theoretical sense to be explained below, (Theorem 1 and the Corollary to Proposition 4). Furthermore we recover the two previously known results when the boundary is of measure 0 and the resulting proof seems conceptually clearer.

Each model set may be viewed as a member of a family whose members are related by translation in physical space and translation of their windows. This family is

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'parameterized' by the compact group that is the quotient of the superspace and the projection lattice (the so-called *torus parameterization*). It is for this parameterized family (with the Haar measure) that we have the almost sure results.

The main result on uniform distribution is a consequence of the Birkhoff ergodic theorem. From this we derive a result that model sets for which uniform distribution holds are also ϵ-almost periodic. Their diffractivity follows then by the main result of [2].

2 Main Results

A *cut and project scheme* is a triple $(G, H, \tilde{L})$ consisting of a pair of σ-compact, locally-compact, Abelian Hausdorff topological groups G, H and a lattice $\tilde{L} \subset G \times H$. Furthermore it is assumed that the projections $\pi_G\colon G \times H \longrightarrow G$ and $\pi_H\colon G \times H \longrightarrow H$ satisfy

1. $\pi_G|_{\tilde{L}}$ is one-to-one and
2. $\pi_H(\tilde{L})$ is dense in H.

We write $L := \pi_G(\tilde{L})$, so L is a subgroup of G, and note that the mapping

$$(\ )^* := \pi_H \circ (\pi_G|_{\tilde{L}})^{-1}\colon L \longrightarrow H \tag{1}$$

has dense image in H. Using this notation, we see that $\tilde{L} = \{(u, u^*) \mid u \in L\}$. The assumption of σ-compactness shows that L is countable.

We will denote by θ and θ_H a fixed pair of Haar measures on G and H.

A set $\varLambda \subset G$ is a *(weak) model set* for the cut and project scheme $(G, H, \tilde{L})$ if there is a measurable and relatively compact set $W \subset H$ (called the *window*) and an $x \in G$ such that

$$\begin{aligned} \varLambda = x + \varLambda(W) &:= x + \{\pi_G(\tilde{u}) \mid \tilde{u} \in \tilde{L}, \pi_H(\tilde{u}) \in W\} \\ &= x + \{u \in L \mid u^* \in W\}. \end{aligned} \tag{2}$$

If, in addition, W has non-empty interior, then $\varLambda$ is a called a *(full) model set* or *cut and project set*. In this case $\varLambda$ is relatively dense in G [4], [5].

In this paper we will use the term model set for either weak or full model sets.

Fix any non-empty measurable and relatively compact subset W of H. Associated with W we have an entire family of model sets $\mathfrak{M} = \mathfrak{M}(G, H, \tilde{L}, W)$ of the form $\varLambda(x, y) := x + \{u \in L \mid u^* \in -y + W\}$, where (x, y) runs over $G \times H$. Two such sets $\varLambda(x, y)$ and $\varLambda(x', y')$ are equal if (x, y) and (x', y') are congruent modulo $\tilde{L}$.[1] Thus we have a type of parameterization of $\mathfrak{M}$ via $\mathbb{T} := (G \times H)/\tilde{L}$ and it is appropriate to write $\varLambda(\xi) = \varLambda(\xi, W)$, where $\xi := (x, y) + \tilde{L}$, instead of $\varLambda(x, y)$. This parameterization is often called the *torus parameterization*, after [1] where it was first introduced. We supply $\mathbb{T}$, which is a compact Abelian group, with Haar measure μ. We assume that the Haar measures on G and H are normalized so that the canonical measure defined on $\mathbb{T}$ by them makes μ a probability measure, *i.e.*, fundamental regions have volume 1.

[1] If in addition W is the closure of its interior and H has no non-trivial compact subgroups then this condition is also necessary for equality.

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Proposition 1 *The natural action of G on $\mathbb{T}$ through*

$$t + \big((x, y) + \tilde{L}\big) = (t + x, y) + \tilde{L}$$

defines a minimal and uniquely ergodic (that is, strictly ergodic) dynamical system.

Proof To show that every G-orbit is dense, it is sufficient to show that the orbit of 0 is dense. However, $(G, 0) + \tilde{L} \supset (0, L^*)$ and, since L^* is dense in H, $\overline{(G, 0) + \tilde{L}} \supset (0, H)$. Now it follows that $\overline{(G, 0) + \tilde{L}} = G \times H$.

