J. Aust. Math. Soc. **77** (2004), 165–174

ON GROUPS WITH ALL SUBGROUPS ALMOST SUBNORMAL

ELOISA DETOMI

(Received 1 May 2001; revised 11 April 2003)

Communicated by R. B. Howlett

Abstract

In this paper we consider groups in which every subgroup has finite index in the nth term of its normal closure series, for a fixed integer n. We prove that such a group is the extension of a finite normal subgroup by a nilpotent group, whose class is bounded in terms of n only, provided it is either periodic or torsion-free.

2000 *Mathematics subject classification*: primary 20E15, 20F19.

A subgroup H of a group G is said to be *almost subnormal* if it has finite index in some subnormal subgroup of G. This occurs when H has finite index in some term $H^{G,n}$, $n \geq 0$, of its normal closure series in G; recall that $H^{G,0} = G$ and $H^{G,n} = H^{H^{G,n-1}}$.

A finite-by-nilpotent group has every subgroup almost subnormal, and for finitely generated groups the converse holds (see [8, 6.3.3]). Note that, if a group G has a finite normal subgroup N such that G/N is nilpotent of class n, then each subgroup H of G has finite index in $H^{G,n}$. For $n = 1$, the converse is settled by a well-known theorem of Neumann [10]: a group G, in which every subgroup H has finite index in its normal closure H^G, is finite-by-abelian. Later, Lennox [7] considered the case in which n is larger than 1 and there is also a bound on the indices. He proved that there exists a function μ such that if $|H^{G,n} : H| \leq c$ for every subgroup H of a group G, where n and c are fixed integer, then the $\mu(n + c)$-th term $\gamma_{\mu(n+c)}(G)$ of the lower central series of G is finite of order at most $c!$. Recall that a theorem by Roseblade states that a group G in which $H = H^{G,n}$ for every subgroup H, is nilpotent and $\gamma_{\rho(n)+1}(G) = 1$, for a well-defined function ρ. Recently, Casolo and Mainardis in [2, 3] gave a description of the structure of groups with all subgroups almost subnormal, proving, in particular, that such groups are finite-by-soluble.

In this paper we consider the class A_n, $n \geq 1$, of groups G in which $|H^{G,n} : H|$ is finite for every subgroup H of G, but no bound on the indices $|H^{G,n} : H|$ is assumed. In particular, we give a generalization of Neumann's theorem to periodic A_n-groups:

THEOREM 1. *There exists a function δ of n, such that if G is a torsion group with the property that $|H^{G,n} : H| < \infty$ for every subgroup H of G, then $\gamma_{\delta(n)}(G)$ is finite.*

We then consider torsion-free groups. By a result due to Casolo and Mainardis [2], torsion-free A_n-groups have every subgroup subnormal and so they turn out to be nilpotent, by a recent result by Smith [14] (see also Casolo [1]). Here, we give a different proof of their nilpotency and, in particular, a bound on their nilpotency class, thus generalizing Neumann's theorem to torsion-free A_n-groups:

THEOREM 2. *There exists a function η of n such that each torsion-free group G in which $|H^{G,n} : H| < \infty$ for every subgroup H, is nilpotent of class at most $\eta(n)$.*

This also gives a different proof of Roseblade's theorem for torsion-free groups with all subgroups subnormal of bounded defect.

Finally, we observe that Smith in [13] gives examples of A_2-groups which are not finite-by-nilpotent. Thus, Theorem 1 and Theorem 2 are no longer true if we drop the assumptions that G is either periodic or torsion-free. Also, Casolo and Mainardis, in [2], construct a non-hypercentral A_2-group. On the other hand, in Proposition 13 we shall prove that locally nilpotent A_n-groups are hypercentral, partially answering the question posed in[8, page 191]. Recall that Heineken-Mohamed groups [6] are example of groups in which every subgroups is almost subnormal but they do not belong to any of the classes A_n.

1. A_n^+-groups

In order to achieve our result on periodic A_n-groups, we find it convenient to study a larger class of groups. We denote by A_n^+ the class of all groups G in which there exists a finite subgroup F with the property that every subgroup H containing F has finite index in the nth term $H^{G,n}$ of its normal closure series. By abuse of notation, we shall denote the above by $(G, F) \in A_n^+$. Note that $A_n \subseteq A_n^+$ but $A_n \neq A_n^+$. Indeed, the group described in [4, Proposition 4] is a periodic A_2^+-group but it is not finite-by-nilpotent, and so, by Theorem 1, it does not belong to A_n.

