

PERIODIC k -GRAPH ALGEBRAS REVISITED

DILIAN YANG

(Received 18 September 2014; accepted 16 February 2015; first published online 20 April 2015)

Communicated by Aidan Sims

Abstract

Let P be a finitely generated cancellative abelian monoid. A P -graph Λ is a natural generalization of a k -graph. A pullback of Λ is constructed by pulling it back over a given monoid morphism to P , while a pushout of Λ is obtained by modding out its periodicity, which is deduced from a natural equivalence relation on Λ . One of our main results in this paper shows that, for some k -graphs Λ , Λ is isomorphic to the pullback of its pushout via a natural quotient map, and that its graph C^* -algebra can be embedded into the tensor product of the graph C^* -algebra of its pushout and $C^*(\text{Per } \Lambda)$. As a consequence, in this case, the cycline algebra generated by the standard generators corresponding to equivalent pairs is a maximal abelian subalgebra, and there is a faithful conditional expectation from the graph C^* -algebra onto it.

2010 *Mathematics subject classification*: primary 46L05; secondary 47L30.

Keywords and phrases: k -graph, P -graph, pullback, pushout, periodicity, maximal abelian subalgebra.

1. Introduction

Let P be a finitely generated cancellative abelian monoid. It was first suggested to study P -graphs in [9], where \mathbb{N}^k -graphs (now known as k -graphs or *higher-rank graphs*) were studied. But this has not been done until very recently. Based on some ideas in [9], Carlsen *et al.* study P -graphs, their pullbacks and associated C^* -algebras in [2]. Making use of them as a tool, they successfully describe the picture of the primitive ideal space in the C^* -algebra of a row-finite and source-free higher-rank graph. We should mention that the primitive ideal space of directed graph C^* -algebras is studied in [5].

In this paper, we study pullbacks and pushouts of P -graphs, and we are particularly interested in their applications to periodic higher-rank graphs. Let P and Q be two finitely generated cancellative abelian monoids, $f : P \rightarrow Q$ be a monoid morphism, and Γ be a row-finite and source-free Q -graph. It is natural to construct a P -graph $f^*\Gamma$, which is called the pullback of Γ via f , from Γ and f . This idea first appeared in [9] and was further explored in [2]. To construct pushouts is a little bit more involved;

The author was partially supported by an NSERC grant.

© 2015 Australian Mathematical Publishing Association Inc. 1446-7887/2015 \$16.00

we make use of an idea from [3, 4], which has been substantially generalized to higher-rank graphs in [2]. Let Λ be a row-finite and source-free P -graph. Define a natural equivalence relation \sim on Λ , and then construct the periodicity $\text{Per } \Lambda$ from \sim . Loosely speaking, under some suitable conditions the pushout Λ/\sim of Λ is a $\mathbf{q}(P)$ -graph, which is obtained by modding out the periodicity of Λ . One key property we observe is that the pushout Λ/\sim is aperiodic. This seems rather natural, but to prove it needs some care. It is shown that Λ is isomorphic to the pullback of the pushout Λ/\sim : $\Lambda \cong \mathbf{q}^*(\Lambda/\sim)$. In this case, $C^*(\Lambda)$ can be embedded into the tensor product of $C^*(\Lambda/\sim)$ with $C^*(\text{Per } \Lambda)$. Actually, it turns out that this embedding is the best one could possibly obtain, in the sense that $C^*(\Lambda)$ could not be isomorphic to $C^*(\Lambda/\sim) \otimes C^*(\text{Per } \Lambda)$ in any canonical way. Along the way, we also give an in-depth study on periodicity of P -graphs.

Our results are applied to the cycline subalgebra \mathcal{M} of a higher-rank graph. By definition, \mathcal{M} is the sub- C^* -algebra of $C^*(\Lambda)$ generated by the standard generators corresponding to equivalent pairs in Λ . It turns out that \mathcal{M} plays an important role in the structure of $C^*(\Lambda)$ due to the following Cuntz–Krieger uniqueness theorem in [1]: a representation of $C^*(\Lambda)$ is injective if and only if so is its restriction onto \mathcal{M} . It was asked in [1] if \mathcal{M} is an abelian core of $C^*(\Lambda)$ [11]. We answer this question affirmatively for a large class of k -graphs. That is, we show that \mathcal{M} is a maximal abelian subalgebra (MASA) of $C^*(\Lambda)$, which is actually isomorphic to the tensor product of the canonical diagonal algebra \mathfrak{D}_Λ with $C^*(\text{Per } \Lambda)$, and that there is also a faithful conditional expectation from $C^*(\Lambda)$ onto \mathcal{M} .

This paper is motivated by [1, 2] and strongly influenced by [2]. In Section 2 some necessary background on P -graphs and the associated C^* -algebras is briefly given. The main result of Section 3 is an embedding theorem, which roughly says that the C^* -algebras of a class of k -graph C^* -algebras, being pullbacks, can be embedded into tensor products of P -graph C^* -algebras with commutative C^* -algebras. In Section 4 we study the periodicity of P -graphs in detail. We give some characterizations of aperiodicity, and provide a (sort of) concrete description for periodicity. Our results generalize and unify what we have known for row-finite and source-free k -graphs in the literature. We finish off this section by showing that all pullbacks are periodic. We believe that the results of this section may be of independent interest and very useful in the future. In Section 5 we reverse the process of Section 3. Loosely speaking, we start with a P -graph Λ whose periodicity $\text{Per } \Lambda$ is a subgroup of the Grothendieck group $\mathcal{G}(P)$ of P . By modding out its periodicity, we obtain an aperiodic pushout, whose pullback is isomorphic to Λ . As an application of our results, in Section 6 we give a partial answer to a question left open in [1].

Notation and conventions. In this paper, all monoids are assumed to be finitely generated, cancellative and abelian. If P is such a monoid, we also regard it as a category with one object. We use $\mathcal{G}(P)$ to denote the Grothendieck group of P , and we always embed P into $\mathcal{G}(P)$.

As usual, for $m, n \in \mathbb{Z}^k$, we use $m \vee n$ and $m \wedge n$ to denote the coordinatewise maximum and minimum of m and n , respectively. For $n \in \mathbb{Z}^k$, we let $n_+ = n \vee 0$ and $n_- = -(n \wedge 0)$. Of course, $n = n_+ - n_-$ with $n_+ \wedge n_- = 0$.

2. P -graphs

Let P be a monoid. P -graphs are a generalization of k -graphs, and share many properties with k -graphs. In this section, we briefly recall some basics on P -graphs which will be needed later. Refer to [2, Section 2] for more details.

A P -graph is a countable small category Γ with a functor $d : \Gamma \rightarrow P$ such that the following factorization property holds: whenever $\xi \in \Gamma$ satisfies $d(\xi) = p + q$, there are unique elements $\eta, \zeta \in \Gamma$ such that $d(\eta) = p$, $d(\zeta) = q$ and $\xi = \eta\zeta$. Clearly, any k -graph is an \mathbb{N}^k -graph. All notions on k -graphs can be generalized to P -graphs. For instance, for $p \in P$, let $\Gamma^p = d^{-1}(p)$, and so Γ^0 is the vertex set of Γ . There are source and range maps $s, r : \Gamma \rightarrow \Gamma^0$ such that $r(\xi)\xi s(\xi) = \xi$ for all $\xi \in \Gamma$. For $v \in \Gamma^0$, $v\Gamma = \{\xi \in \Gamma : r(\xi) = v\}$ and $\Gamma v = \{\xi \in \Gamma : s(\xi) = v\}$. We say that a P -graph Γ is *source-free* (respectively *sink-free*, *row-finite*) if $|v\Gamma^p| > 0$ (respectively $|\Gamma^p v| > 0$, $|v\Gamma^p| < \infty$) for all $v \in \Gamma^0$ and $p \in P$.

Let

$$\Omega_P = \{(p, q) \in P \times P \mid q - p \in P\}.$$

Define $d, s, r : \Omega_P \rightarrow P$ by $d(p, q) = q - p$, $s(p, q) = q$, and $r(p, q) = p$. It is shown in [2, Example 2.2] that Ω_P is a row-finite and source-free P -graph.

