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DISTINGUISHED SUBFIELDS OF INTERMEDIATE FIELDS

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Let L be a finitely generated extension of a field K of characteristic $p \neq 0$. If L/K is algebraic, then there is a unique intermediate field S such that

$$L \subseteq K^{p^{-\infty}} \otimes_K S.$$

S is just the maximal separable extension of K in L. If L/K is not algebraic, then Dieudonne [**4**] showed there exist maximal separable extensions D of K in L such that $L \subseteq K^{p^{-\infty}} \otimes_K D$. In general, not every maximal separable extension of K in L has the property. Those which do have the property are called distinguished. Kraft [**7**] established that a maximal separable extension D of K in L is distinguished if and only if $[L:D]$ is as small as possible. If the minimum of the $[L:D]$ is p^r, r is called the order of inseparability of L/K, denoted $\operatorname{inor}(L/K)$.

Let L_1 be an intermediate field of L/K. If L/K is algebraic, then the maximal separable extension S_1 of K in L_1 is contained in the maximal separable extension S of K in L, and moreover S is separable over S_1. This paper is concerned with the relationship between distinguished subfields D_1 of L_1/K and distinguished subfields D of L/K in the case where L/K is not necessarily algebraic. The exact analogue holds, that is every D_1 is contained in a D with D separable over D_1 if and only if

$$\operatorname{inor}(L/K) = \operatorname{inor}(L_1/K) + \operatorname{inor}(L/L_1).$$

However in view of the nonuniqueness of distinguished subfields and the fact that maximal separable extensions need not be distinguished, the exact analogue of the algebraic situation is quite strong to impose in the general situation. Thus we are led to examine when some (or all) distinguished subfields D_1 of L_1/K are merely contained in a distinguished subfield D of L/K.

Recall that L is modular over K if L^{p^n} and K are linearly disjoint over their intersection for all n. The concept was first introduced by Sweedler to characterize which finite dimensional purely inseparable field extensions can be expressed as a tensor product of simple extensions. It has since been used successfully to investigate arbitrary field extensions [**8**]. One general result is Theorem 2.3; If L/L_1 is modular and $L_1(L^p)$ is

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separable over L_1, then there is a distinguished subfield of L_1/K contained in one of L/K.

Results along this line can be used to determine structural properties of inseparable field extensions. In [**1**] and [**2**] it is shown that there exist unique minimal intermediate fields C^* and L^*, $L \supseteq C^* \supseteq L^* \supseteq K$, such that L/C^* is separable and L^*/K has the same order of inseparability as does L/K. Any intermediate field F such that $\text{inor}(F/K) = \text{inor}(L/K)$ is called a form of L/K. Forms have been characterized by the condition that L^{p^n} and $K(F^{p^n})$ are linearly disjoint over F^{p^n} for all n [**2**, Theorem 1.3, p. 656]. In [**2**, Theorem 2.2, p. 659], it was shown that if F is a form of L/K and D is distinguished for L/K, then $L = FD$. Thus if $D \cap F$ is distinguished in F/K, a degree argument shows $L = D \otimes_{D \cap F} F$. Now, let $L \supseteq C^* \supseteq L^* \supseteq K$ be the unique intermediate fields defined above and assume there exists a distinguished subfield D_1 of L^*/K contained in a distinguished subfield D_2 of C^*/K. Then by a degree argument

$$C^* = L^* \otimes_{D_1} D_2$$

and since L/D_2 is finitely generated with L/C^* separable and C^*/D_2 purely inseparable,

$$L = C^* \otimes_{D_2} S$$

for some separable extension S of D_2 [**8**, Theorem 4, p. 1178]. Thus

$$L = L^* \otimes_{D_1} D_2 \otimes_{D_2} S \approx L^* \otimes_{D_1} S.$$

Theorem 3.9 shows that if C^*/L^* is modular, then we can find such a D_2 and D_1. An example is presented proving that in general such $D_2 \supseteq D_1$ need not exist.

1. Since L/K is finitely generated, there exists an integer n such that $K(L^{p^n})$ is separable over K. The least such n is called the *inseparability exponent* of L over K, denoted $\text{inex}(L/K)$.