Also, we denote by $\mathfrak{U}_n^+$ the class of all groups G in which there exists a finite subgroup F such that every subgroup of G containing F is subnormal of defect at most n in G. Clearly, $\mathfrak{U}_n^+ \subseteq A_n^+$, but $\mathfrak{U}_n^+ \neq A_n^+$, since Smith's groups [13] are locally nilpotent A_2-groups which are not finite-by-nilpotent while, for $\mathfrak{U}_n^+$-groups, the following holds:

https://doi.org/10.1017/S1446788700013549 Published online by Cambridge University Press

THEOREM 3 (Detomi [4]). *There exists a function $\beta(n)$ of n, such that if G belongs to $\mathfrak{U}_n^+$ and it is either a locally nilpotent group or a torsion group with $\pi(G)$ finite, then $\gamma_{\beta(n)}(G)$ is finite. In particular, if G is locally nilpotent, then G is nilpotent and its nilpotency class is bounded by a function depending on n and $|F|$.*

Here $\pi(G)$ denotes the set of primes dividing the orders of the elements of G.

The following are two known result which we include without proofs. If N is a subgroup (normal subgroup) with finite index in G, then we write $N \leq_f G$ ($N \trianglelefteq_f G$).

LEMMA 4. *Let G be a countable residually finite group and let H be a finite subgroup of G. Then $H = \bigcap_{N \trianglelefteq_f G} HN$.*

LEMMA 5. *Let G be a group and let F be a finitely generated subgroup of a subgroup H of G. If $[G,_n V] \leq V$ for every finitely generated subgroup V of H such that $F \leq V$, then $[G,_n H] \leq H$.*

We establish an elementary property of periodic A_n^+-groups:

LEMMA 6. *A periodic A_n^+-group is locally finite and finite-by-soluble.*

PROOF. Let $(G, F) \in A_n^+$. Then $F \leq_f F^{G,n}$ gives that $F^{G,n}$ is finite and that every section $F^{G,i}/F^{G,i+1}$ belongs to A_n. Since, by the already mentioned result by Casolo-Mainardis, every A_n-group is finite-by-soluble, the group G has a finite series in which each factor is finite or soluble.

Let X be a finitely generated subgroup of G. Clearly X has a finite series with finite or soluble factors. Hence, since a finitely generated torsion soluble group is finite and a subgroup with finite index in a finitely generated group is finitely generated, each factor in this series of X is finite, and so X is finite. This proves that G is locally nilpotent.

Now, since G has a finite series with finite or soluble factors, to prove that G is finite-by-soluble, it is sufficient to show that soluble-by-finite periodic A_n^+-groups are finite-by-soluble.

Let $(G, F) \in A_n^+$ be a torsion group and let A be a soluble normal subgroup with finite index in G. We can assume that $A \trianglelefteq G$, since A_G has finite index in G. Let τ be a left transversal to A in G and set $H = \langle \tau, F \rangle$. As H has finite index in $K = H^{G,n}$, K is finitely generated and hence finite, by the local finiteness of G. Note that $G = AK$.

We proceed by induction on the defect d of subnormality of K in G. If K is normal in G, then $G/K \cong A/A \cap K$ is soluble, and we are done. If $d > 1$, then, as K has defect of subnormality bounded by $d-1$ in K^G, we can apply the induction hypothesis to K^G, obtaining that some term of the derived series of K^G is finite (and normal in G). Therefore, as $G/K^G \cong A/A \cap K^G$ is soluble, we get that G is finite-by-soluble, which is the desired conclusion. □

https://doi.org/10.1017/S1446788700013549 Published online by Cambridge University Press

With the same argument as in [4, Lemma 9], it is easy to see that:

LEMMA 7. *Let $G \in A_n^+$ be a locally finite group. If there exists a subgroup A with finite index in G such that $\gamma_{m+1}(A)$ is finite, then $\gamma_{nm+1}(G)$ is finite.*

Roughly speaking, the next proposition says that periodic A_n^+-groups are near to being $\mathfrak{U}_n^+$-groups.