Let Λ and Γ be two P -graphs. A P -graph morphism between Λ and Γ is a functor $x : \Lambda \rightarrow \Gamma$ such that $d_\Gamma(x(\lambda)) = d_\Lambda(\lambda)$ for all $\lambda \in \Lambda$. The *infinite path space* of Γ is defined as

$$\Gamma^\infty = \{x : \Omega_P \rightarrow \Gamma \mid x \text{ is a } P\text{-graph morphism}\}.$$

For $x \in \Gamma^\infty$ and $p \in P$, there is a unique element $\sigma^p(x) \in \Gamma^\infty$ defined by

$$\sigma^p(x)(q, r) = x(p + q, p + r).$$

That is, σ^p is a shift map on Γ^∞ . If $\mu \in \Gamma$ and $x \in s(\mu)\Gamma^\infty$, then μx is defined to be the unique infinite path such that $\mu x(0, p) = \mu \cdot x(0, p - d(\mu))$ for any $p \in P$ with $p - d(\mu) \in P$. If $\sigma^p(x) = \sigma^q(x)$ for some $p \neq q \in P$, x is said to be (eventually) *periodic*.

DEFINITION 2.1. A P -graph Γ is said to be *periodic* if there is $v \in \Gamma^0$ such that every $x \in v\Gamma^\infty$ is periodic. Otherwise, Γ is called *aperiodic*.

For a row-finite and source-free P -graph Γ , we associate to it a universal C^* -algebra $C^*(\Gamma)$ as follows.

DEFINITION 2.2. Let Γ be a row-finite and source-free P -graph. A *Cuntz–Krieger Γ -family* in a C^* -algebra \mathcal{A} is a family $\{S_\lambda : \lambda \in \Gamma\}$ in \mathcal{A} such that:

- (CK1) $\{S_\nu : \nu \in \Gamma^0\}$ is a set of mutually orthogonal projections;
- (CK2) $S_\mu S_\nu = S_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;

- (CK3) $S_\lambda^* S_\lambda = S_{s(\lambda)}$ for all $\lambda \in \Gamma$;
- (CK4) $S_v = \sum_{\lambda \in v\Gamma^p} S_\lambda S_\lambda^*$ for all $v \in \Gamma^0$ and $p \in P$.

The P -graph C^* -algebra $C^*(\Gamma)$ is the universal C^* -algebra among Cuntz–Krieger Γ -families. We usually use s_λ to denote its generators.

It is known that

$$C^*(\Gamma) = \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Gamma\}.$$

One important property is that $C^*(\Gamma)$ and the reduced C^* -algebra $C_r^*(\mathcal{G}_\Gamma)$ are isomorphic, where \mathcal{G}_Γ is the groupoid associated to Γ .

By the universal property of $C^*(\Gamma)$, there is a natural gauge action γ of the dual group $\widehat{\mathcal{G}(P)}$ of $\mathcal{G}(P)$ on $C^*(\Gamma)$ defined by

$$\gamma_\chi(s_\lambda) = \chi(d(\lambda))s_\lambda \quad (\chi \in \widehat{\mathcal{G}(P)}, \lambda \in \Gamma).$$

Averaging over γ gives a faithful conditional expectation Φ from $C^*(\Gamma)$ onto the fixed point algebra $C^*(\Gamma)^\gamma$. It turns out that $C^*(\Gamma)^\gamma$ is an AF algebra and

$$\mathfrak{F}_\Gamma := C^*(\Gamma)^\gamma = \overline{\text{span}}\{s_\mu s_\nu^* : d(\mu) = d(\nu)\}.$$

The (canonical) diagonal algebra \mathfrak{D}_Γ of $C^*(\Gamma)$ is defined as

$$\mathfrak{D}_\Gamma = \overline{\text{span}}\{s_\mu s_\mu^* : \mu \in \Gamma\},$$

which is the canonical MASA of \mathfrak{F}_Γ .

Furthermore, as k -graphs, we also have the following two important uniqueness theorems for $C^*(\Gamma)$: the gauge invariant uniqueness theorem [2, Proposition 2.7] and the Cuntz–Krieger uniqueness theorem [2, Corollary 2.8].

Throughout rest of this paper, all P -graphs are assumed to be row-finite and source-free. An \mathbb{N}^k -graph is simply called a k -graph as in the literature.

3. Pullbacks and an embedding theorem

We begin this section with the notion of pullbacks. Let P and Q be two monoids, and $f : P \rightarrow Q$ be a (monoid) homomorphism. If (Γ, d_Γ) is a Q -graph, the pullback of Γ via f is the P -graph $(f^*\Gamma, d_{f^*\Gamma})$ defined as follows: $f^*\Gamma = \{(\lambda, p) : d_\Gamma(\lambda) = f(p)\}$ with $d_{f^*\Gamma}(\lambda, p) = p$, $s(\lambda, p) = s(\lambda)$ and $r(\lambda, p) = r(\lambda)$. The composition of two paths in $f^*\Gamma$ is given by $(\mu, p)(\nu, q) = (\mu\nu, p + q)$ if $s(\mu) = r(\nu)$.

The pullback $f^*\Gamma$ defined above is indeed a P -graph. The proof is completely similar to that of [2, Lemma 3.2] where $P = \mathbb{N}^k$. The following properties are easily derived from the very definition of pullbacks, and so their proofs are omitted here.

LEMMA 3.1. *Let P and Q be two monoids, and $f : P \rightarrow Q$ be a homomorphism. Suppose that (Γ, d_Γ) is a Q -graph. Then we have the following.*

- (i) *If Γ is source-free, then so is $f^*\Gamma$. The converse holds if f is surjective.*
- (ii) *If $f^*\Gamma$ is row-finite, then so is Γ . The converse holds if f is surjective.*

In the remainder of this section, we focus on morphisms f induced from group homomorphisms on \mathbb{Z}^k . In this case, we can prove the following embedding theorem.

THEOREM 3.2. *Let H be a subgroup of \mathbb{Z}^k , and $\mathbf{q} : \mathbb{Z}^k \rightarrow \mathbb{Z}^k/H$ be the quotient map. Suppose that Γ is a $\mathbf{q}(\mathbb{N}^k)$ -graph. Then there is an injective C^* -homomorphism from $C^*(\mathbf{q}^*\Gamma)$ to $C^*(\Gamma) \otimes C^*(H)$ ¹.*

PROOF. Without loss of generality, we assume that H has rank $r \geq 1$. Then there is a basis $\{f_1, \dots, f_r, \dots, f_k\}$ of \mathbb{Z}^k and $d_1, \dots, d_r \geq 1$ such that d_i divides d_{i+1} for all $1 \leq i \leq r - 1$ and $\{d_1 f_1, \dots, d_r f_r\}$ is a basis of H [14, Theorem 2.12]. Furthermore, it follows from [14, Theorem 2.14] that $\mathbb{Z}^k/H \cong \mathbb{Z}_{d_1} \times \dots \times \mathbb{Z}_{d_r} \times \mathbb{Z}^{k-r}$ under the isomorphism

$$\begin{aligned} \varphi : \mathbb{Z}^k/H &\rightarrow \mathbb{Z}_{d_1} \times \dots \times \mathbb{Z}_{d_r} \times \mathbb{Z}^{k-r}, \\ \left[\sum_{i=1}^k n_i f_i \right] &\mapsto ([n_1]_{d_1}, \dots, [n_r]_{d_r}, n_{r+1}, \dots, n_k). \end{aligned}$$

So we identify \mathbb{Z}^k/H with $\mathbb{Z}_{d_1} \times \dots \times \mathbb{Z}_{d_r} \times \mathbb{Z}^{k-r}$ via φ in what follows.