1.1 LEMMA. *Let $j \geqq \text{inex}(L/K)$. If $Y^{p^j} \subseteq L^{p^j}$ is a relative p-basis of $K(L^{p^j})/K$, then $K(L^{p^j})(Y)$ is a distinguished subfield of L/K.*

Proof. Since $j \geqq \text{inex}(L/K)$, $K(L^{p^j})$ is separable over K. Thus Y^{p^j} is a separating transcendence basis of $K(L^{p^j})$ over K. Thus Y is a separating transcendence basis of $K(L^{p^j})(Y)$ over K and hence $K(L^{p^j})(Y)$ is separable over K. Since

$$K((K(L^{p^j})(Y))^{p^j}) = K(L^{p^{2j}})(Y^{p^j}) = K(L^{p^j}),$$

[**5**, Proposition 1, p. 288] shows $K(L^{p^j})(Y)$ is distinguished.

1.2 THEOREM. *Let L_1 be an intermediate field of L/K and let $n =$ $\text{inex}(L/K)$. Then the following conditions are equivalent.*

(1) *There exists a distinguished subfield* D_1 *of* L_1/K *which is coseparable in a distinguished subfield* D *of* L/K.

(2) *Every distinguished subfield* D_1 *of* L_1/K *is coseparable in a distinguished subfield* D *of* L/K.

(3) $K(L^{p^n})/K(L_1^{p^n})$ *is separable.*

(4) $\operatorname{inor}(L/K) = \operatorname{inor}(L_1/K) + \operatorname{inor}(L/L_1)$.

Proof. (1) implies (4): Let D be separable over D_1. Since L_1/D_1 is purely inseparable, $D \otimes_{D_1} L_1$ is separable over L_1. Since $L \subset K^{p^{-\infty}}(D)$,

$$L \subset L_1^{p^{-\infty}}(D \otimes_{D_1} L_1).$$

So $D \otimes_{D_1} L_1$ is distinguished for L/L_1. Since

$$\begin{aligned}[L:D] &= [L:D \otimes_{D_1} L_1][D \otimes_{D_1} L_1:D] \\ &= p^{\operatorname{inor}(L/L_1)} \cdot [L_1:D_1] \\ &= p^{\operatorname{inor}(L/L_1)} \cdot p^{\operatorname{inor}(L_1/K)},\end{aligned}$$

(4) follows.

(4) implies (3): Let D_2 be distinguished for L/L_1 and let D_1 be distinguished for L_1/K. By [**8**, Theorem 4, p. 1178], $D_2 = S \otimes_{D_1} L_1$ where S is separable over D_1. By (4), S is distinguished for L/K. Thus

$$K(L^{p^n}) = K(S^{p^n}) \quad \text{and} \quad K(L_1^{p^n}) = K(D_1^{p^n}).$$

Since S/D_1 and D_1/K are separable, $K(S^{p^n})/K(D_1^{p^n})$, i.e., $K(L^{p^n})/K(L_1^{p^n})$ is separable.

(3) implies (2): Let D_1 be a distinguished subfield of L_1/K and let Y_1 be a relative p-basis of D_1/K. Then $Y_1^{p^n}$ is a relative p-basis of $K(L_1^{p^n})/K$. Since $K(L^{p^n})/K(L_1^{p^n})$ is separable, there is a subset Y of L such that $Y \supseteq Y_1$ and Y^{p^n} is a relative p-basis of $K(L^{p^n})/K$. By Lemma 1.1, $D = K(L^{p^n})(Y)$ is a distinguished subfield of L/K. Clearly $D_1 \subseteq D$. If B is a p-basis of K, then $B \cup Y_1$ is a p-basis of D_1 and $B \cup Y$ is a p-basis of D. Thus D/D_1 preserves p-independence, i.e., D/D_1 is separable.

(2) implies (1): This is immediate since L/K and L_1/K have distinguished subfields.

1.3 COROLLARY. *For every intermediate field* L_1 *of* L/K, *every distinguished subfield of* L_1/K *is coseparable in a distinguished subfield of* L/K *if and only if* L/K *is algebraic.*

Proof. Suppose L/K is not algebraic. Let $n = \operatorname{inex}(L/K)$ and let $L_1 = K(L^p)$. Since L/K is finitely generated and non-algebraic,

$$K(L^{p^n}) \supsetneq K(L^{p^{n+1}}).$$

Thus $K(L^{p^n})$ is purely inseparable over its proper subfield $K(L_1^{p^n})$ and Theorem 1.2 applies. The converse is easy.