From the minimality, it follows that any G-invariant measure is also $\mathbb{T}$-invariant, and hence a Haar measure (see, for example, [11, Theorem 6.20]). ■

It is well-known that model sets are uniformly discrete. Here we have a uniform version of this result:

Proposition 2 *Let W be a relatively compact neighbourhood of $\{0\}$ in H. Then there exists a compact neighbourhood V of $\{0\}$ in G so that for all $(x, y) \in G \times H$, the sets $z + V$, $z \in \Lambda(x, y)$ are all mutually disjoint.*

Proof We may, replacing W by its closure if necessary, assume that W is compact. Let U be a compact neighbourhood of $\{0\}$ in G. Since $U \times (W - W)$ is compact, $\big(U \times (W - W)\big) \cap \tilde{L}$ is finite, and we may assume, reducing U if necessary, that in fact $\big(U \times (W - W)\big) \cap \tilde{L} = \{(0, 0)\}$. Let V be a compact neighbourhood of $\{0\}$ in G so that $V - V \subset U$.

Now we observe that if, for some $(x, y) \in G \times H$ and $z_1, z_2 \in \Lambda(x, y)$ we have $(z_1 + V) \cap (z_2 + V) \neq \varnothing$, then the preimage of $z_1 - z_2$ in $\tilde{L}$ lies in $\big((V - V) \times (W - W)\big) \cap \tilde{L}$, and so $z_1 = z_2$. ■

In order to talk about uniform distribution we need to have an appropriate averaging sequence. First, a definition: if A and K are compact subsets of G then the *K-boundary* of A is

$$\partial^K A := (K + A) \setminus A^\circ \cup \big((-K + \overline{G \setminus A}) \cap A\big). \tag{3}$$

Ignoring the interior and closure operators in this definition, which assure that $\partial^K A$ is compact, the K-boundary of A consists of the points that have fallen out of A by adding points of K and the points of A that have come into A from outside by adding points of K. For empty K, $\partial^K A$ is the usual boundary of A and for symmetric K (that is, $K = -K$) it contains $\bigcup_{k \in K} A \Delta (k + A)$.

A sequence $\mathcal{A} = \{A_n\}$ of compact subsets of G is called a *van Hove sequence* if $\theta(A_n) > 0$ for all n and

$$\lim_{n \to \infty} \frac{\theta(\partial^K A_n)}{\theta(A_n)} = 0. \tag{4}$$

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In this paper, $\mathcal{A}$ will be assumed to be a fixed van Hove sequence with the additional property that there is a positive constant C so that for all n,

$$\theta(A_n - A_n) \leq C\theta(A_n). \tag{5}$$

Sequences with these two properties can be shown to exist by using the structure theorem for σ-compact locally compact Abelian groups (for details see [7], [9]).

We will assume that such a sequence $\mathcal{A}$ is fixed once and for all, with respect to which all averaging throughout the paper will be taken.

Theorem 1 *Let $W \subset H$ be a measurable relatively compact set. Then*

$$\lim_{n\to\infty} \frac{1}{\theta(A_n)} \operatorname{card}\{\Lambda(\xi) \cap A_n\} = \theta_H(W) \tag{6}$$

for $\xi \in \mathbb{T}$ μ-almost surely. If the boundary of W has Haar measure 0 *then the result holds for all $\xi \in \mathbb{T}$.*

Proof Choose a compact neighbourhood V of $\{0\}$ in G satisfying the disjointness condition of Prop. 2. Let B be a compact neighbourhood of $\{0\}$ in G with $B \subset V$ and $\Lambda(W - W) \cap B = \{0\}$. Define $\Omega := (B \times W) + \tilde{L}/\tilde{L} \subset \mathbb{T}$. Then the quotient mapping $G \times H \longrightarrow \mathbb{T}$ is injective on the measurable set $B \times W$ and it follows from the extended Weil formula [6, Theorem 3.4.6] that $\mu(\Omega) = \theta(B)\theta_H(W)$.

Let ξ be represented by $(x, y) \in G \times H$. Consider the G-orbit of ξ in $\mathbb{T}$. We have $t + \xi \in \Omega$ if and only if $(t + x, y) + (u, u^*) \in B \times W$ for some $u \in L$. That is,

$$t \in -x - u + B$$
$$u^* \in -y + W.$$

The latter is equivalent to $u \in \Lambda(-y + W)$ and so we have

$$t + \xi \in \Omega \Longleftrightarrow t \in -x - \Lambda(-y + W) + B. \tag{7}$$

This means that the part P of the orbit falling into Ω comes from points in G that lie in B-neighbourhoods of the points of the weak model set $-x - \Lambda(-y + W)$, which, in view of the choice of B, are all disjoint. The measure of $P \cap A_n$ is thus a number lying between the total measure of those B-neighbourhoods of points of $-x - \Lambda(-y + W)$ which are contained entirely in A_n and those whose B-neighbourhoods simply intersect A_n. The difference between the two sets of points of $-x-\Lambda(-y+W)$ lies in $\partial^B A_n$.