PROPOSITION 8. *Let G be a countable residually finite torsion group and let $G \in A_n^+$. Then there exists a subgroup A with finite index in G such that $A \in \mathfrak{U}_n^+$.*

PROOF. Assume that the lemma is false and let G be a counterexample. Proceeding recursively we construct

(a) a descending chain $\{K_i \mid i \in \mathbb{N}\}$ of subgroups with finite index in G,
(b) an ascending chain $\{F_i \mid i \in \mathbb{N}\}$ of finitely generated subgroups of $\bigcap_{i=0}^{\infty} K_i$, and
(c) a sequence of elements $\{x_i \in [K_{i-1},_n F_i] \backslash K_i \mid 1 \leq i \in \mathbb{N}\}$.

Set $K_0 = G$ and let F_0 be a finite subgroup of G such that $|H^{G,n} : H| < \infty$ whenever $F_0 \leq H \leq G$.

Suppose we have already defined F_i, K_i, and $x_i \in [K_{i-1},_n F_i] \backslash K_i$. As F_i is a finitely generated subgroup of $K_i \leq_f G$, and as G is a counterexample, there exists a subgroup $F_i \leq H \leq K_i$ which is not subnormal of defect less or equal to n in K_i, that is, $[K_i,_n H] \nleq H$. So, by Lemma 5, there exists a finitely generated subgroup F_{i+1} of H with $F_i \leq F_{i+1}$ and $[K_i,_n F_{i+1}] \nleq F_{i+1}$. Let us fix an element $x_{i+1} \in [K_i,_n F_{i+1}] \backslash F_{i+1}$. Since, by Lemma 6, G is locally finite, we can apply Lemma 4 to the finitely generated, hence finite subgroup F_{i+1}, and so we get that $x_{i+1} \notin F_{i+1}N$ for a suitable subgroup $N \trianglelefteq_f K_i$. Then we set $K_{i+1} = F_{i+1}N$, so that $F_{i+1} \leq K_{i+1} \leq_f G$ and $x_{i+1} \in [K_i,_n F_{i+1}] \backslash K_{i+1}$. Note that K_{i+1} contains all the subgroups $F_0, \ldots, F_{i+1}$.

Now we consider the subgroups $K = \bigcap_{i \in \mathbb{N}} K_i$ and $H = \langle F_i \mid i \in \mathbb{N} \rangle$. Since $H \geq F_0$, by assumption we have that H has finite index in $H^{G,n}$. So, the chain $\{H^{G,n} \cap K_i\}_{i \in \mathbb{N}}$, stretching from $H^{G,n}$ to H, is finite and there exists an integer i such that $H^{G,n} \cap K_i = H^{G,n} \cap K_j$ for every $j \geq i$. But, since $[G,_n H] \leq H^{G,n}$ and $F_{i+1} \leq H \cap K_i$, we get that

$$x_{i+1} \in [K_i,_n F_{i+1}] \leq [K_i,_n H \cap K_i] \leq [G,_n H] \cap K_i$$
$$\leq H^{G,n} \cap K_i = H^{G,n} \cap K_{i+1},$$

that is $x_{i+1} \in K_{i+1}$, in contradiction to our construction. □

THEOREM 9. *There exists a function $\delta(n)$ of n, such that if G is a periodic A_n^+-group and if either G is locally nilpotent or $\pi(G)$ is finite, then $\gamma_{\delta(n)}(G)$ is finite. In particular, if G is locally nilpotent then G is nilpotent.*

PROOF. Set $\delta(1) = 2$ and define recursively $\delta(n) = 2n(\beta(n) - 1) + 2\delta(n-1) + 1$, where β is the function defined in Theorem 3.

Assume first that G is countable. We shall proceed by induction on n. Let F be a finite subgroup of G such that every subgroup H containing F has finite index in $H^{G,n}$.

If $n = 1$ then $|F^G : F| < \infty$ and F^G is finite. Since $G/F^G \in A_1$, the quotient $G'F^G/F^G$ is finite by Neumann's theorem. Hence $G' = \gamma_2(G)$ is finite.

