M. Glasner
Nagoya Math. J.
Vol. 45 (1971), 129–138

MANIFOLDS WITHOUT GREEN'S FORMULA*

MOSES GLASNER

Recently attention has been focused on manifolds that carry covariant tensors that are merely bounded measurable. In terms of these tensors global differential equations are defined and their weak solutions are called harmonic functions. Nakai [6] initiated the classification of these manifolds with respect to the global properties of the harmonic functions that they carry.

The classical Green's formula $\int_{\Omega}(\nabla v\cdot\nabla u + v\Delta u)dv = \int_{\partial\Omega} v\frac{\partial u}{\partial u}dS$ is no longer meaningful due to illusiveness of the tensor on sets of measure zero. Previously, a great many appeals to Green's formula were made for the purpose of establishing orthogonality in the Dirichlet inner product. The very definition of (weak) harmonicity on manifolds of this sort makes these appeals unnecessary. This observation already allows one to reproduce a considerable amount of the theory (cf. [5], [6], [7], [2]).

On the other hand, Green's formula has been used to give more detailed information and in this paper we present a substitute. Essentially, it is fabricated from the capacitary measure and its relation to the Dirichlet inner product introduced by Stampacchia [11]. We then apply it to generalize Sario's principal function theorem, as well as the construction of the operators L_0 and L_1, and to establish the Royden-Nakai decomposition theorem in this setting (cf. [10], [1], [3], [9], [8]).

These are some of the basic tools of the classification theory of Riemann surfaces and Riemannian manifolds and using them one should be able to develop a theory for the manifolds studied here.

It should also be pointed out that although Nakai (cf. [8, p. 304]) and Walsh [12] have established the principal function theorem in Brelot's harmonic spaces, their notions of flux on the manifolds considered in this paper can be easily computed only by using our observations.

Received October 10, 1970.
* Supported in part by the National Science Foundation, Grant GP 14133.

1. We consider a C^1, orientable, connected, separable noncompact m-manifold R. We shall call R a *Riemannian manifold* if it carries a symmetric covariant tensor (g_{ij}) which is Lebesgue measurable and essentially bounded in parametric balls. We further assume that there exists a covering of R by parametric balls $\mathscr{B}=\{B\}$ in which the following ellipticity condition is met: there exists a $\varkappa$ such that

$$\varkappa^{-1}|\xi|^2 \leqq \xi(g_{ij}(x))\xi^t \leqq \varkappa|\xi|^2, \tag{1}$$

for every vector $\xi \in \boldsymbol{R}^m$, almost every $x \in B$ and every $B \in \mathscr{B}$.

The usual definition of the Hodge star operator gives an isomorphism of the exterior algebra of measurable forms over R. For an open set $U \subset R$ we consider the set $\mathscr{T}(U)$ of Tonelli functions on U, i.e. the real-valued continuous functions on U with weak exterior derivatives and finite Dirichlet integrals $D_K(f) = \int_K df \wedge {}^*df < +\infty$ for compact sets $K \subset U$. The mixed Dirichlet integral of e, $f \in \mathscr{T}(U)$ is $D_K(e, f) = \int_K de \wedge {}^*df$. In a parametric ball with local coordinates x the Dirichlet integral of e, $f \in \mathscr{T}(B)$ is

$$D_K(e, f) = \int_{x(K)} \sqrt{g}\, g^{ij} e_{x_i} f_{x_i} dx, \tag{2}$$

where (g^{ij}) is the inverse and g the determinant of the matrix (g_{ij}) and e_{x_i}, f_{x_i} are weak partial derivatives.