Define a ‘projection’ j from \mathbb{Z}^k/H onto the torsion-free part via

$$j : \mathbb{Z}^k/H \rightarrow \mathbb{Z}^k, \quad ([n_1]_{d_1}, \dots, [n_r]_{d_r}, n_{r+1}, \dots, n_k) \mapsto (0, \dots, 0, n_{r+1}, \dots, n_k).$$

Clearly, j is a homomorphism. Implicitly identifying the above basis $\{f_i\}_{i=1}^k$ with the standard basis $\{e_i\}_{i=1}^k$ of \mathbb{Z}^k , one has that $H = \bigoplus_{i=1}^r H_i$ with $H_i = \langle d_i f_i \rangle$ in the i th factor of $\mathbb{Z}^k = \bigoplus_{i=1}^k \mathbb{Z}$. So $j \circ \mathbf{q}$ is the projection from \mathbb{Z}^k onto the last $k - r$ factors:

$$j \circ \mathbf{q}(n) = (0, \dots, 0, n_{r+1}, \dots, n_k) \quad \forall n \in \mathbb{Z}^k.$$

For $1 \leq i \leq r$, let $h_i := d_i f_i$ and V_i be the unitary generators in the group C^* -algebra $C^*(H)$ corresponding to h_i . Define an action $\Theta : \mathbb{T}^k \rightarrow \text{Aut}(C^*(\Gamma) \otimes C^*(H))$ via

$$\Theta_t(s_\mu \otimes V^n) = t^{j \circ d(\mu)} s_\mu \otimes t^n V^n \quad \forall t \in \mathbb{T}^k, \tag{3.1}$$

where $n = (n_1, \dots, n_r, 0, \dots, 0) \in \mathbb{Z}^k$, and we use the multi-index notation $V^n = \prod_{i=1}^r V_i^{n_i}$. One may think of the action Θ as the tensor product of a natural action of $H^\perp \subset C^*(\Gamma)$ on $C^*(\Gamma)$ and the gauge action of \widehat{H} on $C^*(H)$.

Notice that, for any $(\mu, n) \in \mathbf{q}^*\Gamma$, one has

$$n - j \circ d(\mu) = n - j \circ \mathbf{q}(n) = (n_1, \dots, n_r, 0, \dots, 0).$$

Then one can verify that $\{s_\mu \otimes V^{n-j \circ d(\mu)} : d(\mu) = \mathbf{q}(n)\}$ is a Cuntz–Krieger $\mathbf{q}^*\Gamma$ -family. To this end, let $t_{(\mu,n)} := s_\mu \otimes V^{n-j \circ d(\mu)}$. Obviously (CK1) and (CK3) hold true. For (CK2), let $s(\mu) = s(\nu)$ and $(\mu, n), (\nu, m) \in \mathbf{q}^*\Gamma$. Then

$$\begin{aligned} t_{(\mu,n)} t_{(\nu,m)} &= (s_\mu \otimes V^{n-j \circ d(\mu)})(s_\nu \otimes V^{m-j \circ d(\nu)}) \\ &= s_\mu s_\nu \otimes V^{n+m-j \circ d(\mu)-j \circ d(\nu)} \\ &= s_{\mu\nu} \otimes V^{n+m-j \circ (d(\mu)+d(\nu))} \\ &= t_{(\mu\nu, n+m)} = t_{(\mu,n)(\nu,m)}, \end{aligned}$$

¹Since the group C^* -algebra $C^*(H)$ is abelian, it does not matter which C^* -tensor product one chooses.

where the third ‘=’ used the property (CK2) for $\{s_\mu : \mu \in \Gamma\}$ and the property that j is a homomorphism. To verify (CK4), for $v \in (\mathbf{q}^*\Gamma)^0 = \Gamma^0$ and $n \in \mathbb{N}^k$, we compute

$$\begin{aligned} \sum_{(\mu,n) \in v(\mathbf{q}^*\Gamma)^n} t_{(\mu,n)} t_{(\mu,n)}^* &= \sum_{(\mu,n) \in v(\mathbf{q}^*\Gamma)^n} (s_\mu \otimes V^{n-j \circ d(\mu)})(s_\mu \otimes V^{n-j \circ d(\mu)})^* \\ &= \sum_{(\mu,n) \in v(\mathbf{q}^*\Gamma)^n} s_\mu s_\mu^* \otimes I \\ &= \sum_{\mu \in v\Gamma^{\mathbf{q}(n)}} s_\mu s_\mu^* \otimes I \\ &= s_v \otimes I \end{aligned}$$

due to property (CK4) for $\{s_\mu : \mu \in \Gamma\}$.

By the universal property of $C^*(\mathbf{q}^*\Gamma)$, there is a (unique) $*$ -homomorphism π determined by

$$\begin{aligned} \pi : C^*(\mathbf{q}^*\Gamma) &\rightarrow C^*(\Gamma) \otimes C^*(H), \\ s_{(\mu,n)} &\mapsto s_\mu \otimes V^{n-j \circ d(\mu)}, \end{aligned} \tag{3.2}$$

where $(\mu, n) \in \mathbf{q}^*\Gamma$. It is easy to check that π is equivariant between the gauge action γ of \mathbb{T}^k on $C^*(\mathbf{q}^*\Gamma)$ and the action Θ on $C^*(\Gamma) \otimes C^*(H)$ defined by (3.1). In fact, for $t \in \mathbb{T}^k$ and $(\mu, n) \in \mathbf{q}^*\Gamma$, we have

$$\begin{aligned} \Theta_t \circ \pi(s_{(\mu,n)}) &= \Theta_t(s_\mu \otimes V^{n-j \circ d(\mu)}) \\ &= t^{j \circ d(\mu)} s_\mu \otimes t^{n-j \circ d(\mu)} V^{n-j \circ d(\mu)} \\ &= t^n s_\mu \otimes V^{n-j \circ d(\mu)} \\ &= \pi \circ \gamma_t(s_{(\mu,n)}). \end{aligned}$$

By the gauge invariant uniqueness theorem of k -graphs [9, Theorem 3.4], π is injective. □

One naturally wonders if π defined in (3.2) is also surjective. Unfortunately, this is not the case in general, as the following simple example shows. Actually, Example 5.5 in Section 6 shows that Theorem 3.2 is the best one could have.

EXAMPLE 3.3. Let $H = 2\mathbb{Z}$ and consider the $\mathbb{N}/H = \mathbb{Z}_2$ -graph Γ : $\Gamma^0 = \{v\}$, and $\Gamma^1 = \{e\}$. So $\mathbf{q}^*\Gamma$ is a 1-graph: $\mathbf{q}^*\Gamma^0 = \{v\}$, and $(\mathbf{q}^*\Gamma)^1 = \{(e, 1)\}$. That is, $\mathbf{q}^*\Gamma$ is a single-vertex directed graph with one edge. So $C^*(\mathbf{q}^*\Gamma) = C^*(H) = C(\mathbb{T})$, while $C^*(\Gamma) \otimes C^*(H) = C^*(\mathbb{Z}_2) \otimes C^*(H)$.

We will return to this example again in Section 5.

In some special cases, the above embedding could become an isomorphism. The following generalizes [9, Corollary 3.5(iii)].

COROLLARY 3.4. *If \mathbb{Z}^k/H is torsion-free, then $C^*(\mathbf{q}^*\Gamma) \cong C^*(\Gamma) \otimes C^*(H)$.*

PROOF. Keep the same notation as in the proof of Theorem 3.2, including the choice of basis $\{f_1, \dots, f_k\}$ and the implicit identification of the standard basis for \mathbb{Z}^k with this basis. By the definition of π in (3.2), one can see that the range of π is the C^* -algebra

$$\begin{aligned} \mathfrak{A} &:= \pi(C^*(\mathbf{q}^*\Gamma)) = C^*(s_\mu s_\nu^* \otimes V^n : d(\mu) - d(\nu) = \mathbf{q}(n - j \circ \mathbf{q}(n))) \\ &= C^*(s_\mu s_\nu^* \otimes V^n : d(\mu)_i - d(\nu)_i = [n_i]_{d_i} : 1 \leq i \leq r). \end{aligned}$$

Since \mathbb{Z}^k/H is torsion-free, then $d_i = 1$ and $d(\mu)_i = d(\nu)_i = [n_i] = 0$ for $1 \leq i \leq r$. Thus $s_\mu s_\nu^* \otimes V^n$ is in \mathfrak{A} for all $\mu, \nu \in \Gamma$ and $n \in H$. Therefore, $\mathfrak{A} = C^*(\Gamma) \otimes C^*(H)$, as desired. \square

One important consequence of Theorem 3.2 is given below. But a lemma first.