Let now $n > 1$ and let X be a finitely generated subgroup of G with $X \geq F$. Because G is locally finite, X is finite. Observe that, for every subgroup H of X^G containing X, we have $H^G = X^G$ and so $|H^{X^G,n-1} : H| < \infty$. Thus X^G belongs to A^+_{n-1} and by the inductive hypothesis we get that $\gamma_{\delta(n-1)}(X^G)$ is finite. Now, by a theorem of Hall it follows that $\zeta_{2\delta(n-1)-2}(X^G)$ has finite index in X^G. Thus, the index of $C_G(X^G/\zeta_{2\delta(n-1)-2}(X^G))$ in G is finite and, denoting by $R = \bigcap_{N \trianglelefteq_f G} N$ the finite residual of G, we obtain that $[R, X^G] \leq \zeta_{2\delta(n-1)-2}(X^G)$. In particular,

$$[R,_{2\delta(n-1)} X^G] \leq \left[R, X^G,_{2\delta(n-1)-2} X^G\right] = 1.$$

Therefore, if we take $s = 2\delta(n-1)$ elements in G, say $x_1, \dots, x_s$, and we consider the finitely generated subgroup $X = \langle x_1, \dots, x_s, F\rangle$, then we get $[R, x_1, \dots, x_s] \leq [R,_s X^G] = 1$, which implies $R \leq \zeta_s(G)$.

Now, as $G/R \in A^+_n$ is a countable residually finite torsion group, by Proposition 8 it follows that there exists a subgroup A with finite index in G, such that $A/R \in \mathfrak{U}^+_n$. Also, A/R satisfies the assumptions of Theorem 3 and so $\gamma_{\beta(n)}(A/R)$ is finite. By Lemma 7 it follows that $\gamma_{n(\beta(n)-1)+1}(G/R)$ is finite and then Hall's theorem gives that $\zeta_{2n(\beta(n)-1)}(G/R)$ has finite index in G/R. Therefore, as $R \leq \zeta_s(G)$, clearly $\zeta_{2n(\beta(n)-1)+s}(G)$ has finite index in G and, by a theorem of Baer (see [12, 14.5.1]), we conclude that $\gamma_{2n(\beta(n)-1)+s+1}(G)$ is finite. This proves that $\gamma_{\delta(n)}(G)$ is finite, for every countable group G satisfying the assumption of the theorem.

For the general case, we assume, contrary to our claim, that there exists a group G, satisfying the assumption of the theorem, such that $\gamma_{\delta(n)}(G)$ is not finite.

Let T be a countable and not finite subset of $\gamma_{\delta(n)}(G)$. Then we can find a countable set of commutators $x_i = [y_{1,i}, \dots, y_{\delta(n),i}]$, $i \in \mathbb{N}$, $y_{j,i} \in G$, such that $T \leq \langle x_i \mid i \in \mathbb{N}\rangle$. Let $Y = \langle F, y_{j,i} \mid j = 1, \dots, \delta(n),\ i \in \mathbb{N}\rangle$. As Y is a countable A^+_n-group, by the first part of the proof, $\gamma_{\delta(n)}(Y)$ is finite. Thus $T \subseteq \gamma_{\delta(n)}(Y)$ is finite, against our assumption.

Finally, if G is locally nilpotent, since every finite normal subgroup is contained in some term of the upper central series (by a theorem of Mal'cev and McLain [12, 12.1.6]), it follows that G is nilpotent, and the proof is complete. □

As a consequence, we get the announced result on periodic A_n-groups:

PROOF OF THEOREM 1. Let G be a periodic A_n-group. By a result of Casolo and

https://doi.org/10.1017/S1446788700013549 Published online by Cambridge University Press

Mainardis [3], there exists a finite normal subgroup N of G such that G/N has every subgroup subnormal. In particular, G/N is locally nilpotent. Now Theorem 9 gives that $\gamma_{\delta(n)}(G/N)$ is finite and, as N is finite, the result follows. □

2. Torsion-free A_n-groups

First we observe some basic properties of isolators in locally nilpotent groups. Recall that the *isolator* of a subgroup H in a group G is defined to be the set $I_G(H) = \{x \in G \mid x^n \in H \text{ for some } 1 \leq n \in \mathbb{N}\}$. If G is a locally nilpotent group then $I_G(H)$ is a subgroup of G and if G is also torsion-free then $\gamma_n(I_G(H)) \leq I_G(\gamma_n(H))$ (see, for example, [5, 9]).

