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RANGE SETS AND BMO NORMS OF ANALYTIC FUNCTIONS

SHOJI KOBAYASHI

1. Introduction. In this paper we are concerned with the space BMOA of analytic functions of bounded mean oscillation for Riemann surfaces, and it is shown that for any analytic function on a Riemann surface the area of its range set bounds the square of its BMO norm, from which it is seen as an immediate corollary that the space BMOA includes the space AD of analytic functions with finite Dirichlet integrals.

Let R be an open Riemann surface which possesses a Green's function, i.e., $R \notin O_G$, and f be an analytic function defined on R. The *Dirichlet integral* $D_R(f) = D(f)$ of f on R is defined by

$$(1.1) \quad D_R(f) = D(f) = \frac{1}{\pi} \iint_R |f'(z)|^2 dxdy,$$

and we denote by $\mathrm{AD}(R)$ the space of all functions f analytic on R for which $D(f) < +\infty$. We denote by $\pi A(f)$ the area of the range set $f(R)$ of f, that is,

$$(1.2) \quad A_R(f) = A(f) = \frac{1}{\pi} \text{ Area } \{f(R)\}.$$

Then it is trivial that

$$(1.2) \quad A(f) \leqq D(f)$$

and that equality holds in (1.2) if and only if f is univalent on R.

Let U denote the unit disc and T the unit circle in the complex plane $\mathbf{C}$. For $0 < p < \infty$, the *Hardy class* $H_p(U)$ is the space of functions f analytic in U whose H_p norm

$$(1.3) \quad \|f\|_p = \sup_{0<r<1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p}$$

is finite, We denote by $\mathrm{BMOA}(U)$ the space of analytic functions in U which are of class $H_1(U)$ and whose boundary values belong to the space $\mathrm{BMO}(T)$ of John and Nirenberg [**10**]. The space $\mathrm{BMOA}(U)$ has been investigated by many people (e.g. [**2**, **3**, **4**]) and noticed to be defined in an

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equivalent way which makes it conformally invariant. More precisely, it is known that for a function f analytic in U the following are equivalent (see [**1**] or [**6**]):

(a) $f \in \mathrm{BMOA}(U)$;

(b) $\sup_{a \in U} \|f_a\|_p < +\infty$, $0 < p < \infty$, where

$$f_a(z) = f\left(\frac{z + a}{1 + \bar{a}z}\right) - f(a);$$

(c) There exist functions f_1 and f_2 analytic in U with bounded real parts such that $f = f_1 + if_2$;

(d) $\displaystyle \sup_{a \in U} \iint_U |f'(z)|^2 \log \left|\frac{1 - \bar{a}z}{z - a}\right| dxdy < +\infty.$

Following [**13**], we define BMOA for Riemann surfaces by a condition similar to (d), that is, we denote by $\mathrm{BMOA}(R)$ the space of functions f analytic on R for which

$$(1.4) \quad B_R(f) = B(f) = \sup_{a \in R} \frac{2}{\pi} \iint_R |f'(z)|^2 g(z, a) dxdy < +\infty,$$

where $g(z, a)$ denotes a Green's function of R with logarithmic singularity at a. We call $B(f)^{1/2}$ the BMO *norm* of f on R.

Note that the condition (c) is not equivalent to (1.4) unless R is simply connected. As for the condition (b), however, we can consider a similar condition for general Riemann surfaces by using the least harmonic majorants, which is equivalent to (1.4) (see [**11**]).

Metzger [**13**] showed that $\mathrm{BMOA}(R)$ includes $\mathrm{AD}(R)$, analogously to the case of the unit disc. His proof, however, depends deeply on a striking result on BMOA and omitted values by Hayman and Pommerenke [**7**], and on a Merberg's theorem on Green's functions of covering surfaces. Stegenga [**15**] showed that if f is a function analytic in U then

$$(1.5) \quad B(f) \leqq cA(f)$$

holds for some constant c, as a corollary to a theorem similar to that of Hayman and Pommerenke [**7**], which he obtained independently.

In this paper we prove the following theorem, which obviously implies both the Metzger's result $\mathrm{AD}(R) \subset \mathrm{BMOA}(R)$ and the Stegenga's one (1.5).

MAIN THEOREM. $B(f) \leqq A(f)$.

COROLLARY 1. $B(f) \leqq D(f)$.

COROLLARY 2. $\mathrm{AD}(R) \subset \mathrm{BMOA}(R)$.

For the proof we use only Green's formula and Littlewood's subordination principle, so (the author thinks that) our proof is more elementary than theirs. In Section 2, we obtain an expression of the BMO norm using the least harmonic majorants, from which we see that it is invariant under the pull back by a universal covering map. In Section 3, we prove the inequality stated in Corollary 1 above, by using Green's formula. In Section 4, we prove our Main Theorem by using results which are established in Sections 2 and 3. Finally, in Section 5, we state some remarks on equality conditions of Main Theorem and Corollary 1.