LEMMA 3.5 [2, Proposition 3.3]. *We retain the conditions of Theorem 3.2. Then, to any $h \in H$, there corresponds a unitary W_h in the centre of $\mathcal{M}(C^*(\mathbf{q}^*\Gamma))$, the multiplier algebra of $C^*(\mathbf{q}^*\Gamma)$, which is given by*

$$W_h = s\text{-}\lim_F \sum_{v \in F} \sum_{\lambda \in v\Gamma^{\mathbf{q}(h_+)}} s_{(\lambda, h_+)} s_{(\lambda, h_-)}^*.$$

Here the limit is taken in the strict topology as F increases over finite subsets of Γ^0 . Furthermore, one has

$$s_{(\lambda, h_+)} = W_h s_{(\lambda, h_-)} = s_{(\lambda, h_-)} W_h \quad \forall \lambda \in \Gamma^{\mathbf{q}(h_+)}.$$

Keep the same notation as in the proof of Theorem 3.2. Suppose that the rank of H is r (≥ 1 without loss of generality) and $H = \langle h_i := d_i f_i \mid 1 \leq i \leq r \rangle$. To simplify our writing, for $1 \leq i \leq r$, we use $W_i := W_{h_i}$ to denote the central unitaries determined by h_i in Lemma 3.5, and also use the multi-index notation $W^n = \prod_{i=1}^r W_i^{n_i}$ for $n = (n_1, \dots, n_r) \in \mathbb{Z}^r$.

COROLLARY 3.6. *We retain the conditions of Theorem 3.2. Then:*

- (i) $\mathfrak{D}_{\mathbf{q}^*\Gamma} \cong \mathfrak{D}_\Gamma$ and $\mathfrak{F}_{\mathbf{q}^*\Gamma} \cong \mathfrak{F}_\Gamma$;
- (ii) $\mathfrak{F}_{\mathbf{q}^*\Gamma} C^*(W_h : h \in H) \cong \mathfrak{F}_{\mathbf{q}^*\Gamma} \otimes C^*(W_h : h \in H)$.

PROOF. Keep the same notation as in the proof of Theorem 3.2.

(i) This follows from the definition of π (cf. (3.2)) as $\pi(s_{(\mu, m)} s_{(\nu, m)}^*) = s_\mu s_\nu^* \otimes I$ for all $(\mu, m), (\nu, m) \in \mathbf{q}^*\Gamma$.

(ii) Let $\mathfrak{A} := \mathfrak{F}_{\mathbf{q}^*\Gamma} C^*(W_h : h \in H)$. We first claim that

$$\pi(\mathfrak{A}) = \mathfrak{F}_\Gamma \otimes C^*(V_i^{d_i} : 1 \leq i \leq r).$$

Notice that

$$\begin{aligned} \pi(s_{(\lambda, h_{i_+})} s_{(\lambda, h_{i_-})}^*) &= (s_\lambda \otimes V^{h_{i_+} - j \circ d(\lambda)})(s_\lambda^* \otimes (V^*)^{h_{i_-} - j \circ d(\lambda)}) \\ &= (s_\lambda s_\lambda^*) \otimes V^{h_{i_+} - j \circ d(\lambda) - h_{i_-} + j \circ d(\lambda)} \\ &= s_\lambda s_\lambda^* \otimes V^{h_i} \\ &= s_\lambda s_\lambda^* \otimes V_i^{d_i}. \end{aligned} \tag{3.3}$$

Now arbitrarily take a standard element $s_\mu s_\nu^* \in \mathfrak{F}_\Gamma$ and $n = (n_1, \dots, n_r, 0, \dots, 0) \in \mathbb{Z}^k$. Consider $s_{(\mu,m)} s_{(\nu,m)}^* W^n \in \mathfrak{A}$ where $\mathbf{q}(m) = d(\mu) (= d(\nu))$. We derive from Lemma 3.5 and (3.3) that¹

$$\begin{aligned} &\pi(s_{(\mu,m)} s_{(\nu,m)}^* W^n) \\ &= s\text{-}\lim_F \sum_{v \in F} \left\{ (s_\mu \otimes V^{m-j\mathbf{q}(m)})(s_\nu \otimes V^{m-j\mathbf{q}(m)})^* \prod_i \left(\sum_{\lambda \in \Gamma^{\mathbf{q}(h_{i+})}} s_\lambda s_\lambda^* \otimes V_i^{d_i} \right)^{n_i} \right\} \\ &= s\text{-}\lim_F \sum_{v \in F} \left\{ (s_\mu s_\nu^* \otimes I) \prod_i \left(\sum_{\lambda \in \Gamma^{\mathbf{q}(h_{i+})}} s_\lambda s_\lambda^* \right)^{n_i} \otimes V_i^{d_i n_i} \right\} \\ &= s\text{-}\lim_F \sum_{v \in F} \{ s_\mu s_\nu^* s_\nu \otimes V^{dn} \} \\ &= \left\{ s\text{-}\lim_F \sum_{v \in F} s_\mu s_\nu^* s_\nu \right\} \otimes V^{dn} \\ &= s_\mu s_\nu^* \otimes V^{dn}. \end{aligned}$$

The last ‘=’ above used the fact that $(\sum_{v \in F} s_\nu)_F$ is strictly convergent to the identity of the multiplier algebra of $C^*(\Gamma)$. This fact can be proved by a standard argument (cf. [2, proof of Proposition 3.3]). Therefore, we have proved our claim.

Combining the injectivity of π thanks to Theorem 3.2 with $\mathfrak{F}_\Gamma \cong \mathfrak{F}_{\mathbf{q}^*\Gamma}$ from (i), one now has $\mathfrak{A} \cong \mathfrak{F}_{\mathbf{q}^*\Gamma} \otimes C^*(V_i^{d_i} : 1 \leq i \leq r)$. From (3.3), in the multiplier algebra $\mathcal{M}(C^*(\mathbf{q}^*\Gamma))$, one can verify that $C^*(W_i : 1 \leq i \leq r) \cong C^*(V_i^{d_i} : 1 \leq i \leq r)$. Hence $\mathfrak{A} \cong \mathfrak{F}_{\mathbf{q}^*\Gamma} \otimes C^*(W_h : h \in H)$, which ends the proof. \square

Let us remark that Corollary 3.6(i) can also be seen directly from the following identity, which generalizes the one given in Lemma 3.5.

If $(\mu, m) \in \mathbf{q}^*\Gamma$ and $n \in \mathbb{N}^k$ such that $h := n - m \in H$, then

$$s_{(\mu,n)} = W_h s_{(\mu,m)} = s_{(\mu,m)} W_h.$$

In fact, assume that $h = h_+ - h_-$. Then $n + h_- = m + h_+$. Let $\lambda \in \Gamma$ satisfying $r(\lambda) = s(\mu)$ and $d(\lambda) = h_-$. So (λ, h_+) also belongs to $\mathbf{q}^*\Gamma$. Then by Lemma 3.5,

$$s_{(\mu,n)} s_{(\lambda,h_-)} = s_{(\mu,\lambda,n+h_-)} = s_{(\mu,\lambda,m+h_+)} = s_{(\mu,m)} s_{(\lambda,h_+)} = s_{(\mu,m)} W_h s_{(\lambda,h_-)}.$$

Hence

$$s_{(\mu,n)} \sum_{\lambda \in s(\mu)\Gamma^{h_-}} s_{(\lambda,h_-)} s_{(\lambda,h_-)}^* = s_{(\mu,m)} W_h \sum_{\lambda \in s(\mu)\Gamma^{h_-}} s_{(\lambda,h_-)} s_{(\lambda,h_-)}^*,$$

which implies $s_{(\mu,n)} = W_h s_{(\mu,m)}$.

An important and interesting application of Theorem 3.2 and Corollary 3.6 will be exhibited in Section 6, after we investigate pushouts of P -graphs. But we need to study periodicity of P -graphs first. This is done in the next section.