LEMMA 10. *Let G be a locally nilpotent group and let $H \leq G$. Then*

(1) *if $I_G(H)$ is finitely generated, then $|I_G(H) : H| < \infty$;*
(2) *if G is torsion-free and H is cyclic, then $I_G(H)$ is locally cyclic.*

PROOF. (1) As $K = I_G(H)$ is a finitely generated nilpotent group, H is subnormal in K, say $H = H^{K,n}$ for an integer n, and every section $H^{K,i}/H^{K,i+1}$ is finitely generated and nilpotent, for $i = 1, \dots, n-1$. Furthermore, by definition of $I_G(H)$, each $H^{K,i}/H^{K,i+1}$ is periodic and hence finite. Thus, H has finite index in K.

(2) Let K be a finitely generated subgroup of $I_G(H)$. As H is cyclic, we can assume that $H \leq K$. Since K is torsion-free and nilpotent, it has a central series with infinite cyclic factors (see [12, 5.2.20]). So, if K is not cyclic, there is a cyclic normal subgroup N of K with infinite index in K. Now, since, by (1), H has finite index in K, then $H \cap N \neq 1$. Therefore, as H is cyclic, $|K/N| \leq |NH/N| = |H/H \cap N|$ is finite, a contradiction. □

We state now a consequence of a well-known argument by Robinson (see [12, 5.2.5]). Recall that the Hirsch length of a polycyclic group G is the number of infinite factors in a series of G with cyclic factors.

LEMMA 11. *Let H be a nilpotent group of class c. If H/H' can be generated by r elements, then the Hirsch length h of H is bounded by a function $g(c, r)$ of c and r.*

The already mentioned theorem of Mal'cev and McLain [12, 12.1.6] states that each principal factor of a locally nilpotent group is central. The following consequence is well known, but we include the easy proof for the convenience of the reader:

LEMMA 12. *Let G be a locally nilpotent group and let N be a finitely generated normal subgroup of G. Then there exists an integer n such that $N \leq \zeta_n(G)$. Moreover, if N is torsion-free with Hirsch length h, then $N \leq \zeta_h(G)$.*

PROOF. The theorem of Mal'cev and McLain implies that if N is finite then it is contained in $\zeta_m(G)$ for an integer m bounded by the composition length of N. Also, when N is torsion-free with Hirsch length h, we get that N/N^p is finite and so $N/N^p \leq \zeta_h(G/N^p)$ for every prime p; therefore $[N,_h G] \leq \bigcap_p N^p = 1$ by a residual property of torsion-free finitely generated nilpotent groups (see for example [11, page 170]). Since the torsion subgroup of a finitely generated normal subgroup of G is finite, the lemma follows. □

PROPOSITION 13. *Let G be a locally nilpotent A_n-group. Then G is hypercentral.*

PROOF. By an already cited result of Casolo and Mainardis, A_n-groups are finite-by-soluble and so G is soluble. It is sufficient to prove that G has a non trivial centre. We proceed by induction on the derived length of G. Let A be the centre of G'; by inductive assumption, $A \neq 1$. Let H be a finitely generated subgroup of G. As $|H^{G,n} : H|$ is finite, $H^{G,n}$ is finitely generated and so nilpotent; in particular, $[A,_n H]$ is finitely generated. Since $A = \zeta(G')$, $[A,_n H]^g = [A,_n H^g] \leq [A,_n H[H, \langle g\rangle]] = [A,_n H]$ for $g \in G$, and so $[A,_n H]$ is normal in G. Thus Proposition 12 gives that $[A,_n H] \leq \zeta_k(G)$ for some $k \geq 1$. So, if $[A,_n H] \neq 1$, then $\zeta(G) \neq 1$. Otherwise, $[A,_n H] = 1$ for any finitely generated subgroup of G; thus $A \leq \zeta_n(G)$ and we again conclude that $\zeta(G) \neq 1$. □

A group G is said *n-Engel* if $[x,_n y] = 1$ for all $x, y \in G$. We recall that a torsion-free soluble n-Engel group G with positive derived length d is nilpotent of class at most n^{d-1} (see [11, 7.36]).

Our interest on Engel groups is motivated by the following fact:

LEMMA 14. *A torsion-free A_n-group is $(n+1)$-Engel.*

PROOF. Let G be a torsion-free A_n-group and let $1 \neq x \in G$. By the definition of the class A_n, $\langle x\rangle$ has finite index in $\langle x\rangle^{G,n}$, so that $\langle x\rangle^{G,n}$ is a finitely generated subgroup of $I_G(\langle x\rangle)$. By the already mentioned result in [2], every subgroup of G is subnormal, so that G is locally nilpotent. Thus, by Lemma 10, $\langle x\rangle^{G,n}$ is cyclic, so that $\langle x\rangle$ char $\langle x\rangle^{G,n}$, and hence $\langle x\rangle$ is subnormal of defect at most n in G, that is $[G,_n x] \leq \langle x\rangle$. Therefore, $[G,_{n+1} x] = [G,_n x, x] = 1$, as claimed. □

Now we are in a position to prove the announced result on torsion-free A_n-groups.