1.4 PROPOSITION. *For every intermediate field L_1 of L/K, every distinguished subfield of L_1/K is contained in a distinguished subfield of L/K if and only if every maximal separable intermediate field of L/K is distinguished.*

Proof. Let S be a maximal separable intermediate field of L/K which is not distinguished. Then $L_1 = S$ does not have a distinguished subfield which is contained in one of L/K. For the converse, it suffices to show that every intermediate field of L/K which is separable over K is extendable to a maximal separable intermediate field of L/K. That this is true follows easily from Zorn's Lemma.

When every maximal separable intermediate field of L/K is distinguished is examined in [**3**].

2. In this section we concentrate on a single subfield L_1 of L/K and examine when a (or every) distinguished subfield of L_1 is contained in one of L/K.

2.1 PROPOSITION. *Let L_1 be an intermediate field of L/K and let $n =$ inex(L/K). If every distinguished subfield of L_1/K is contained in one of L/K, then*

$$K^{p^{-n}}(L^{p^i}) \cap L_1 \subseteq K(L^{p^i}) \cup K^{p^{-n}}(L_1^{\,p}) \quad \textit{for} \quad 0 \leqq i \leqq n.$$

Proof. The conclusion is immediate for $i = 0$. Assume the conclusion is not true for all i, $0 \leqq i \leqq n$, and let i be the least integer such that there exists

$$\theta \in K^{p^{-n}}(L^{p^{i+1}}) \cap L_1 \backslash K(L^{p^{i+1}}) \cup K^{p^{-n}}(L_1^{\,p}).$$

Then

$$\theta^{p^n} \in K(L_1^{\,p^n}) \backslash K(L_1^{\,p^{n+1}}),$$

so by Lemma 1.1 θ is part of a separating transcendence basis of a distinguished subfield D_1 of L_1/K. Now

$$\theta \in K^{p^{-n}}(L^{p^i}) \cap L_1 \subseteq K(L^{p^i}) \cup K^{p^{-n}}(L_1^{\,p})$$

by the minimality of i. But $\theta \notin K^{p^{-n}}(L_1^{\,p})$, so $\theta \in K(L^{p^i})$. From above,

$$\theta \in K^{p^{-n}}(L^{p^{i+1}}) \cap L_1 \backslash K(L^{p^{i+1}}).$$

As in the proof of [**3**, Theorem 1], θ is not in any distinguished subfield of L/K, and hence D_1 cannot be contained in any distinguished subfield of L/K.

The following result shows that the necessary condition of Proposition 2.1 is sufficient in a special situation.

2.2 THEOREM. *Let* L_1 *be an intermediate field of* L/K. *Suppose* $\text{inex}(L/K) = 1$ *and the transcendence degree of* L/K *is* 1. *Then*

$$K^{p^{-1}}(L^p) \cap L_1 \subseteq K(L^p) \cup K^{p^{-1}}(L_1^p)$$

if and only if every distinguished subfield of L_1/K *is contained in one of* L/K.

Proof. Let D_1 be a distinguished subfield of L_1/K. If D_1 is algebraic over K, then D_1 is in every distinguished subfield of L/K. Thus suppose D_1 is not algebraic over K and let t be a separating transcendence basis of D_1/K. Either $t \in K^{p^{-1}}(L^p)$ or $t \notin K^{p^{-1}}(L^p)$. If $t \in K^{p^{-1}}(L^p)$, then by assumption

$$t \in K(L^p) \cup K^{p^{-1}}(L_1^p).$$

Since t is a separating transcendence basis of D_1 over K and $K(L_1^p) = K(D_1^p)$ ($\text{inex}(L/K) = 1$), $t^p \notin K(L_1^{p^2})$ i.e., $t \notin K^{p^{-1}}(L_1^p)$. Thus $t \in K(L^p)$. But any element of D_1 which is separable algebraic over $K(t)$ is also separable algebraic over $K(L^p)$, and hence in $K(L^p)$. Thus $D_1 \subseteq K(L^p)$ and since $\text{inex}(L/K) = 1$, D_1 is contained in every distinguished subfield of L/K. If $t \notin K^{p^{-1}}(L^p)$, i.e., $t^p \notin K(L^{p^2})$, t is a separating transcendence basis of a distinguished subfield D of L/K by Lemma 1.1. Clearly $D \supseteq D_1$.