¹Below we use the usual convention: for an operator A , $A^k = (A^*)^{-k}$ if $k < 0$.

4. Periodicity

It is known that the periodicity of higher-rank graphs plays a very important role in the structure of their C^* -algebras. See, for example, [2–4, 7, 10, 12, 13, 15]. This section provides a detailed analysis of the periodicity of P -graphs. We give some natural characterizations of the aperiodicity defined in [9] (via infinite paths), in terms of the triviality of $\text{Per } \Lambda$ (coinciding with our notion of aperiodicity) and/or local periodicities. Applying those results, we prove that all pullbacks are periodic. We believe that the results in this section may be of independent interest and useful in studying P -graphs in the future.

Let P be a monoid and Λ be a P -graph. Define an equivalence relation \sim on Λ as follows:

$$\xi \sim \eta \iff s(\xi) = s(\eta) \quad \text{and} \quad \xi x = \eta x \quad \forall x \in s(\xi)\Lambda^\infty. \tag{4.1}$$

If $\xi \sim \eta$, obviously one also has $r(\xi) = r(\eta)$ automatically. So \sim respects sources and ranges.

Associate to \sim two important sets: the *periodicity of Λ* ,

$$\text{Per } \Lambda = \{d(\xi) - d(\eta) : \xi, \eta \in \Lambda, \xi \sim \eta\} \subseteq \mathcal{G}(P),$$

and

$$\Lambda_{\text{Per}}^0 = \left\{ v \in \Lambda^0 \mid \begin{array}{l} \text{any } \xi \in v\Lambda, p \in P \text{ with } d(\xi) - p \in \text{Per } \Lambda \\ \implies \text{there is } \eta \in v\Lambda^p \text{ such that } \xi \sim \eta \end{array} \right\}.$$

Here are some versions of ‘local periodicity’:

$$\begin{aligned} \Sigma_v &= \{(m, n) \in P \times P : \sigma^m(x) = \sigma^n(x) \forall x \in v\Lambda^\infty\} \quad (v \in \Lambda^0), \\ \text{Per}_v &= \{m - n : (m, n) \in \Sigma_v\} \quad (v \in \Lambda^0), \\ \Sigma_\Lambda &= \bigcup_{v \in \Lambda^0} \Sigma_v. \end{aligned}$$

The above notion of periodicity, $\text{Per } \Lambda$, was first introduced in [3, 4] to study the representation theory of single-vertex k -graph algebras. It has been substantially generalized in [2] to all k -graphs (cf. also [7]). The set Λ_{Per}^0 was used in [2] to construct a subgraph of Λ , while Σ_v is related to a notion of local periodicity of Λ at v in [10, 13, 15].

4.1. Characterizations of aperiodicity. In this subsection, let us fix a monoid P and a P -graph Λ .

LEMMA 4.1. *Let λ, μ in Λ be such that $\lambda \sim \mu$ and $d(\mu) = d(v)$. Then $\lambda = \mu$.*

PROOF. Fix $x \in s(\lambda)\Lambda^\infty$. Since $\lambda \sim \mu$, we have $\lambda x = \mu x$. This implies $\lambda = (\lambda x)(0, d(\lambda)) = (\lambda x)(0, d(\mu)) = (\mu x)(0, d(\mu)) = \mu$. □

LEMMA 4.2. *Let $v \in \Lambda^0$ and $(m, n) \in \Sigma_v$. Then $x(m, m+n) \sim x(n, m+n)$ for all $x \in v\Lambda^\infty$.*

PROOF. Let $x \in v\Lambda^\infty$, $v := x(m, m+n)$, and $\mu = x(n, m+n)$. Obviously, $s(\mu) = s(v)$.

In the sequel, we first show

$$x(0, m)v = x(0, n)\mu. \quad (4.2)$$

Indeed, by the factorization property, there are $v' \in v\Lambda^n$ and $\mu' \in s(v')\Lambda^m$ such that

$$x(0, m+n) = x(0, m)v = v'\mu'.$$

But

$$v'\mu' = x(0, m+n) = x(0, n)\mu'$$

implies $v' = x(0, n)$. So

$$x(0, m)v = x(0, n)\mu'.$$

Thus $x(0, m)v\sigma^{m+n}(x) = x(0, n)\mu'\sigma^{m+n}(x)$. One now has $v\sigma^{m+n}(x) = \mu'\sigma^{m+n}(x)$ as $(m, n) \in \Sigma_v$. But $\sigma^m(x) = v\sigma^{m+n}(x)$ and $\sigma^m(x) = \sigma^n(x)$, so one has $\sigma^n(x) = \mu'\sigma^{m+n}(x)$, implying

$$\mu' = (\sigma^n(x))(0, m) = x(n, n+m) = \mu.$$

This proves (4.2).

Now let $y \in s(\mu)\Lambda^\infty$. Then it follows from $(m, n) \in \Sigma_v$ and (4.2) that

$$\mu y = \sigma^n(x(0, n)\mu y) = \sigma^m(x(0, n)\mu y) = \sigma^m(x(0, m)v y) = v y.$$

This proves $\mu \sim v$. □

The following result describes the relation between the periodicity of infinite paths and that induced from the equivalence relation \sim defined by (4.1). Notice that it was proved in [2] under the condition that Λ^0 is a maximal tail.

THEOREM 4.3. *Let $m, n \in P$. Then $(m, n) \in \Sigma_\Lambda \iff m - n \in \text{Per } \Lambda$.*

PROOF. ‘ \Rightarrow ’: Let $(m, n) \in \Sigma_\Lambda$. Then $(m, n) \in \Sigma_v$ for some $v \in \Lambda^0$. By Lemma 4.2, there are $\mu, v \in \Lambda$ such that $d(\mu) = m$, $d(v) = n$, and $\mu \sim v$. Thus $m - n \in \text{Per } \Lambda$.

‘ \Leftarrow ’: Assume that $m, n \in P$ such that $m - n \in \text{Per } \Lambda$. So, by definition, there are $\mu, v \in \Lambda$ satisfying $\mu \sim v$ and $m - n = d(\mu) - d(v)$. Without loss of generality, we assume that $m \neq n$. Set $m' := d(\mu)$ and $n' := d(v)$.

Let $x \in s(\mu)\Lambda^\infty$. Then from $\mu \sim v$ and $m + n' = m' + n$, on the one hand, we have

$$\begin{aligned} \mu x(0, n) &= (\mu x)(0, m' + n) = (v x)(0, m' + n) \\ &= (v x)(0, m + n') = v x(0, m). \end{aligned}$$

On the other hand, $\mu \sim v$ gives

$$\mu x(0, n)\sigma^n(x) = \mu x = v x = v x(0, m)\sigma^m(x).$$

Therefore, combining the above identities yields $\sigma^m(x) = \sigma^n(x)$. This implies $(m, n) \in \Sigma_{s(\mu)} \subseteq \Sigma_\Lambda$, as desired. □

The first corollary below is immediate from the above theorem.

COROLLARY 4.4. $(m, n) \in \Sigma_\Lambda \iff (m - p, n - p) \in \Sigma_\Lambda$ for any $p \in \mathcal{G}(P)$ such that $m - p, n - p \in P$.

As expected, one has the following result.

COROLLARY 4.5. $\text{Per } \Lambda = \bigcup_{v \in \Lambda^0} \text{Per}_v$.

PROOF. Assume $m - n \in \text{Per } \Lambda$ with $m, n \in P$. Then $(m, n) \in \Sigma_v$ for some $v \in \Lambda^0$ by Theorem 4.3. So $m - n \in \text{Per}_v$, proving $\text{Per } \Lambda \subseteq \bigcup_v \text{Per}_v$.

Conversely, let $m, n \in P$ be such that $m - n \in \text{Per}_v$ for some $v \in \Lambda^0$. Then by definition $m - n = m' - n'$ for some $(m', n') \in \Sigma_v$. Applying Theorem 4.3 again yields $m - n \in \text{Per } \Lambda$. Thus $\bigcup_v \text{Per}_v \subseteq \text{Per } \Lambda$. □

Slightly differently from [10, 13, 15], we call Per_v the *local periodicity of Λ at v* . So Corollary 4.5 tells us that the (global) periodicity of Λ is the union of its all local periodicities. If $\text{Per}_v = \{0\}$ for all $v \in \Lambda^0$, then we say that Λ *has no local periodicity*.