2. Universal covering and BMO norm. For a function f analytic on R and $a \in R$, we denote by h_a the least harmonic majorant of the subharmonic function $u(z) = |f(z) - f(a)|^2$ on R, where, for convention, we set $h_a(z) = +\infty$ if u admits no harmonic majorants.

THEOREM 1. $B(f) = \sup_{a \in R} h_a(a)$.

Proof. The theorem is an easy consequence of the following lemma, which is proved by a similar way to that of obtaining a formula for the solution of the Dirichlet problem in terms of a Green's function (cf. [**9**, pp. 399-405]).

LEMMA 1. *For any* $a \in R$

$$(2.1) \quad h_a(a) = \frac{2}{\pi} \iint_R |f'(z)|^2 g(z, a) dxdy.$$

Proof. By considering a regular exhaustion of R, it is sufficient to prove (2.1) under the condition that R is a finite Riemann surface and that f is analytic on the closure $\bar{R}$ of R.

Let S be an interior of a compact bordered Riemann surface $\bar{S}$ and Γ be its boundary. If u and v are C^2 functions on $\bar{S}$, then Green's formula states that

$$(2.2) \quad \iint_S (v\Delta u - u\Delta v) dxdy = \int_\Gamma \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) ds,$$

where Δ denotes the Laplacian, $\dfrac{\partial}{\partial n}$ differentiation in the inner normal direction and ds arc length measure on Γ.

Apply (2.2) with $u(z) = |f(z) - f(a)|^2$ and $v(z) = g(z, a)$ in the domain obtained by deleting from R a small disc centered at a, and let the disc shrink, then, noting $\Delta u = 4|f'(z)|^2$ and $\Delta v = 0$, we obtain (2.1), since the integrals along the boundary of the small disc approach to zero and

$$\frac{1}{2\pi} \frac{\partial g(z, a)}{\partial n} ds$$

is the harmonic measure on the boundary ∂R of R at a with respect to R.

COROLLARY. 3. $f \in \mathrm{BMOA}(R)$ *if and only if* $\sup_{a \in R} h_a(a) < +\infty$.

Let $\nu: U \to R$ denote a universal covering map of R. It is known that the least harmonic majorant of a subharmonic function is preserved by ν, more precisely, that if u is a function subharmonic on R with the least harmonic majorant h, then $h \circ \nu$ coincides with the least harmonic majorant of $u \circ \nu$ in U (see [**12**, p. 316, Lemma 1]). Combining this with the theorem, we obtain the following:

COROLLARY 4. *Let* $g = f \circ \nu$, *then* $B_R(f) = B_U(g)$.

Proof. For $b \in U$ let H_b denote the least harmonic majorant of the subharmonic function $v(z) = |g(z) - g(b)|^2$ in U, then we see by the fact mentioned above that

$$H_b(z) = (h_{\nu(b)} \circ \nu)(z) \tag{2.3}$$

holds for $z \in U$, since $v(z) = (u \circ \nu)(z)$ if $a = \nu(b)$. Therefore we see by the theorem and (2.3)

$$\begin{aligned} B_R(f) &= \sup_{a \in R} h_a(a) = \sup_{b \in U} (h_{\nu(b)} \circ \nu)(b) \\ &= \sup_{b \in U} H_b(b) = B_U(g), \end{aligned}$$

as asserted.

COROLLARY. 5. $f \in \mathrm{BMOA}(R)$ *if and only if* $f \circ \nu \in \mathrm{BMOA}(U)$.

Remark. (1) Corollary 3 holds even if we set h_a to be the least harmonic majorant of $|f(z) - f(a)|^p$ for $0 < p < \infty$. This corresponds to the condition (b) for the case of the unit disc, which we mentioned in the previous section (cf. [**11**]).

(2) Corollary 4 and 5 were proved by Metzger [**13**, p. 1257, Proposition 2] by using a Myrberg's theorem on Green's functions of covering surfaces.

3. Dirichlet integral and BMO norm. In this section we prove the inequality which was stated in Corollary 1 in Section 1.

THEOREM 2. $B(f) \leqq D(f)$.

For the proof we need a lemma.