2.3 THEOREM. *Let* L_1 *be an intermediate field of* L/K *and suppose every distinguished subfield of* L_1/K *is contained in one of* L/K. *If* T *is relatively* p*-independent in* L/K *and is part of a separating-transcendence basis of a distinguished subfield of* L_1/K, *then* T *is part of a separation transcendence basis of a distinguished subfield of* L/K. *If* $\text{inex}(L/K) = 1$ *the converse is also true.*

Proof. The proof is by induction on $|T|$. Let $n = \text{inex}(L/K)$. We have

$$K^{p^{-n}}(L^p) \cap L_1 \subseteq K(L^p) \cup K^{p^{-n}}(L_1^p)$$

by Proposition 2.1. Suppose $T = \{t\}$. Then

$$t \notin K(L^p) \cup K^{p^{-n}}(L_1^p)$$

and $t \in L_1$. Hence $t \notin K^{p^{-n}}(L^p)$. Thus t is part of a separating transcendence basis for a distinguished subfield of L/K. Suppose $|T| = m > 1$ and that the result is true for field extensions of exponent n such that $|T| = m - 1$. Let $T = \{t_1, \ldots, t_m\}$. As above t_1 is part of a separating transcendence basis of a distinguished subfield of L_1/K and L/K. Thus every distinguished subfield of $L_1/K(t_1)$ and $L/K(t_1)$ is also one of L_1/K and L/K respectively. Hence every distinguished subfield of $L_1/K(t_1)$ is contained in one of $L/K(t_1)$. Now $\{t_2, \ldots, t_m\}$ is relatively p-independent in $L/K(t_1)$ and is part of a separating transcendence basis for a distinguished subfield of $L_1/K(t_1)$. Thus by the induction hypothesis, $\{t_2, \ldots, t_m\}$ is part of a separating transcendence basis of a distinguished subfield of $L/K(t_1)$. Hence T is part of a separating transcendence basis of a distinguished subfield of L/K.

Now assume $\operatorname{inex}(L/K) = 1$. Let T be a separating transcendence basis of a distinguished subfield D_1 of L_1/K. Let $T' \subseteq T$ be maximal such that T' is relatively p-independent in L/K. Then by assumption T' is in a distinguished subfield D of L/K. Hence

$$T \subseteq K(L^p)(T') = K(D^p)(T') \subseteq D.$$

Thus $D_1 \subseteq D$.

We now show the necessary conditions in Proposition 2.1 and Proposition 2.3 are not sufficient when $\operatorname{inex}(L/K) > 1$.

2.4 *Example.* Let

$$K = P(x, y) \quad \text{and} \quad L = K(z, zx^{p^{-2}} + y^{p^{-2}})$$

where P is a perfect field of characteristic $p \neq 0$ and $\{x, y, z\}$ is algebraically independent over P. Every set of one relatively p-independent element of L/K is (part of) a separating transcendence basis of L/K [**3**, Example 13]. Hence for every intermediate field L_1 of L/K, any set which is relatively p-independent in L/K and which is part of a separating transcendence basis for a distinguished subfield of L_1/K is also part of a separating transcendence basis of a distinguished subfield of L/K. Also,

$$K^{p^{-2}}(L^{p^i}) \cap L = K(L^{p^i}), \quad i = 1, 2.$$

Hence for every intermediate field L_1 of L/K,

$$K^{p^{-2}}(L^{p^i}) \cap L_1 \subseteq K(L^{p^i}) \cup K^{p^{-2}}(L_1^{\,p}), \quad i = 1, 2.$$

That is, every intermediate field L_1 of L/K satisfies the necessary conditions of Propositions 2.1 and 2.3. However, L/K has a maximal separable subfield S which is not distinguished in L/K [**3**, Example 13]. Let $L_1 = S$. Then no distinguished subfield of L_1/K is contained in a distinguished subfield of L/K.

In the following example we show that the assumption of transcendence degree 1 in Theorem 2.3 is needed. The example provides us with extensions $L/L_1/K$ for which the necessary condition in Proposition 2.1 holds, but not the one in Theorem 2.3.