Denote by $\mathscr{D}(U)$ the C^1 functions with compact supports in U. A function u is called harmonic at a point $x \in R$ if there is a neighborhood U of x such that $u \in \mathscr{T}(U)$ and if for every open set V with $\bar{V} \subset U$ we have $D_V(u, \varphi) = 0$ for every $\varphi \in \mathscr{D}(V)$. For an open set Ω we denote by $H(\Omega)$ the space of harmonic functions on Ω, i.e. $u \in H(\Omega)$, if u is harmonic at every point of Ω. If x is a coordinate system on a parametric ball $B \in \mathscr{B}$, then a function $u \in H(B)$ is a weak solution of the uniformly elliptic equation

$$(\sqrt{g}\, g^{ij} u_{x_i})_{x_j} = 0$$

in $x(B)$. The sheaf $\{(u, \Omega) | \Omega \text{ open}, u \in H(\Omega)\}$ forms a harmonic space in the sense of Brelot (cf. [4], [5], [2]) and we shall use the results of the axiomatic theory freely.

2. The Dirichlet integral over R of functions e, $f \in \mathscr{T}(R)$ is defined by $D(e, f) = \lim_{\Omega \to R} D_\Omega(e, f)$. This limit exists for all pairs of functions with

$D(e) = \lim_{\Omega \to R} D_\Omega(e) < +\infty$ and $D(f) < +\infty$. The Royden algebra M of R is the set of all $f \in \mathscr{T}(R)$ which are bounded and $D(f) < +\infty$. For a sequence $\{f_n\}$ of functions we use the notations $f = C\text{-}\lim f_n$ to indicate f_n converges uniformly to f on compact subsets of R, $f = B\text{-}\lim f_n$ to indicate $f = C\text{-}\lim f_n$ and $\{f_n\}$ is bounded and $f = D\text{-}\lim f_n$ to indicate $\lim D(f - f_n) = 0$. We write $f = BD\text{-}\lim f_n$, for example, to indicate two modes of convergence.

LEMMA 2.1. *The Royden algebra M is an algebra and a lattice under the operations $\cap, \cup$ of pointwise min and max. If $\{f_n\} \subset M$, $f = B\text{-}\lim f_n$ and $\{f_n\}$ is D-Cauchy, then $f = BD\text{-}\lim f_n$ and $f \in M$.*

LEMMA 2.2. *Let Ω be a relatively compact open set in R and $f \in M$ with* supp $f \subset \bar{\Omega}$, *then there exists $\{\varphi_n\} \subset C^1(R)$ such that* supp $\varphi_n \subset \Omega$ *and* $f = BD\text{-}\lim \varphi_n$.

For the proofs of these lemmas see [7] (also cf. [9]). It is relations (1) and (2) that is the key.

COROLLARY 2.3. *If Ω is a relatively compact open set in R and $h \in M \cap H(\Omega)$, then $D(h, f) = 0$, for every $f \in M$ with* supp $f \subset \bar{\Omega}$.

Indeed for every φ_n approximating f in the sense of the lemma we can see that $D(h, \varphi_n) = 0$ from the definition of harmonicity and the existence of a partition of unity subordinate to any finite open cover of supp φ_n.

3. A relatively compact open set Ω will be called *regular* if $\partial\Omega$ is a C^1 submanifold of R. For regular open sets the Dirichlet problem is solvable.

THEOREM 3.1. *Let Ω_0, Ω be regular regions with $\bar{\Omega} \subset \Omega_0$. There exists a positive measure μ on $\partial\Omega$ such that*

$$\int \varphi \, d\mu = D(\varphi, u)$$

for all $\varphi \in M$ with supp $\varphi \subset \bar{\Omega}_0$, *where $u \in M$ such that $u | \bar{\Omega} = 1$,* supp $u \subset \bar{\Omega}_0$ *and $u \in H(\bar{\Omega}_0 \backslash \bar{\Omega})$.*

This is merely Stampacchia's result [11, Théorème 3.9]. His hypothesis, the coercivity of $D_{\Omega_0}(\cdot, \cdot)$ on the completion of the space $\mathscr{D}(\Omega_0)$ with respect to the norm $\int_{\Omega_0} |f|^2 {*1} + D_{\Omega_0}^{1/2}(f)$, is verified easily in view of (1) and (2). This has also been remarked by Maeda [5].