LEMMA 2. *For every positive integer* k *and any* $a \in R$,

$$\frac{k}{k+1} h_a(a) \leqq \frac{1}{\pi} \iint_R |f'(z)|^2 (1 - e^{-2kg(z,a)}) dxdy. \tag{3.1}$$

Proof. Similarly to the case for Lemma 1, we may assume that R is a finite Riemann surface and that f is analytic on $\bar{R}$. Let $g^*(z, a)$ be a harmonic conjugate of $g(z, a)$, which is locally defined up to an additive constant, and let

$$P(z) = g(z, a) + ig^*(z, a),$$

then it is easily seen that $P'(z)dz$ is a globally well-defined differential which is analytic on R except at a, where it has a simple pole. For $t > 0$, let

$$\Gamma_t = \{z \in R:g(z, a) = t\},$$

then it is known that Γ_t consists of a finite number of analytic Jordan curves. Since $g(z, a)$ is constant on Γ_t, we see that

$$(3.2) \quad iP'(z)dz = -\frac{\partial g^*(z, a)}{\partial s}ds = \frac{\partial g(z, a)}{\partial n}ds$$

along Γ_t, where ds is the arc length and $\dfrac{\partial}{\partial n}$ is the inner normal derivative with respect to the domain $R_t = \{z \in R:g(z, a) > t\}$.

We apply Green's formula (2.2) with

$$u(z) = |f(z) - f(a)|^2 \text{ and } v(z) = 1 - e^{-2kg(z,a)} \text{ in } R.$$

Simple calculations show that

$$(3.3) \quad \Delta u = 4|f'(z)|^2,$$

$$(3.4) \quad \Delta v = -4k^2 |P'(z)|^2 e^{-2kg(z,a)}$$

and that for $z \in \Gamma$

$$(3.5) \quad \frac{\partial v}{\partial n} = 2k \frac{\partial g(z, a)}{\partial n}.$$

Substituting (3.3), (3.4) and (3.5) into (2.2), we see

$$(3.6) \quad \iint_R |f'(z)|^2(1 - e^{-2kg(z,a)})dxdy + k^2 \iint_R |f(z) - f(a)|^2 |P'(z)|^2 e^{-2kg(z,a)}dxdy = \frac{k}{2} \int_{\partial R} |f(z) - f(a)|^2 \frac{\partial g(z, a)}{\partial n}ds.$$

Since

$$|f(z) - f(a)|^2 e^{2g(z,a)} = |\{f(z) - f(a)\}e^{P(z)}|^2$$

is subharmonic on R and it coincides with $h_a(z)$ on the boundary ∂R of R, we see by the maximum principle that

(3.7) $\quad |f(z) - f(a)|^2 e^{2g(z,a)} \leqq h_a(z)$

holds for any $z \in R$. Using (3.7), we can estimate the integral in the second term of (3.6), in fact, we see by (3.2) and (3.7)

$$(3.8) \quad \iint_R |f(z) - f(a)|^2 |P'(z)|^2 e^{-2kg(z,a)} dxdy$$

$$\leqq \iint_R h_a(z) e^{-2(k+1)g(z,a)} |P'(z)|^2 dxdy$$

$$= \iint_R h_a(z) e^{-2(k+1)g(z,a)} \left(\frac{\partial g(z, a)}{\partial n}\right)^2 dsdn$$

$$= \int_0^\infty \int_{\Gamma_t} h_a(z) e^{-2(k+1)t} \frac{\partial g(z, a)}{\partial n} dsdt$$

$$= 2\pi h_a(a) \int_0^\infty e^{-2(k+1)t} dt$$

$$= \frac{\pi}{k+1} h_a(a),$$

since $\dfrac{1}{2\pi} \dfrac{\partial g(z, a)}{\partial n} ds$ is the harmonic measure on Γ_t at a with respect to R_t, where we have put $t = g(z, a)$ and hence

$$dt = \frac{\partial g(z, a)}{\partial n} dn.$$

Since the right side of (3.6) coincides with $k\pi h_a(a)$, we obtain (3.1) by substituting (3.8) into (3.6). This completes the proof of Lemma 2.

By letting $k \to \infty$ in (3.1) and applying Lebesgue's monotone convergence theorem, we obtain the following corollary, which, combined with Theorem 1, gives Theorem 2.

COROLLARY 6. *For any* $a \in R$

$$h_a(a) \leqq \iint_R |f'(z)|^2 dxdy.$$

As for plane domains we obtain the following inequality, which seems to be interesting in itself.

COROLLARY 7. *If G is a plane domain with Green's function $g(z, a)$, then for any $a \in G$*

$$\iint_G g(z, a) dxdy \leqq \tfrac{1}{2} \text{Area}(G).$$

Proof. Consider the case where $R = G$ and $f(z) \equiv z$ in the theorem.

Remark. Corollary 2 immediately follows from Theorem 2, so we used only Green's formula in order to obtain $AD(R) \subset BMOA(R)$.

4. Proof of main theorem. Let G be the range set of f, i.e., $G = f(R)$, and let $\rho: U \rightarrow G$ denote a universal covering map of G.

LEMMA 3.

(4.1) $\quad B_R(f) \leqq B_U(\rho)$.