2.5 *Example.* Let $K = P(x, y_1, y_2)$ and

$$L = K(z, w, x^{p^{-1}}z + y_1^{\,p^{-1}}, x^{p^{-1}}w + y_2^{\,p^{-1}})$$

where P is a perfect field of characteristic $p \neq 0$ and $\{x, y_1, y_2, w, z\}$ is algebraically independent over P. Then

$$K^{p^{-1}}(L^p) \cap L = K(L^p)$$

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and hence for every intermediate field L_1 of L/K, the necessary condition of Proposition 2.1 holds. It is shown that L/K has a maximal separable intermediate field S which is not distinguished [**3**, Example 12]. Let $L_1 = S$. Then no distinguished subfield of L_1/K is contained in one of L/K. Hence the necessary condition of Theorem 2.3 cannot hold since $\operatorname{inex}(L/K) = 1$.

[**3**, Example 11] provides an extension L/K of transcendence degree >1 which has every maximal separable subfield distinguished. Hence every maximal intermediate field L_1 has every distinguished subfield in a distinguished subfield of L/K by Proposition 1.4.

2.6 THEOREM. *Let L_1 be an intermediate field of L/K. If L/L_1 is modular and* $\operatorname{inex}(L/L_1) = 1$, *then there exists a distinguished subfield of L_1/K contained in one of L/K.*

Proof. Since L/L_1 is modular, $L = S \otimes_{L_1} J$ where S and J are intermediate fields of L/L_1 such that S/L_1 is separable and J/L_1 is purely inseparable of exponent 1 [**8**, Theorem 4, p. 1178]. Since L/J is separable, every distinguished subfield of J/K is contained in one of L/K by Theorem 1.2. Thus it suffices to prove the theorem for $L^p \subseteq L_1$. Let $n = \operatorname{inex}(L/K)$. Let $Y_3 \subseteq L^{p^{n+1}} \subseteq L_1^{p^n}$ be a relative p-basis of $K(L^{p^{n+1}})/K(L_1^{p^{n+1}})$. Then $Y_3^{p^{-1}}$ is a relative p-basis of $K^{p^{-1}}(L^{p^n})/K^{p^{-1}}(L_1^{p^n})$ and hence is relatively p-independent in $K(L^{p^n})/K(L_1^{p^n})$. Extend $Y_3^{p^{-1}}$ to a relative p-basis $Y_1 \subseteq L^{p^n}$ of $K(L^{p^n})/K(L_1^{p^n})$. Let $Y_2 \subseteq L_1^{p^n}$ be a relative p-basis of $K(L_1^{p^n})/K(L^{p^{n+1}})$. Then $Y_2 \cup Y_3$ is a relative p-basis of $K(L_1^{p^n})/K$ and $Y_1 \cup Y_2$ is a relative p-basis of $K(L^{p^n})/K$. By Lemma 1.1,

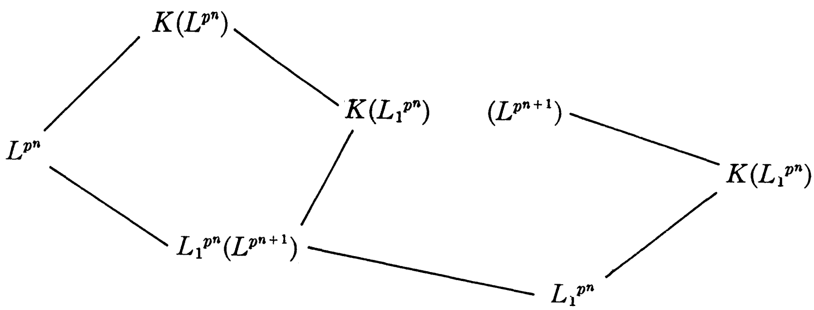
$$D_1 = K(L_1^{p^n})(Y_2^{p^{-n}}, Y_3^{p^{-n}}) \quad \text{and} \quad D = K(L^{p^n})(Y_1^{p^{-n}}, Y_2^{p^{-n}})$$

are distinguished subfields of L_1/K and L/K respectively. Clearly $D_1 \subseteq D$.

3. In this section we examine containment relations for distinguished subfields in the case where L_1 is a form of L/K. Recall L_1 is a form if $\operatorname{inor}(L/K) = \operatorname{inor}(L_1/K)$. Forms L_1 have been characterized by the condition L^{p^n} and $K(L_1^{p^n})$ are linearly disjoint over $L_1^{p^n}$ for all n [**2**, Theorem 1.3, p. 656].