From the observation that the theorem depends only on the behavior

https://doi.org/10.1017/S0027763000014707 Published online by Cambridge University Press

of the functions on $\bar{\Omega}_0\backslash\Omega$ we obtain the following.

COROLLARY 3.2. *There exists a positive measure* ν *on* $\partial\Omega$ *such that* $\int\varphi\, d\nu = D(\Psi, u_0)$ *for all* $\Psi\in M$ *with* $\operatorname{supp}\Psi\subset R\backslash\Omega$ *where* $u_0\in M$ *such that* $u_0|\bar{\Omega}=0$, $u_0|R\backslash\Omega_0=1$ *and* $u_0\in H(\Omega_0\backslash\bar{\Omega})$.

Note that u and u_0 above are related by $u=1-u_0$. Thus $D(f, u) = -D(f, u_0)$ for any $f\in M$. Moreover, given any $f\in M$ we can write it as $f=\varphi+\Psi$, where $\varphi, \Psi\in M$, $\operatorname{supp}\varphi\subset\Omega_0$ and $\operatorname{supp}\Psi\subset R\backslash\Omega$. The following generalized Green's formula now follows.

COROLLARY 3.3 *For any* $f\in M$

$$D(f, u) = \int f\, d\mu - \int f\, d\nu.$$

If $h\in M\cap H(\Omega_0)$, then by Corollary 2.3 $D(h, u)=0$ and therefore we have the following.

COROLLARY 3.4. *For* $h\in M\cap H(\Omega_0)$, $\int h\, d\mu = \int h\, d\nu$. *In particular* $\int d\mu = \int d\nu$.

4. A regular boundary neighborhood W of R is the complement of a regular region Ω. We now turn to the principal function problem. Given $s\in H(W)$, when is it possible to find a $p\in H(R)$ which "imitates" s on W? In order to describe the mode of imitation we introduce the following definition of normal operators. Let $L: M(\alpha)\to H^C(W)$, where $\alpha=\partial W$, $M(\alpha)$ is M restricted to α and $H^C(W)$ are the functions in $H(W)$ with continuous extensions to α.

DEFINITION. *The operator* L *is called normal if*

(a) $Lf|\alpha = f$

(b) L *is linear*

(c) $\min_\alpha f \leq Lf \leq \max_\alpha f$

(d) $D(Lf, u) = 0$.

Here u has the same meaning as in No. 3. i.e. for a regular region Ω_0, with $\bar{\Omega}\subset\Omega_0$, $\Omega=R\backslash W$, u is the function in M with $u|\bar{\Omega}=1$, $u|R\backslash\Omega_0=0$, $u\in H(\Omega_0\backslash\bar{\Omega})$. Condition (d) is independent of the particular choice of Ω_0 but for the sake of clarity we keep it, as well as Ω (and W), fixed. In fact if Ω_0' were another regular region containing $\bar{\Omega}$ and u' the corresponding function, then $\operatorname{supp} u - u'$ is a compact subset of W and consequently Corollary 2.3 gives $D(Lf, u - u') = 0$.

THEOREM 4.1. *Given $s \in H(W) \cap M$ and L a normal operator. There exists a $p \in H(R)$ such that $p|W = s + L(p - s|\alpha)$ if and only if $D(s, u) = 0$.*

As above the truth of $D(s, u) = 0$ is independent of the particular choice of Ω_0.

For the necessity we observe that $p \in H(R)$ implies that $D(p, u) = 0$ by Corollary 2.3.

To establish the sufficiency we employ the following well-known fact (cf. [8]).

LEMMA 4.2. *There exists a $q \in (0, 1)$ such that $q \sup_W |h| \geq \sup_{\alpha_0} |h|$ for all $h \in H(W)$ which change sign on $\alpha_0 = \partial\Omega_0$.*

Assume now that $D(s, u) = 0$. There is no loss in generality in assuming that $s|\alpha = 0$. For if we can show the sufficiency under this additional assumption, then we replace s by $s' = s - L(s|\alpha)$ and the resulting p satisfies $p|W = s' + L(p - s'|\alpha) = s + L(p - s|\alpha)$.