Proof. As in Section 2, let $\nu: U \rightarrow R$ denote a universal covering map of R. By considering $g = f \circ \nu$ instead of f and applying Corollary 4, it is sufficient to prove the lemma for the case where f is an analytic function defined in U. Let $a \in U$ be arbitrarily fixed. By the monodromy theorem, we can determine a single-valued branch ψ of $\rho^{-1} \circ f$ in U, so we see that ψ is a bounded analytic function in U with $|\psi(z)| \leqq 1$, $z \in U$, for which

(4.2) $\quad f = \rho \circ \psi$

holds. Set $b = \psi(a)$, then by (4.2)

(4.3) $\quad f(a) = \rho(b)$.

Let h_a be the least harmonic majorant of

$$u(z) = |f(z) - f(a)|^2$$

and H_b be that of

$$v(z) = |\rho(z) - \rho(b)|^2.$$

Since (4.2) and (4.3) mean that $u(z)$ is subordinate to $v(z)$, we see by Littlewood's subordination principle (see, e.g., [**9**, p. 421]) that

(4.4) $\quad h_a(z) \leqq (H_b \circ \psi)(z)$

holds for $z \in U$. Putting $z = a$ in (4.4), we see by Theorem 1

$$h_a(a) \leqq (H_b \circ \psi)(a) = H_b(b) \leqq B_U(\rho),$$

from which we obtain (4.1) again by Theorem 1, as asserted.

Now we are in a position where we can complete the proof of the Main Theorem. Let I denote the identity map defined in G, i.e., $I(z) = z$, $z \in G$. Applying Corollary 4 with $R = G$ and $f = I$, we see

(4.5) $\quad B_G(I) = B_U(I \circ \rho) = B_U(\rho)$.

By Lemma 3, (4.5) and Theorem 2 we see

$$B_R(f) \leqq B_U(\rho) = B_G(I) \leqq D_G(I) = A(I) = A(f),$$

which completes the proof.

5. Conjectures on equality conditions. Up to this time, we are regrettably unable to settle equality conditions for Main Theorem and Corollary 1. We conclude this paper by presenting two conjectures on these equality problems, which are suggested from our proofs.

CONJECTURE 1. *$B(f) = A(f)$ if and only if*

$$f \circ \nu = c\,\phi + d, \tag{5.1}$$

where c and d are constants and ϕ is an inner function.

CONJECTURE 2. *$B(f) = D(f)$ if and only if R is a Riemann surface which is obtained from a simply-connected W by deleting at most a set of capacity zero and f is (extended to) a conformal map of W onto a disc.*

The if parts are almost trivial for both Conjectures 1 and 2. In fact, as for Conjecture 1, a theorem of Frostman [**5**] asserts that if ϕ is an inner function, then ϕ assumes every point in U except at most a set of capacity zero, so we see $A(\phi) = 1$. On the other hand, let h_a be the least harmonic majorant of $|\phi(z) - \phi(a)|^2$, then we easily see

$$\begin{aligned} h_a(a) &= \frac{1}{2\pi}\int_0^{2\pi} \left|\phi\left(\frac{e^{i\theta} + a}{1 + \bar{a}e^{i\theta}}\right) - \phi(a)\right|^2 d\theta \\ &= 1 - |\phi(a)|^2, \end{aligned}$$

since $|\phi(e^{i\theta})| = 1$ a.e. on T. Since we can take $a \in U$ so that $|\phi(a)|$ is arbitrarily small, we see $B(\phi) = 1$, and hence $B(\phi) = A(\phi)$. Therefore, applying Corollary 4, we see that $B(f) = A(f)$ holds if f satisfies (5.1). As for Conjecture 2, if R and f are as stated in the conjecture, we easily see that $f \circ \nu$ is of form of (5.1), since the least harmonic majorant of $|z - \alpha|$ in $G = f(R)$ is constant, where α is the center of the disc. Therefore we see $B(f) = A(f)$ by the if part of Conjecture 1, and hence $B(f) = D(f)$, since the univalence of f guarantees $A(f) = D(f)$.

If Conjecture 1 is true then so is Conjecture 2. In fact, suppose that Conjecture 1 be true and that $B(f) = D(f)$ holds, then we see that f satisfies (5.1) and that f is univalent on R, since $B(f) = D(f)$ implies $B(f) = A(f) = D(f)$ by (1.3). Since a set of capacity zero is removable for H_p (see e.g. [**8**] or [**14**]), and hence so is for BMOA, we see that R and f should be as mentioned in Conjecture 2.

In any case we do not know yet whether each of the only if parts of Conjectures 1 and 2 is true or not.

Added in proof. In a paper to appear in Kodai Math. J. the author recently settled the conjectures mentioned above, the one negatively and the other affirmatively.

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Technological University of Nagaoka,
Niigata, Japan