The most important consequence of Theorem 4.3 is probably the characterizations of aperiodicity/periodicity given below, which unify and generalize what we have known for row-finite and source free k -graphs in the literature.

THEOREM 4.6. *Let Λ be a P -graph. Then the following are equivalent:*

- (i) Λ is aperiodic;
- (ii) $\text{Per } \Lambda = \{0\}$;
- (iii) Λ has no local periodicity.

PROOF. (ii) \iff (iii) follows immediately from Corollary 4.5.

(i) \implies (ii): Suppose that Λ is aperiodic. To the contrary, assume that $\text{Per } \Lambda \neq \{0\}$. Then there are $\mu \neq \nu \in \Lambda$ such that $\mu \sim \nu$. Since Λ is aperiodic, there is an aperiodic $x \in s(\mu)\Lambda^\infty$. But we also have $\mu x = \nu x$. So $\sigma^{d(\nu)}(x) = \sigma^{d(\mu)+d(\nu)}(\mu x) = \sigma^{d(\mu)+d(\nu)}(\nu x) = \sigma^{d(\mu)}(x)$. This contradicts x being aperiodic.

(iii) \implies (i): This can be proved after applying some obvious modifications to the proof of [13, Lemma 3.2] (iii) \implies (i) (for example, by replacing $m \vee n$ there with $m + n$). □

One can also obtain some properties of Λ_{Per}^0 by making use of Theorem 4.3.

COROLLARY 4.7. $\Lambda_{\text{Per}}^0 \subseteq \{v \in \Lambda^0 : \Sigma_v = \Sigma_\Lambda\}$.

PROOF. Let $(m, n) \in \Sigma_\Lambda$, $v \in \Lambda_{\text{Per}}^0$, $x \in v\Lambda^\infty$, and $\mu = x(0, m)$. Since $v \in \Lambda_{\text{Per}}^0$, due to Theorem 4.3 there is $\nu \in v\Lambda$ such that $d(\nu) = n$ and $\mu \sim \nu$. Then one has

$$\mu\sigma^m(x) = \nu\sigma^n(x).$$

This implies $\nu = x(0, n)$. Thus

$$\mu\sigma^m(x) = x = x(0, n)\sigma^n(x) = \nu\sigma^n(x).$$

Comparing the above identities gives $\nu\sigma^m(x) = \nu\sigma^n(x)$, and so $\sigma^m(x) = \sigma^n(x)$. Thus $(m, n) \in \Sigma_v$, proving the desired inclusion. □

We do not know if $\Lambda^0_{\text{Per}} = \{v \in \Lambda^0 : \Sigma_v = \Sigma_\Lambda\}$ in general. We expect so, but have not yet found a proof. The corollary below is quite strong, but notice that all strongly connected k -graphs studied in [7] satisfy its requirements.

COROLLARY 4.8. *If Λ is sink-free and $\Sigma_v = \Sigma_\Lambda$ for every $v \in \Lambda^0$, then $\Lambda^0_{\text{Per}} = \Lambda^0$.*

PROOF. Let $v \in \Lambda^0$. Suppose that $\mu \in v\Lambda^m$ and $n \in \mathbb{N}^k$ satisfies $m - n \in \text{Per } \Lambda$. Arbitrarily choose $x \in \Lambda^\infty$ such that $\mu = x(n, m + n)$. This can be done as Λ is also sink-free. By Theorem 4.3, $(m, n) \in \Sigma_\Lambda$. Since $\Sigma_{r(x)} = \Sigma_\Lambda$, one also gets $(m, n) \in \Sigma_{r(x)}$. Applying Lemma 4.2 gives $\mu \sim x(m, m + n)$. Thus $v \in \Lambda^0_{\text{Per}}$, and we are done. \square

4.2. Pullbacks are periodic. In this subsection, we apply the main results of the previous one to prove that all pullbacks are periodic.

Let Γ be a P -graph. Notice that $x \in \Gamma^\infty$ is uniquely determined by $x(0, p)$ for all $p \in P$. Indeed, for $p, q \in P$ with $q - p \in P$, $x(p, q)$ is unique determined by the (unique) factorization $x(0, q) = x(0, p)x(p, q)$.

The following result is in the same vein as [9, Proposition 2.9] and [1, Proposition 2.4].

PROPOSITION 4.9. *Let P, Q be two monoids, $f : P \rightarrow Q$ be a surjective homomorphism, and Γ be a Q -graph. Then f induces a homeomorphism $f_* : (f^*\Gamma)^\infty \rightarrow \Gamma^\infty$ by*

$$f_*(x)(0, f(n)) = \mathbf{p}_1(x(0, n)) \quad \forall x \in (f^*\Gamma)^\infty, n \in P,$$

where $\mathbf{p}_1 : f^*\Gamma \rightarrow \Gamma$ is the projection $\mathbf{p}_1(\mu, n) = \mu$ for all $(\mu, n) \in f^*\Gamma$.

PROOF. The proof is similar to that of [1, Proposition 2.4]. Here we only prove that f_* is well defined. That is, we need to show the following: if $n, n' \in P$ satisfy $f(n) = f(n')$, then

$$\mathbf{p}_1(x(0, n)) = \mathbf{p}_1(x(0, n')) \quad \forall x \in (f^*\Gamma)^\infty.$$

Notice that $x(0, n)x(n, n + n') = x(0, n')x(n', n + n')$ implies

$$(\mathbf{p}_1(x(0, n)), n)(\mathbf{p}_1(x(n, n + n')), n') = (\mathbf{p}_1(x(0, n')), n')(\mathbf{p}_1(x(n', n + n')), n).$$

Taking \mathbf{p}_1 on both sides gives

$$\mathbf{p}_1(x(0, n))\mathbf{p}_1(x(n, n + n')) = \mathbf{p}_1(x(0, n'))\mathbf{p}_1(x(n', n + n')).$$

But $d(\mathbf{p}_1(x(0, n))) = f(n) = f(n') = d(\mathbf{p}_1(x(0, n')))$. So one has $\mathbf{p}_1(x(0, n)) = \mathbf{p}_1(x(0, n'))$ by the factorization property of Γ . \square

The identity in the following lemma turns out to be very handy.

LEMMA 4.10. *Let P, Q be two monoids, $f : P \rightarrow Q$ be a homomorphism, and Γ be a Q -graph. If $m, n \in P$ such that $f(m) = f(n)$, then*

$$(\mu, m)(w, p)(v, n) = (\mu, n)(w, p)(v, m)$$

for all $(\mu, m), (w, p), (v, n) \in f^*\Gamma$ with $w \in s(\mu)\Gamma r(v)$.

PROOF. This follows immediately from the definition of composition:

$$(\mu, m)(w, p)(v, n) = (\mu wv, m + p + n) = (\mu wv, n + p + m) = (\mu, n)(w, p)(v, m). \quad \square$$

THEOREM 4.11. *Let P be a monoid, H be a nonzero subgroup of $\mathcal{G}(P)$, $\mathbf{q} : \mathcal{G}(P) \rightarrow \mathcal{G}(P)/H$ be the quotient map, and Γ be a P/H -graph. Then $\mathbf{q}^*\Gamma$ is a periodic P -graph.*

PROOF. In what follows, we show a little more than is needed: if m, n in P such that $m - n \in H$, then

$$\sigma^m(x) = \sigma^n(x) \quad \forall x \in \Lambda^\infty.$$

Once this is done, we obtain $H \subseteq \text{Per } \mathbf{q}^*\Gamma$ by Theorem 4.3. Thus $\mathbf{q}^*\Gamma$ is periodic by Theorem 4.6.