Let K be the Dirichlet operator for, Ω_0, i.e. $K : C(\alpha_0) \to H^C(\Omega_0)$ such that $Kf|\alpha_0 = f$. Also let $T : C(\alpha_0) \to C(\alpha_0)$ be the linear operator defined by $Tf = L(Kf|\alpha)|\alpha_0$. The problem can be reduced to finding a $\tilde{p} \in C(\alpha_0)$ such that

$$\tilde{p} = T\tilde{p} + s|\alpha_0. \tag{3}$$

For then the problem is solved by defining p by

$$p|\Omega_0 = K\tilde{p}, \quad p|W = s + L(K\tilde{p}|\alpha).$$

Indeed the maximum principle together with (3) shows that p is well-defined on $\bar{\Omega}_0 \cap W$.

To solve (3) we need to show that

$$\sup |T^k(s|\alpha_0)| \leq q^k \sup_{\alpha_0} |s|, \quad \text{for all } k \tag{4}$$

since $\tilde{p} = \sum_0^\infty T^k(s|\alpha_0)$ would be the solution. We note that the hypotheses on s together with Corollary 3.3 give $\int s \, d\nu = 0$. By applying Corollary 3.4

https://doi.org/10.1017/S0027763000014707 Published online by Cambridge University Press

we obtain $\int Ks\, d\mu = \int s\, d\nu = 0$. Property ($d$) of L implies that $0 = D(LKs, u) = \int Ks\, d\mu - \int LKs\, d\nu = -\int LKs\, d\nu$. Since ν is a positive measure on α_0 we conclude that $L(K(s|\alpha_0)|\alpha)$ changes sign on α_0. Thus by Lemma 4.2, properties (a) and (c) of normal operators and the maximum principle we have

$$\begin{aligned} \sup |T|(s|\alpha_0)| &= \sup_{\alpha_0} |L(K(s|\alpha_0)|\alpha)| \\ &\leq q \sup |L(K(s|\alpha_0)|\alpha)| \\ &= q \sup_{\alpha} |K(s|\alpha_0)| \leq q \sup_{\alpha_0} |s|. \end{aligned}$$

Inequality (4) follows by repeating the argument k times.

5. We now turn to the task of demonstrating the existence of normal operators. We shall construct operators L_0 and L_1 following the procedure given in [1] which uses the Royden ideal boundary theory. As shown in [1], on Riemannian manifolds with C^1-Hölder metric tensors the procedure results in operators which coincide with Sario's L_0 and L_1 (cf. [10]).

The Royden compactification R^* of R is the compact Hausdorff space which contains R as an open dense subset such that the functions of M extend to R^* continuously and separate the points of R^*. Let M_Δ be the BD-closure of the functions in M with compact supports. The Royden harmonic boundary is the subset of $R^*\backslash R$ given by

$$\Delta = \{p \in R^* | f(p) = 0 \text{ for every } f \in M_\Delta\}.$$

THEOREM 5.1. *There exists a linear mapping $\pi : M \to H(W) \cap M$ such that*

$$f = \pi f \text{ on } \Delta \cup \Omega, \tag{5}$$

$$D(\pi f) \leq D(f). \tag{6}$$

For every $h \in H(W) \cap M$

$$\min_{\alpha \cup \Delta} h \leq h|W \leq \max_{\alpha \cup \Delta} h. \tag{7}$$

This can be established using the techniques of [2] (also cf. [3], [9]).

6. It will be convenient to interpret $L : M(\alpha) \to H^C(W)$ as acting on functions $f \in M$ and having the property that $Lf = Lf'$ whenever $f|\alpha = f'|\alpha$.