3.1 PROPOSITION. *Let L_1/K be a form of L/K. If B is a relative p-basis of L/L_1, then B is part of a separating transcendence basis of a distinguished subfield of L/K. Conversely, if B is a separating transcendence basis of a distinguished subfield of L/K, then B contains a relative p-basis of L/L_1.*

Proof. Suppose B is a relative p-basis of L/L_1. Since L_1/K is a form of L/K, we have linear disjointness in the following diagram by [**6**, Lemma, p. 162], where $n = \operatorname{inex}(L/K)$.



Clearly then B^{p^n} is relatively p-independent in $K(L^{p^n})/K$. By Lemma 1.1, B is part of a separating transcendence basis of a distinguished subfield.

Conversely, suppose B is a separating transcendence basis of a distinguished subfield D. Then every element of D is separable algebraic over $K(B)$. Hence $D \subseteq L_1(L^p)(B)$. By [**2**, Theorem 2.2, p. 659], $L = DL_1$, so $L = L_1(L^p)(B)$.

Thus, the number of elements of a relative p-basis of L over a form L_1 is bounded above by the transcendence degree of L/K. For L/K finitely generated, insep(L/K) equals the number of elements in a relative p basis of L over K less the transcendence degree of L/K.

3.2 Corollary. *Let L_1/K be a form of L/K and let $n =$* inex(L/K). *Then the following conditions are equivalent.*

(1) $K(L_1^{p^n}) \subseteq K(L^{p^{n+1}})$.

(2) *transcendence degree of* $(L/K) = \log_p[L:L(L_1^p)]$

(3) *B is a relative p-basis of L/L_1 if and only if B is a separating transcendence basis of a distinguished subfield of L/K.*

(4) insep(L/L_1) = *transcendence degree of L_1/K.*

Proof. The equivalence of the conditions (1), (2), and (3) follows easily from Proposition 3.1 and the diagram there. The equivalence of (2) and (4) follows easily from the definition of insep(L/K).

3.3 Theorem. *Let L_1/K be a form of an inseparable extension L/K. Every distinguished subfield of L_1/K is contained in one of L/K if and only if L/L_1 is separable.*

Proof. Suppose every distinguished subfield of L_1/K is contained in one of L/K. We prove L/L_1 is separable by induction on the transcendence degree of L/K. Suppose it is 1. Then by Proposition 3.1 either $L = L_1(L^p)$ or $[L:L_1(L^p)] = p$. If $L = L_1(L^p)$, then since L/L_1 is finitely generated, L/L_1 is separable algebraic. Suppose $[L:L_1(L^p)] = p$. Then by Corollary 3.2, $K(L_1^{p^n}) \subseteq K(L^{p^{n+1}})$ so $L_1 \subseteq K^{p^{-n}}(L^p)$. Thus

$$L_1 \subseteq K(L^p) \cup K^{p^{-n}}(L_1^p)$$

by Proposition 2.1. Now $L_1 \not\subseteq K(L^p)$ since L_1/K is a form of L/K and

L/K is not separable. Hence $L_1 \subseteq K^{p^{-n}}(L_1{}^p)$. Thus $K(L_1{}^{p^n}) = K/(L_1{}^{p^{n+1}})$ and so L_1/K is algebraic. Thus the transcendence degree of L/L_1 is one, and since $[L:L_1(L^p)] \leqq 1$ by Proposition 3.1, L/L_1 is separable. Now suppose the transcendence degree of $L/K = d > 1$. Let $D_1 \subseteq D$ be distinguished subfields of L_1/K and L/K respectively. Assume $K(D^p) \not\supseteq D_1$. Then there is an $x \in D$ such that x is part of a separating transcendence of D_1/K and D/K. Now $L_1/K(x)$ is a form of $L/K(x)$ and if $D_1{}^*$ is a distinguished subfield of $L_1/K(x)$ it is also one of L_1/K. Hence $D_1{}^*$ is contained in a distinguished subfield D^* of L/K, and D^* is also a distinguished subfield of $L/K(x)$. That is, every distinguished subfield of $L_1/K(x)$ is in one of $L/K(x)$. Thus L/L_1 is separable by the induction hypothesis. Assume $K(D^p) \supseteq D_1$. Then

$$L_1 \subseteq K^{p^{-n}}(D^p) \subseteq K^{p^{-n}}(L^p).$$

Hence by Proposition 2.1,

$$L_1 \subseteq K(L^p) \cup K^{p^{-n}}(L_1{}^p).$$

As above, we conclude L/L_1 is separable. The converse follows from Theorem 1.2.