To this end, let a_1, \dots, a_k be fixed generators of P (recall that P is finitely generated). Fix $m, n \in P$ such that $m - n \in H$. Then choose $\ell \in P$ satisfying $m + n + \ell - (a_1 + \dots + a_k) \in P$. (The existence of such ℓ is clear.) Let $x \in \Lambda^\infty$. Since Λ is source-free, one can always write x as

$$\begin{aligned} x &= (\mu_1, m)(v_1, n)(w_1, \ell)(\mu_2, m)(v_2, n)(w_2, \ell) \cdots \\ &= \prod_{i=1}^\infty (\mu_i, m)(v_i, n)(w_i, \ell), \end{aligned}$$

where $\mu_i, v_i, w_i \in \Gamma$ with $d(\mu_i) = \mathbf{q}(m)$, $d(v_i) = \mathbf{q}(n)$, $d(w_i) = \mathbf{q}(\ell)$.

Then, on the one hand,

$$\sigma^m(x) = (v_1, n)(w_1, \ell)(\mu_2, m)(v_2, n)(w_2, \ell) \cdots = \prod_{i \geq 2} (v_{i-1}, n)(w_{i-1}, \ell)(\mu_i, m).$$

On the other hand, since $m - n \in H$, one repeatedly applies Lemma 4.10 to obtain

$$\begin{aligned} \sigma^n(x) &= \sigma^n((\mu_1, n)(v_1, m)(w_1, \ell)(\mu_2, m)(v_2, n)(w_2, \ell) \cdots) \quad (\text{by Lemma 4.10}) \\ &= (v_1, m)(w_1, \ell) \prod_{i \geq 2} (\mu_i, m)(v_i, n)(w_i, \ell) \\ &= (v_1, m)(w_1, \ell) \prod_{i \geq 2} (\mu_i, n)(v_i, m)(w_i, \ell) \quad (\text{by Lemma 4.10}) \\ &= (v_1, m)(w_1, \ell)(\mu_2, n) \cdot (v_2, m)(w_2, \ell)(\mu_3, n) \cdots \cdots \quad (\text{by rearrangement}) \\ &= (v_1, n)(w_1, \ell)(\mu_2, m) \cdot (v_2, n)(w_2, \ell)(\mu_3, m) \cdots \cdots \quad (\text{by Lemma 4.10}) \\ &= \prod_{i \geq 2} (v_{i-1}, n)(w_{i-1}, \ell)(\mu_i, m). \end{aligned}$$

Therefore $\sigma^m(x) = \sigma^n(x)$. \square

It is probably worth remarking that, in general, $H \subsetneq \text{Per } \mathbf{q}^*\Gamma$. For example, let $H = 2\mathbb{Z}$ and Γ be the \mathbb{Z}_2 -graph consisting of one vertex v and one edge e . Then $\text{Per } \mathbf{q}^*\Gamma = \mathbb{Z}$.

5. Pushouts

In this section, we reverse the pulling-back process of Section 3: we start with a P -graph Λ , and construct an aperiodic graph from Λ by modding out its periodicity. In order to achieve this goal, we need $\text{Per } \Lambda$ to be a subgroup of $\mathcal{G}(P)$. To this end, we have to impose the condition that Λ^0 is a maximal tail.

DEFINITION 5.1. We say that Λ^0 is a maximal tail if for any $u, v \in \Lambda^0$ there is $w \in \Lambda^0$ such that $u\Lambda w \neq \emptyset$ and $v\Lambda w \neq \emptyset$.

The general notion of a maximal tail is given in [2] (cf. also [8]). For Λ^0 , the conditions (b) and (c) required in [2] are redundant. It is very easy to check that if a P -graph Λ is cofinal¹, then Λ^0 is a maximal tail. It is also noteworthy that if Λ is strongly connected (that is, $v\Lambda w \neq \emptyset$ for all $v, w \in \Lambda^0$), then Λ^0 is a maximal tail.

Recall that the equivalence relation \sim defined in (4.1) respects sources and ranges, and so Λ/\sim is a category: for all $\xi, \eta \in \Lambda$,

$$r([\xi]) = [r(\xi)], \quad s([\xi]) = [s(\xi)], \quad [\xi][\eta] = [\xi\eta].$$

The following theorem is an analogue of [2, Theorem 4.2]. Actually, Lemma 4.5 and Corollary 4.6 in [2] have been generalized in Corollary 4.4 and Theorem 4.3 in Section 4 above to all P -graphs.

THEOREM 5.2. Let Λ be a P -graph such that Λ^0 is a maximal tail. Then the following hold true.

- (i) $\text{Per } \Lambda$ is a subgroup of $\mathcal{G}(P)$.
- (ii) If Λ^0_{Per} is nonempty, then it is a hereditary subset of Λ^0 .
- (iii) Let $\mathbf{q} : \mathcal{G}(P) \rightarrow \mathcal{G}(P)/\text{Per } \Lambda$ be the quotient map. Then $\Lambda^0_{\text{Per}}\Lambda/\sim$ is a $\mathbf{q}(P)$ -graph with degree map $\tilde{d} := \mathbf{q} \circ d$.
- (iv) $\Lambda^0_{\text{Per}}\Lambda$ is isomorphic to the pullback $\mathbf{q}^*(\Lambda/\sim)$ via $\lambda \mapsto ([\lambda], d(\lambda))$.

The group $\text{Per } \Lambda$ is called the *periodicity group* of Λ , and the above $\mathbf{q}(P)$ -graph $\Lambda^0_{\text{Per}}\Lambda/\sim$ is called the *pushout* of Λ .

5.1. The aperiodicity of Λ/\sim . By Theorem 5.2(iii), for simplicity, from now on we assume that $\Lambda^0_{\text{Per}} = \Lambda^0$. The main result of this subsection is that Λ/\sim is aperiodic. Intuitively, this is very natural and simple: Λ/\sim is obtained by removing *all* periods of Λ , and so Λ/\sim should have the trivial periodicity only.

THEOREM 5.3. Let Λ be a P -graph such that Λ^0 is a maximal tail and $\Lambda^0_{\text{Per}} = \Lambda^0$. Then the pushout Λ/\sim is an aperiodic $\mathbf{q}(P)$ -graph.

PROOF. Let $\Gamma = \Lambda/\sim$. By Theorems 5.2 and 4.6, it suffices to show that $\text{Per } \Gamma = \{0\}$.

¹Recall that Λ is cofinal if, for every $x \in \Lambda^\infty$ and $v \in \Lambda^0$, there is $n \in P$ such that $v\Lambda x(n) \neq \emptyset$.

Let $\mathbf{q} : \mathcal{G}(P) \rightarrow \mathcal{G}(P)/\text{Per } \Lambda$ be the quotient map, and identify Λ with $\mathbf{q}^*\Gamma$ by Theorem 5.2(iv). Then, completely similarly to Proposition 4.9, one can see that \mathbf{q} induces the homeomorphism

$$\mathbf{q}_* : \Lambda^\infty \rightarrow \Gamma^\infty, \\ x \mapsto \dot{x} := \mathbf{q}_*(x) : (0, \mathbf{q}(n)) \mapsto [x(0, n)].$$

Also, the projection $\mathbf{p}_1 : \Lambda \rightarrow \Gamma$ is now given by

$$\mathbf{p}_1(\lambda) = [\lambda] \quad \forall \lambda \in \Lambda.$$

For convenience, let \mathbf{q}_2 be the quotient map from Ω_P onto $\Omega_{\mathbf{q}(P)}$ defined by $\mathbf{q}_2(0, n) = (0, \mathbf{q}(n))$. Let $x \in \Lambda^\infty$ and note that, for all $n \in P$, one has $x(0, n) = (\mathbf{p}_1(x(0, n)), n) = ([x(0, n)], n)$. Then one gets the following commuting diagram:

$$\begin{array}{ccc} \Omega_P & \xrightarrow{x} & \Lambda \\ \mathbf{q}_2 \downarrow & & \downarrow \mathbf{p}_1 \\ \Omega_{\mathbf{q}(P)} & \xrightarrow{\dot{x}} & \Gamma. \end{array}$$

We now suppose that $\mu, \nu \in \Lambda$ such that $[\mu]$ and $[\nu]$ are equivalent in Γ :

$$[\mu] \sim_\Gamma [\nu].$$

In what follows, we show that μ and ν are actually equivalent in Λ :

$$\mu \sim_\Lambda \nu.$$

Once this is done, we have that $[\mu] = [\nu]$, which proves that $\text{Per } \Gamma = \{0\}$.