THEOREM 6.1. *For a given $f \in M$ consider $F = \{g \in M | g|\Omega = f\}$. There exists a unique function $h \in F$ such that $D(h) = \min_{g \in F} D(g)$. Moreover, the mapping*

$L_0 : f \to h|W$ *gives a normal operator.*

If we replace the given f by

$$f = \begin{cases} f \text{ on } \Omega \\ ((\min_\alpha f) \cup f) \cap \max_\alpha f \text{ on } W, \end{cases}$$

then the family F is not disturbed. Therefore we may assume at the outset that f satisfies

(8) $$\min_\alpha f \leq f|\Delta \leq \max_\alpha f.$$

Denote by G the family $\{\pi g | g \in F\}$. By virtue of (5) we have $G \subset F$ and then by (7) we see that $\min_\alpha f \leq h|W \leq \max_\alpha f$ for any $h \in G$.

Set $d' = \inf_G D(h)$ and $d = \inf_F D(g)$. Clearly $d' \leq d$. On the other hand, for every $\varepsilon > 0$ there is a $g_\varepsilon \in F$ such that $d + \varepsilon \geq D(g_\varepsilon)$. Since $\pi g_\varepsilon \in G$, we conclude by (6) that $d + \varepsilon \geq D(g_\varepsilon) \geq D(\pi g_\varepsilon) \geq d'$. Hence $d = d'$. We can therefore choose a sequence $\{h_n\} \subset G$ such that $\lim D(h_n) = d$. Since $\{h_n\}$ is bounded there exists a subsequence, again denoted by $\{h_n\}$ with $h = B\text{-}\lim h_n$ and $h \in H^C(W)$. The function $(h_n + h_{n+p})/2$ being in F implies that $D(h_n + h_{n+p}) \geq 4d$. Thus by the parallelogram law

$$D(h_n - h_{n+p}) \leq 2D(h_n) + 2D(h_{n+p}) - 4d$$

and consequently $\{h_n\}$ is D-Cauchy. We conclude by Lemma 2.1 that $h = BD\text{-}\lim h_n \in F$.

If $\varphi \in M$ and $\varphi | \Omega = 0$, then $h + r\varphi \in F$ for all $r \in \boldsymbol{R}$. Since $D(h + r\varphi) = D(h) + 2rD(h, \varphi) + r^2 D(\varphi) \geq D(h)$ we must have

(9) $$D(h, \varphi) = 0.$$

If h' were another minimizing function in F, then $D(h, h - h') = 0$. This would give $0 \leq D(h - h') = D(h) - D(h') = 0$. Thus $h = h'$ and the first assertion is valid.

Now suppose $f, f' \in M$ with $f|\alpha = f'|\alpha$ and h, h' are the corresponding minimizing functions. Let

$$\bar{h} = \begin{cases} f \text{ on } \Omega \\ h' \text{ on } W. \end{cases}$$

Then $\bar{h} \in F$ and is also a minimizing function. We conclude that L_0 is well defined and clearly satisfies properties (a) and (c) of normal operators. From (9) we obtain $D(h, 1 - u) = 0$, which is (d). To show the linearity of

L_0 set $\Psi = L_0 f + rL_0 f' - L_0(f + rf')$, for $r \in \boldsymbol{R}$ and $f, f' \in M$. Since Ψ vanishes on Ω, (9) again gives

$$D(\Psi) = D(L_0 f, \Psi) + D(rL_0 f', \Psi) - D(L_0(f + rf'), \Psi) = 0,$$

which means that $\Psi = 0$.

7. The operator L_1 is characterized in the following.

THEOREM 7.1. *To each $f \in M$ there corresponds a constant c and a function $v \in M$ such that $v|\Omega = f$, $v|\Delta = c$ and the mapping $L_1 : f \to v|W$ is a normal operator.*