Let $\mathbf{1}_\Omega$ be the indicator function on Ω. Then, since $\mathcal{A}$ is van Hove,

(8)
$$\lim_{n\to\infty} \frac{1}{\theta(A_n)} \int_{A_n} 1_\Omega(t + \xi)\, d\theta(t) = \lim_{n\to\infty} \frac{\theta(B)}{\theta(A_n)} \operatorname{card}\big\{ \big(-x - \Lambda(-y + W)\big) \cap A_n \big\}$$

(if the limit exists).

On the other hand, by the Birkhoff ergodic theorem (see [3]—this is where the property (5) is used),

$$\lim_{n\to\infty} \frac{1}{\theta(A_n)} \int_{A_n} 1_\Omega(t+\xi)\, d\theta(t) = \int_{\mathbb{T}} 1_\Omega\, d\mu = \theta(B)\theta_H(W)$$

for $\xi \in \mathbb{T}$, μ-almost surely. Comparing, we obtain the first part of the theorem.

Suppose that the boundary of W has measure 0. We can certainly assume the same of B, whence also Ω has boundary of measure 0. Then families of compactly supported continuous functions f_i, g_i with $f_i \le 1_\Omega \le g_i$ and converging in the L^1-norm to 1_Ω can found. Since $\mathbb{T}$ is uniquely ergodic, the conclusion of the Birkhoff theorem applies to these functions for all ξ. The proof of this well-known fact depends only on the van Hove property of the averaging sequence, see for example [11]. Thus we can bound the left hand side of (8) between two quantities converging to $\theta(B)\theta_H(W)$. ∎

We continue to assume that we have a relatively compact and measurable window W. Let $\xi \in \mathbb{T}$ and let $z \in G$. The $\mathcal{A}$-autocorrelation coefficient $\eta(z,\xi)$ of $\Lambda(\xi)$ is defined by

$$\eta(z) = \eta(z,\xi) = \lim_{n\to\infty} \frac{1}{\theta(A_n)} \sum_{u,v\in\Lambda(\xi)\cap A_n, u-v=z} 1, \tag{9}$$

if this limit exists. Clearly $0 \le \eta(z) \le \eta(0) = \operatorname{den}\big(\Lambda(\xi)\big)$ and $\eta(z) = 0$ unless $z \in L$.

Proposition 3 *The autocorrelation coefficient $\eta(z,\xi)$ exists and is equal to $\theta_H\big(W \cap (z^* + W)\big)$ for all $z \in L$ for $\xi \in \mathbb{T}$, μ-almost surely.*

Proof Let $\xi = (x,y) + \tilde{L}$. Let $z \in L$ and let $u, v \in G$ be any two points satisfying $u - v = z$. We have

$$\begin{aligned} u, v \in x + \Lambda(-y+W) &\iff (u-x)^* \in (-y+W) \cap (z^* - y + W) \\ &\iff u \in x + \Lambda\Big(-y + \big(W \cap (z^* + W)\big)\Big). \end{aligned}$$

Notice that the sets being counted in

$$\lim_{n\to\infty} \frac{1}{\theta(A_n)} \operatorname{card}\Big\{ x + \Lambda\Big(-y + \big(W \cap (z^* + W)\big)\Big) \cap A_n \Big\}$$

differ from the corresponding sets of the definition (9) of the autocorrelation coefficient at z only on the set $A_n \Delta (-z + A_n)$ whose contribution is negligible as $n \to \infty$. However, by Theorem 1, the former is $\theta_H\big(W \cap (z^* + W)\big)$ for $\xi \in \mathbb{T}$ almost surely. Since L is countable, this is true for all $z \in L$ simultaneously. ∎

An element $x \in G$ is called an ϵ-*almost period* of $\Lambda(\xi)$ if $\eta(0) - \eta(x) < \epsilon$. We say that $\Lambda(\xi)$ is ϵ-*almost periodic* if for all $\epsilon > 0$ the set of ϵ-almost periods is relatively dense in G. If ϵ is small enough and η is not everywhere 0, the almost periods of ω are in L.