3.4 COROLLARY. *The following conditions are equivalent for an inseparable extension of L/K.*

(1) *For every form L_1/K of L/K, every distinguished subfield of L_1/K is coseparable in one of L/K.*

(2) *For every form L_1/K of L/K, every distinguished subfield of L_1/K is contained in one of L/K.*

(3) *L/L^* is separable algebraic where L^* is the unique minimal form of L/K.*

Proof. Clearly (1) implies (2). Assume (2). Then L/L^* is separable. For any field L_1, $L \supseteq L_1 \supseteq L^*$,

$$\operatorname{inor}(L/K) \geqq \operatorname{inor}(L_1/K) \geqq \operatorname{inor}(L^*/K)$$

[**2**, Theorem 1.2, p. 656], so L_1 is a form of L/K. Thus $L^*(L^p)$ is a form of L/K and by Theorem 3.3, $L/L^*(L^p)$ is separable. Thus $L = L^*(L^p)$, i.e., L is separable algebraic over L^*. Assume (3). Then L is separable algebraic over any form L_1. Thus if D_1 is a distinguished subfield of L_1/K, $L = L_1 \otimes_{D_1} S$ where S/D_1 is separable algebraic. By a degree argument S is distinguished for L/K.

3.5 COROLLARY. *The following conditions are equivalent for a form L_1/K of an inseparable extension L/K.*

(1) *Every distinguished subfield of L_1/K is coseparable in one of L/K.*

(2) *There exists a distinguished subfield of L_1/K which is coseparable in one of L/K.*

(3) *Every distinguished subfield of* L_1/K *is contained in one of* L/K.

(4) L/L_1 *is separable.*

Proof. The equivalence of (3) and (4) is Theorem 3.3. The equivalence of (1) and (2) follows from Theorem 1.2. Let $n = \text{inex}(L/K)$. Since L_1/K is a form of L/K, L^{p^n} and $K(L_1^{p^n})$ are linearly disjoint over $L_1^{p^n}$. Thus L/L_1 is separable if and only if $K(L^{p^n})/K(L_1^{p^n})$ is separable. Hence (1) and (4) are equivalent by Theorem 2.2.

We now show it is possible for a form of L/K to have no distinguished subfield contained in one of L/K. We need the following result.

3.6 PROPOSITION. *Suppose* L_1/K *is a form of* L/K *such that* L/L_1 *is not separable and suppose* $\text{inex}(L/K) = 1 =$ *the transcendence degree of* L/K.

(1) *Let* D_1 *be a distinguished subfield of* L_1/K. *Then* D_1 *is contained in a distinguished subfield* D *of* L/K *if and only if* $D_1 \subseteq K(L^p)$.

(2) *No distinguished subfield* D_1 *of* L_1/K *is contained in* $K(L^p)$ *if and only if*

$$L_1 \cap K(L^p) = K(L_1^{(1)}) \cap K(L^p),$$

where

$$K(L_1^{(1)}) = \{x \in L_1 | x^p \in K(L_1^{p^2})\}.$$

Proof. (1) Suppose $D_1 \subseteq D$, but $D_1 \not\subseteq K(L^p)$. Since $\text{inex}(L/K) = 1$, $K(L^p) = K(D^p)$. Thus if $\{x\}$ is a separating transcendence basis of D_1/K, $\{x\}$ is also a separating transcendence basis of D/K. Thus D/D_1 is separable algebraic and L/L_1 is separable (Corollary 3.5), a contradiction. Conversely, if $D_1 \subseteq K(L^p)$, then D_1 is in every distinguished subfield of L/K.

(2) Now $x \in L_1 \backslash K(L_1^{(1)})$ if and only if $x^p \notin K(L_1^{p^2})$, i.e., x is a separating transcendence basis of a distinguished subfield of L_1/K. Hence no distinguished subfield of L_1/K is in $K(L^p)$ if and only if

$$(L_1 \backslash K(L_1^{(1)})) \cap K(L^p) = \emptyset,$$

i.e., if and only if

$$L_1 \cap K(L^p) \subseteq K(L_1^{(1)}) \cap K(L^p).$$

3.7 *Example.* $\text{inex}(L/K) = 1 =$ the transcendence degree of L/K and there exists a form L_1 of L/K such that no distinguished subfield of L_1/K is contained in one of L/K: Let