We choose an exhaustion $\{R_n\}_0^\infty$ of R by regular regions such that $R_0 = \Omega$ and $R_1 = \Omega_0$. Apply Theorem 3.1 and Corollary 3.2 with R_0 and R_n playing the roles of Ω and Ω_0 respectively. Denote the resulting u, μ, ν by u_n, μ_n, ν_n respectively. Take $v_n \in M$ with $v_n|\bar{\Omega} = f$, $v_n|R \backslash R_n = c_n$, $v_n \in H(R_n \backslash \bar{\Omega})$. The constant c_n is chosen so that $D(v_n, u_n) = 0$, i.e. $\int f\, d\mu_n - \int c_n\, d\nu_n = 0$. Since $\int d\mu_n = \int d\nu_n$, the constant c_n is in the interval $[\min_\alpha f, \max_\alpha f]$. Then $\{v_n\}$ is bounded by $\sup_\alpha |f|$ and consequently there is a subsequence with $v = B\text{-}\lim v_n \in H^c(W)$. In addition

$$D(v_{n+p}, v_n - v_{n+p}) = (c_n - c_{n+p}) D(v_{n+p}, 1 - u_{n+p}) = 0.$$

Hence $\{v_n\}$ is D-Cauchy and Lemma 2.1 gives $v = BD\text{-}\lim v_n \in M$. Choose a convergent subsequence of $\{c_n\}$ with limit $c \in [\min_\alpha f, \max_\alpha f]$. Then $v - c = BD\text{-}\lim v_n - c_n \in M_\Delta$ which implies that $v|\Delta = c$ and in turn that the original sequence $\{c_n\}$ is convergent. From (7) we deduce that $v|W$ depends only on $f|\alpha$ and consequently L_1 is well-defined. Moreover (7) gives property (c) of normal operators. Properties (a) and (b) follow trivially from the construction. Finally note that $D(v_n, u_n - u_1) = 0$, since $\operatorname{supp} u_n - u_1 \subset \bar{R}_n \cap W$ and conclude that $D(v_n, u_1) = D(v_n, u_n) = 0$ which gives $D(v, u_1) = 0$, i.e. property (d).

8. Another application of the generalized Green's formula is the Royden-Nakai decomposition which we proceed to describe. A Riemannian manifold R is called *parabolic* or *hyperbolic* according as $\Delta = \phi$ or $\Delta \neq \phi$. As in No. 7 we consider an exhaustion of R by regular regions $\{R_n\}_0^\infty$ such that $\Omega = R_0$ and we use the symbols u_n, μ_n for the u and μ that result from applying Theorem 3.1 with R_0 and R_n playing the roles of Ω and Ω_0 respectively. By the maximum principle we see that $\{u_n\}$ forms a decreasing

https://doi.org/10.1017/S0027763000014707 Published online by Cambridge University Press

sequence and again by Corollary 2.3 we have $D(u_{n+p}, u_{n+p} - u_n) = 0$. Thus $u_\infty = BD\text{-}\lim u_n$ exists, $u_\infty \in H^c(W)$ and is either strictly less than 1 on the interior of W or identically 1. It can easily be seen that (cf. [6], [3])

LEMMA 8.1. *R is parabolic if and only if $D(u_\infty) = 0$.*

The norms of the measures μ_n on α are given by $D(u_n)$ which are bounded. Thus there exists a subsequence of $\{\mu_n\}$ converging in the weak* sense to a nonnegative measure μ_∞ on α.

LEMMA 8.2. *For $g \in M_\Delta$, $D(g, u_\infty) = \int g \, d\mu_\infty$.*

For the proof take a sequence $\{g_k\} \subset M$ with compact supports such that $g = BD\text{-}\lim g_k$. For a fixed k take n_k so large that $\operatorname{supp} g_k \subset \operatorname{supp} u_{n_k}$. Then for every $n \geq n_k$ we have $D(g_k, u_n) = \int g_k \, d\mu_n$ and letting $n \to \infty$ gives $D(g_k, u_\infty) = \int g_k \, d\mu_\infty$. Now letting $k \to \infty$ gives the assertion.

COROLLARY 8.3. $D(u_\infty) = \int d\mu_\infty$. *In particular, if R is hyperbolic, then μ_∞ is nonzero.*

This follows by noting that $u_\infty \in M_\Delta$. The Royden-Nakai decomposition now follows.