Proposition 4 *The model set $\Lambda(\xi, W)$ is almost ϵ-periodic, for μ-almost all $\xi \in \mathbb{T}$. If W has boundary of measure 0 it is true without restriction on ξ.*

Proof Except for $\xi \in \mathbb{T}$ on a set of measure 0, we have, for all $z \in L$,

$$\begin{aligned}\eta(0,\xi) - \eta(z,\xi) &= \theta_H(W) - \theta_H\big(W \cap (z^* + W)\big) \\ &= \theta_H(W) - 1_W * \widetilde{1_W}(z^*).\end{aligned}$$

Now, $\theta_H(W) - 1_W * \widetilde{1_W}(u)$ is uniformly continuous in u and vanishes at 0 [7], and so, given any $\epsilon > 0$, there exists an open neighbourhood V of 0 in H such that $z \in \Lambda(\xi, V) \Rightarrow \eta(0) - \eta(z) < \epsilon$. Since $\Lambda(\xi, V)$ is relatively dense and consists of ϵ-almost periods, we are done. ■

Since $\Lambda(\xi) - \Lambda(\xi)$ is contained in the model set $\Lambda(W - W)$ which is uniformly discrete, we can define, almost surely, a translation bounded measure, the *autocorrelation measure*, by $\gamma_\xi := \sum_{z \in L} \eta(z,\xi)\delta_z$, for $\Lambda(\xi)$. The diffraction of $\Lambda_(\xi)$ is, by definition, the Fourier transform of γ_ξ. The diffraction is thus another measure, and it is a positive measure because the autocorrelation measure is positive-definite. The set $\Lambda(\xi)$ is said to be *pure point* (resp. *singular continuous, absolutely continuous*) if the diffraction measure has the corresponding property.

Corollary 1 *Let $W \subset H$ be a measurable relatively compact set. Then $\Lambda(\xi, W)$ is a pure point diffractive set (with diffraction measure independent of ξ) for μ-almost all $\xi \in \mathbb{T}$. If W has boundary of measure 0 this holds for all $\xi \in \mathbb{T}$ without restriction.*

Proof Referring to [2] we need only verify the four axioms listed there:

- $\sum_{x \in \Lambda(\xi, W)} \delta_x$ is translation bounded;
- the autocorrelation exists for all z;
- $\Lambda(\xi, W) - \Lambda(\xi, W)$ is uniformly discrete;
- $\Lambda(\xi)$ is ϵ-almost periodic. ■

3 Comments

The uniform distribution property in the context of general model sets was proved in [8], but depends on the Riemann integral, and so requires boundaries of measure 0.

When the boundary of the window does not have measure 0 there is a certain paradoxical nature to the uniform distribution property. This is best illustrated by an example. Consider a cut and project scheme in which the physical and internal spaces are both the real line. Let $I := [0, 1]$ in internal space and let $W \subset I$ be an open set of measure strictly between 0 and 1 whose closure is I. Such a set can be constructed by choosing a sequence of positive numbers λ_i, $i = 0, 1, \ldots$ satisfying $\sum_{i=0}^{\infty} 2^i \lambda_i = w < 1$. One can then construct a generalized Cantor set, removing from the middle of I an open interval of length λ_0, then removing open intervals of length λ_1 from the middles of the pair of remaining pair of closed intervals, then open intervals of length λ_2 from the remaining 4 intervals, and so on. Let W denote the union of the removed intervals and B its complement in I. Now I is the closure of W, B is the boundary of W, and W has Lebesgue measure w.

Consider the model sets $\Lambda(x+I)$ as x varies over $\mathbb{R}$. These model sets satisfy the uniform distribution property and all have density equal to 1. On the other hand, $\Lambda(x+W)$ is a model set whose Lebesgue density μ-almost surely exists and equals the Lebesgue measure of W, namely w. Using the Baire category theorem, it is easy to see that the set of x for which the boundary of $x+W$ contains points from the projection of the lattice of the cut and project scheme is a meagre set (since the lattice is a countable set). So for a set of x of the *second category*, $x+W$ has *no* boundary points which are projections of lattice points. For such x, it follows that $\Lambda(x+W) = \Lambda(x+I)$ and thus has density 1, instead of the μ-almost sure density w. Of course one is comparing measures on $\mathbb{T}$ and on the internal space, and in any case second category sets can have Lebesgue measure 0, but still the result seems counter-intuitive.

The boundary of measure 0 version of the diffraction theorem was originally proved by M. Schlottmann in [9]. His original proof also depends on the Birkhoff ergodic theorem, but proceeds via the dynamical spectrum of the model set. Another approach to the pure point diffraction through dynamical systems and connections with almost periodicity is to be found in Solomyak's paper [10].

In [2] we give a proof of the pure point nature of the diffraction via almost periodicity for the boundary of measure 0 case based on the same argument as here, except using the Weyl theorem on uniform distribution.

But when the wind falls,
The trader sinks with his ship

— from the *Ashtavakra Gita*, tr. Thomas Byrom

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