PROOF OF THEOREM 2. Let $G \in A_n$ be a torsion-free group. As already noted, by a result in [2], G is locally nilpotent.

Note that, if there exists a function $\eta(n)$ such that $\gamma_{\eta(n)+1}(H) = 1$, for every finitely generated subgroup H of G, then $\gamma_{\eta(n)+1}(G) = 1$. Hence, without loss of generality, we can assume that G is a finitely generated group. In particular, we get that G is nilpotent and every subgroup of G is finitely generated.

Proceeding by induction on n, we prove that there exists a function $\eta(n)$ such that every torsion-free finitely generated A_n-group has nilpotency class at most $\eta(n)$.

If $n = 1$, then Neumann's theorem gives that G' is finite. Hence, since G is torsion-free, G is abelian, and so we can set $\eta(1) = 1$.

Let now $n > 1$ and let H be a subgroup of G. Set $H^{G,i} = H_i$ for every i, so that, by the definition of the class A_n, we have

$$H \leq_f H_n \trianglelefteq H_{n-1} \trianglelefteq \cdots \trianglelefteq H_2 \trianglelefteq H_1 \trianglelefteq G.$$

Note that, for every subgroup K such that $H \leq K \leq H_1$, we get $K^G = H^G = H_1$ and $K \leq_f K^{K^G,n-1}$. Hence $H_1/H_2 \in A_{n-1}$. With the same argument it is easy to see that the factor H_i/H_{i+1}, for $i = 1, \dots, n-1$, belongs to A_{n-i}. By the induction hypothesis, the factor $H_i/I_{H_i}(H_{i+1})$, being a finitely generated torsion-free A_{n-i}-group, has nilpotency class at most $\eta(n-i)$; hence,

$$\gamma_{\eta(n-i)+1}(H_i) \leq I_{H_i}(H_{i+1}) \leq I_G(H_{i+1}).$$

Thus,

$$\gamma_{\eta(n-i)+1}(I_G(H_i)) \leq I_G(\gamma_{\eta(n-i)+1}(H_i)) \leq I_G(I_G(H_{i+1})) = I_G(H_{i+1}),$$

for every i, so that

$$\gamma_{\eta(n-1)+1}(I_G(H_1)) \leq I_G(H_2),$$
$$\gamma_{\eta(n-2)+1}\left(\gamma_{\eta(n-1)+1}(I_G(H_1))\right) \leq \gamma_{\eta(n-2)+1}(I_G(H_2)) \leq I_G(H_3),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\gamma_{\eta(1)+1}\left(\gamma_{\eta(2)+1}\left(\cdots\left(\gamma_{\eta(n-1)+1}(I_G(H_1))\right)\cdots\right)\right) \leq \gamma_{\eta(1)+1}(I_G(H_{n-1})) \leq I_G(H_n) = I_G(H),$$

where the last equality is due to the fact that $H \leq_f H_n \leq I_G(H)$.

In particular, for $k = k(n) = \sum_{i=1}^{n-1}(\eta(i)+1)$, the kth term $H_1^{(k)}$ of the derived series of H_1 is a subgroup of $I_G(H)$, so that $I_G(H_1^{(k)}) \leq I_G(H)$. Now, by Lemma 14, $H_1/I_G(H_1^{(k)})$ is a soluble torsion-free $(n+1)$-Engel group and so $H_1/I_G(H_1^{(k)})$ is nilpotent of class at most $(n+1)^{k-1}$. Thus, for $c = c(n) = (n+1)^{k-1}+1$, we get that $\gamma_c(H_1) \leq I_G(H_1^{(k)}) \leq I_G(H)$. This proves that $\gamma_c(H^G) \leq I_G(H)$, for every subgroup H of G.