THEOREM 8.4. *Suppose that R is hyperbolic and $f \in \mathscr{T}(R)$ with $D(f) < +\infty$. There exists a unique pair h, g such that $f = h + g$, $h \in H(R)$ and there exists a sequence $\{g_n\} \subset M$ with compact supports with $g = CD\text{-}\lim g_n$.*

For the proof we consider first the positive part f^+ of f. Taking our exhaustion $\{R_n\}_0^\infty$ of R we let h'_n be the continuous function on R such that $h'_n = f^+$ on $R \backslash R_n$ and $h'_n \in H(R_n)$. Set $g'_n = f^+ - h'_n$. Then as in preceding arguments we can see that $D(f^+) = D(h'_n) + D(g'_n)$ and $D(h'_{n+p} - h'_n) = D(h'_n) - D(h'_{n+p})$.

Since $\{h'_n\}$ is eventually positive harmonic on compact sets a subsequence converges uniformly on compact sets to a harmonic function $h' \in H(R)$ or to $+\infty$. Assume the latter alternative. By Lemma 8.2 and the Schwarz inequality we have

$$\int h'_n - f \, d\mu_\infty = D(-g'_n, u_\infty) \leq D^{1/2}(g'_n) D^{1/2}(u_\infty) \leq D^{1/2}(f) D^{1/2}(u_\infty).$$

This is a contradiction in view of the facts that f is finite on α, $D(u_\infty)$ is nonzero and μ_∞ is nontrivial. Thus $h' \in H(R)$ and also $g' = CD\text{-}\lim g'_n$ exists.

The above procedure gives $f^- = h'' + g''$ with the same properties. Then $f = (h' - h'') + (g' - g'')$ is the desired decomposition. If $f = h_0 + g_0$ were another decomposition of this sort, then $h_0 - h = g - g_0$ would be the CD-limit of a sequence $\{\varphi_n\} \subset M$ with compact supports. The harmonicity of $h_0 - h$ gives $D(h_0 - h, \varphi_n) = 0$ and consequently $D(h_0 - h) = 0$, i.e. $h_0 - h$ is a constant k. But then k is the BD-limit of the sequence of functions $\{(\varphi_n \cap |k|) \cup (-|k|)\}$ with compact supports. Thus $k \in M_\Delta$. Since R is hyperbolic, $\Delta \neq \phi$ and k must be 0.

References

[1] M. Glasner, *The principal operators L_0 and L_1 the Royden boundary*, J. Analyse Math. **24** (1971), 163–172.

[2] M. Glasner, *Dirichlet mappings of Riemannian manifolds and the equation $\Delta u = Pu$*, J. Differential Equations, **9** (1971), 390~404.

[3] M. Glasner—R. Katz, *The Royden boundary of a Riemannian manifold*, Ill. J. Math., (1970).

[4] R.M. Hervé, *Quelques propriétés des sursolutions et sursolutions locales d'une équation uniformemént elliptique de la forme $Lu = -\sum_i (\sum a_{ij} u_{x_j})_{x_i} = 0$*, Ann. Inst. Fourier, Grenoble **16** (1966), 241–267.

[5] F–Y. Maeda, *Introduction to a potential theory on a differentiable manifold*, Lecture notes, Kyoto University, October 1968.

[6] M. Nakai, *On parabolicity and Royden compactifications of Riemannian manifolds*, Proc. International Congress Functional Analysis, 1969.

[7] M. Nakai, *Royden algebra and quasi-isometries of Riemannian manifolds*, Pacific J. Math. (to appear).

[8] B. Rodin—L. Sario, *Principal functions*, D. Van Nostrand Co., 1968.

[9] L. Sario—M. Nakai, *Classification of Riemann surfaces*, Springer-Verlag, 1970.

[10] L. Sario—M. Schiffer—M. Glasner, *The span and principal functions in Riemannian spaces*, J. Analyse Math. **15** (1965), 115–134.

[11] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier, Grenoble 15 (1965), 189–258.

[12] B. Walsh, *Flux in axiomatic potential theory*, Inventiones Math. 8 (1969) 175–221.

California Institute of Technology
Pasadena, California 91109