$$K = P(x), \quad L_1 = K((z^p + x^{p-1})^2, x^{p-1})$$

and

$$L = K(z, x^{p-1})$$

where P is a perfect field of characteristic $p > 2$ and $\{x, z\}$ is algebraically independent over P. By Proposition 3.6, it suffices to show

$$L_1 \cap K(L^p) = K(L_1^{(1)}) \cap K(L^p)$$

since L/L_1 is clearly not separable. It follows easily that

$$K(L_1^{(1)}) \cap K(L^p) = K((z^{p^2} + x)^2).$$

Suppose

$$L_1 \cap K(L^p) \supsetneqq K((z^{p^2} + x)^2).$$

Then $K(L^p)/(L_1 \cap K(L^p))$ is separable algebraic and

$$(L_1 \cap K(L^p))/K((z^{p^2} + x)^2)$$

is purely inseparable. Thus

$$K(L^p) = (L_1 \cap K(L^p)) \otimes_{K(z^{p^2}+x)^2)} S$$

where $S/K((z^{p^2} + x)^2)$ is separable. Now z^p is a root of the irreducible polynomial

$$t^{2p} + 2xt^p - (z^{2p^2} + 2z^{p^2}x)$$

in an indeterminate t over $K((z^{p^2} + x)^2)$. Hence by [**10**, Lemma 3.7, p. 102], $x^{p-1} \in K(z^p)$, a contradiction.

Proposition 3.6 can also be used to establish conditions for a form of L/K to have a distinguished subfield contained in one of L/K. Recall that an inseparable algebraic extension L/K is *exceptional* if $K^{p-1} \cap L = K$.

3.8 THEOREM. *Let L_1/K be a form of L/K such that L/L_1 is not exceptional. If* $\mathrm{inex}(L/K) = 1 =$ *transcendence degree of L/K, then there exists a distinguished subfield of L_1/K contained in one of L/K.*

Proof. If L_1 is algebraic over K, then the unique distinguished subfield for L_1 is in every distinguished subfield of L/K. Thus we may assume L/L_1 is algebraic. Suppose no distinguished subfield of L_1/K is contained in one of L/K. Then

$$L_1 \cap K(L^p) = K(L_1^{(1)}) \cap K(L^p)$$

by Proposition 3.6. Since $\mathrm{inex}(L/K) = 1 =$ transcendence degree of L/K if

$$K(L_1^{(1)}) \cap K(L^p) \supsetneqq K(L_1^p)$$

then $K(L_1^{(1)}) \cap K(L^p)$ is distinguished for L_1/K since $K(L_1^{(1)}) \cap K(L^p)$ is separable over K. However, this is impossible since $K(L_1^{(1)}) \cap$

$K(L^p)$ is in every distinguished subfield of L/K. Thus

$$K(L_1^{(1)} \cap K(L^p) = K(L_1^p).$$

Hence $L_1 \cap K(L^p) = K(L_1^p)$. Since L^p and $K(L_1^p)$ are linearly disjoint over L_1^p,

$$L^p \cap K(L_1^p) = L_1^p.$$

Thus $L^p \cap L_1 = L_1^p$. Since L/L_1 cannot be separable, L/L_1 must be exceptional, a contradiction.

3.9 THEOREM. *Let L_1/K be a form of L/K. If L/L_1 is modular, then there exists a distinguished subfield of L_1/K contained in one of L/K.*

Proof. As in the proof of Theorem 2.6, it suffices to assume L/L_1 is purely inseparable. Let $\{x_1, \ldots, x_s\}$ be a subbasis of L/L_1 and let n_i be the exponent of x_i over L_1 and let n be the inseparability exponent of L/K. Since $L^{p^{n+1}}$ and $K(L_1^{p^{n+1}})$ are linearly disjoint over $L_1^{p^{n+1}}$, $\{x^{p^{n+n_1}}, \ldots, x_s^{p^{n+n_s}}\}$ is relatively p-independent over $K(L_1^{p^{n+1}})$ since it is such over $L_1^{p^{n+1}}$. Thus $\{x_1^{p^{n_1}}, \ldots, x_s^{p^{n_s}}\}$ is part of a separating transcendence basis of a distinguished subfield D_1 of L_1/K (Lemma 1.1). Thus

$$L = D_1(x_1, \ldots, x_s) \otimes_{D_1} L_1$$

by a degree argument. Hence $D_1(x_1, \ldots, x_s)$ is a distinguished subfield of L/K.

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