Now take c elements of G, say $x_1, \dots, x_c$, and consider the subgroup $H = \langle x_1, \dots, x_c\rangle$. Clearly we can write $H_1 = H^G$ as a product of the c normal subgroups $\langle x_i\rangle^G$. Since $\gamma_c(\langle x_i\rangle^G) \leq I_G(\langle x_i\rangle)$ and, by Lemma 10, $I_G(\langle x_i\rangle)$ is a cyclic group, then $[\gamma_c(\langle x_i\rangle^G), x_i] = 1$. Moreover $[\gamma_c(\langle x_i\rangle^G), x_i^g] = 1$ for every $g \in G$. Thus $\gamma_c(\langle x_i\rangle^G) \le \zeta(\langle x_i\rangle^G)$ and $\langle x_i\rangle^G$ has nilpotency class at most c. Therefore H_1 is generated by c normal nilpotent subgroups of class at most c, and by Fitting's theorem it follows that H_1 is nilpotent with class $\mathrm{cl}(H_1) \le c^2$.

Now, since H is a c-generated torsion-free nilpotent group of class $\mathrm{cl}(H) \le \mathrm{cl}(H_1) \le c^2$, Lemma 11 implies that the Hirsch length $h(H)$ of H is bounded by

$$g_1 = g(c^2, c) = \frac{c^{c^2+1} - 1}{c - 1}.$$

Also, by Lemma 10, $|I_G(H) : H| < \infty$, so that $h(I_G(H)) = h(H) \le g_1$.

Therefore, $\gamma_c(H_1)$ is a finitely generated normal subgroup of G with Hirsch length $h(\gamma_c(H_1)) \le h(I_G(H)) \le g_1$ and so, by Proposition 12, $\gamma_c(H_1) \le \zeta_{g_1}(G)$. In particular, $[x_1, \dots, x_c, y_1, \dots, y_{g_1}] = 1$ for every $y_1, \dots, y_{g_1}$ in G, so that

$$\gamma_{c+g_1}(G) = 1.$$

Finally, since $c = c(n)$ and $g_1 = g_1(n)$ depend only on n, the result follows on defining $\eta(n) = c + g_1 - 1$. □

Acknowledgement

This paper presents some of the results of my doctoral dissertation, which was directed by Carlo Casolo. I am indebted to him for his advice and patience during the preparation of my thesis. This work was partially supported by MURST research program 'Teoria dei gruppi e applicazioni'.

References

[1] C. Casolo, 'Torsion-free groups in which every subgroup is subnormal', *Rend. Circ. Mat. Palermo (2)* **50** (2001), 321–324.

[2] C. Casolo and M. Mainardis, 'Groups in which every subgroup is f-subnormal', *J. Group Theory* **4** (2001), 341–365.

[3] ——, 'Groups with all subgroups f-subnormal', in: *Topics in infinite groups*, Quad. Mat. 8 (Aracne, Rome, 2001) pp. 77–86.

[4] E. Detomi, 'Groups with many subnormal subgroups', *J. Algebra* **264** (2003), 385–396.

[5] P. Hall, *The Edmonton notes on nilpotent groups*, Queen Mary College Mathematics Notes (Queen Mary College, London, 1968).

[6] H. Heineken and I. J. Mohamed, 'A group with trivial centre satisfying the normalizer condition', *J. Algebra* **10** (1968), 368–376.

[7] J. C. Lennox, 'On groups in which every subgroup is almost subnormal', *J. London Math. Soc.* **15** (1977), 221–231.

[8] J. C. Lennox and S. E. Stonehewer, *Subnormal subgroups of groups* (Oxford Univ. Press, Oxford, 1987).

[9] W. Möhres, 'Torsionfreie Gruppen, deren Untergruppen alle subnormal sind', *Math. Ann.* **284** (1989), 245–249.

[10] B. H. Neumann, 'Groups with finite classes of conjugate subgroups', *Math. Z.* **63** (1955), 76–96.

[11] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups* (Springer, Berlin, 1972).

[12] ——, *A course in the theory of groups* (Springer, Berlin, 1982).

[13] H. Smith, 'Hypercentral groups with all subgroups subnormal', *Bull. London Math. Soc.* **15** (1983), 229–234.

[14] ——, 'Torsion-free groups with all subgroups subnormal', *Arch. Math. (Basel)* **76** (2001), 1–6.

Dipartimento di Matematica
Facoltà di Ingegneria
Università degli Studi di Brescia
via Valotti 9
25133 Brescia
Italia
e-mail: detomi@ing.unibs.it