To this end, arbitrarily choose $x \in s(\mu)\Lambda^\infty$. Note that $s(\mu) = s(\nu)$ as $s([\mu]) = s([\nu])$. So $\dot{x} \in s([\mu])\Gamma^\infty$. Thus one obtains the following consecutive implications:

$$\begin{aligned} & [\mu] \sim_\Gamma [\nu] \\ & \Rightarrow [\mu]\dot{x} = [\nu]\dot{x} \\ & \Rightarrow ([\mu]\dot{x})(\mathbf{q}_2(0, n)) = ([\nu]\dot{x})(\mathbf{q}_2(0, n)) \\ & \quad (\forall n \in P \text{ such that } \mathbf{q}(n) - \tilde{d}([\mu]), \mathbf{q}(n) - \tilde{d}([\nu]) \in \mathbf{q}(P)) \\ & \Rightarrow \dot{x}(0, q(n) - \tilde{d}([\mu])) = \dot{x}(0, q(n) - \tilde{d}([\nu])) \\ & \Rightarrow \dot{x}(0, \tilde{d}([\nu])) = \dot{x}(0, \tilde{d}([\mu])) \quad (\text{by taking } \mathbf{q}(n) = \tilde{d}([\mu]) + \tilde{d}([\nu])) \\ & \Rightarrow [x(0, d(\nu))] = [x(0, d(\mu))] \quad (\text{by the definition of } \dot{x}). \end{aligned}$$

Thus we have $d(\mu) - d(\nu) \in \text{Per } \Lambda$ by the very definition of $\text{Per } \Lambda$. So there is a unique $\nu' \in \Lambda$ such that

$$\mu \sim \nu' \quad \text{and} \quad d(\nu') = d(\nu) \tag{5.1}$$

by Lemma 4.1 and our assumption $\Lambda_{\text{Per}}^0 = \Lambda^0$. Hence $\mu x = \nu' x$, and so $[\mu]\dot{x} = [\nu']\dot{x}$. Combining this with $[\mu]\dot{x} = [\nu]\dot{x}$ (see the first implication above) gives

$$[\nu]\dot{x} = [\nu']\dot{x}.$$

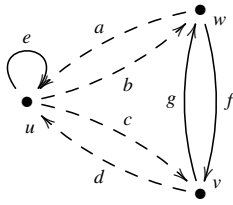


FIGURE 1. Λ : a 2-graph.

But we have $\tilde{d}([v]) = \tilde{d}([v'])$ from $d(v) = d(v')$. Applying Lemma 4.1 to the $\mathbf{q}(P)$ -graph Γ yields $[v] = [v']$, and to the P -graph Λ again gives $v = v'$ as $d(v) = d(v')$. From (5.1), we prove $\mu \sim_\Lambda v$, as desired. \square

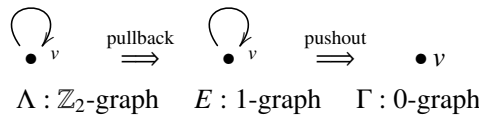
The following corollary is an immediate consequence of Theorem 3.2, by letting $H := \text{Per } \Lambda$, and Theorems 5.2 and 5.3.

COROLLARY 5.4. *Let Λ be a P -graph such that Λ^0 is a maximal tail and $\Lambda^0_{\text{Per}} = \Lambda^0$. Then the pushout Λ/\sim is an aperiodic $\mathbf{q}(P)$ -graph, and we have the embedding*

$$\pi : C^*(\Lambda) \hookrightarrow C^*(\Lambda/\sim) \otimes C^*(\text{Per } \Lambda), \quad s_\lambda \mapsto s_{[\lambda]} \otimes V^{d(\lambda)-jod(\lambda)}.$$

Here, as before, the V_i are unitary generators of $C^*(\text{Per } \Lambda)$.

We should notice that the pullback and pushout processes are not mutually inverse. Consider the 1-graph E consisting of one vertex v and one edge. Then its periodicity group is \mathbb{Z} , and accordingly, the pushout $\Gamma = E/\sim = \{v\}$. Combining Example 3.3 gives



However, unlike Example 3.3, one does obtain $C^*(E) \cong C^*(\Gamma) \otimes C^*(\text{Per } E)$. Notice that, in Example 3.3, $H = 2\mathbb{Z}$ is a proper subgroup of $\text{Per } q^*\Gamma = \mathbb{Z}$.

So it seems more appropriate to expect the injection π defined in Corollary 5.4 to be surjective. But the following example, due to Aidan Sims, shows that our Theorem 3.2 is the best one could possibly get.

EXAMPLE 5.5. Consider the 2-graph Λ drawn in Figure 1. One can check that $\text{Per } \Lambda = \{(2n, 0) : n \in \mathbb{Z}\}$, and

$$ee \sim u, \quad fg \sim v, \quad gf \sim w.$$

So Λ/\sim is a $\mathbb{Z}_2 \times \mathbb{N}$ -graph as drawn in Figure 2, which looks like Λ , but the dotted edges now have ‘torsion’.

We claim that there is no any ‘naturally canonical’ isomorphism between $C^*(\Lambda)$ and $C^*(\Lambda/\sim) \otimes C^*(\mathbb{T})$. The idea is sketched as follows. Suppose that there is such an

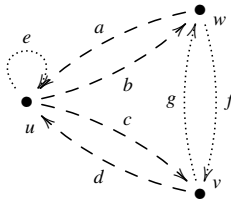


FIGURE 2. Λ/\sim : a $\mathbb{Z}_2 \times \mathbb{N}$ -graph.

isomorphism π . On one hand, consider the 2-cycle subgraph $C_2 := \{v, w, f, g\}$ in Λ . By [12],

$$\pi(s_f) = s_{[f]} \otimes z, \quad \pi(s_g) = s_{[g]} \otimes 1$$

(or *vice versa*). On the other hand, since the dashed 1-graph has no periodicity, it is reasonable to have

$$\pi(s_\alpha) = s_{[\alpha]} \otimes 1 \quad \text{for all dashed paths } \alpha.$$

Then one has

$$\begin{aligned} \pi(s_e) &= \pi(s_e s_c^* s_c) = \pi(s_b^* s_g s_c) = (s_{[b]}^* \otimes 1)(s_{[g]} \otimes 1)(s_{[c]} \otimes 1) \\ &= s_{[b]}^* s_{[g]} s_{[c]} \otimes 1 = s_{[e]} s_{[c]}^* s_{[c]} \otimes 1 = s_{[e]} \otimes 1 \end{aligned}$$

and

$$\begin{aligned} \pi(s_e) &= \pi(s_e s_b^* s_b) = \pi(s_c^* s_f s_b) = (s_{[c]}^* \otimes 1)(s_{[f]} \otimes z)(s_{[b]} \otimes 1) \\ &= s_{[c]}^* s_{[f]} s_{[b]} \otimes z = s_{[e]} s_{[b]}^* s_{[b]} \otimes 1 = s_{[e]} \otimes z, \end{aligned}$$

a contradiction.

6. A distinguished MASA

In this section, we exhibit an interesting application of our results in Sections 3 and 5. But because of Theorem 5.2(iv), we have to restrict ourselves to the following class of k -graphs Λ : Λ^0 is a maximal tail and $\Lambda_{\text{Per}}^0 = \Lambda^0$. This class includes the class of cofinal, sink-free P -graphs as special examples by Corollary 4.8, which exhausts all strongly connected k -graphs recently studied in [7]. Our application answers the questions asked in [1] in this context.

Let Λ be a k -graph satisfying our assumptions. Let

$$\mathcal{M} := C^*(s_\mu s_\nu^* : \mu \sim \nu),$$

which is called the *cycline subalgebra* of $C^*(\Lambda)$ in [1]. It turns out that \mathcal{M} plays a central role in a generalized Cuntz–Krieger uniqueness theorem for Λ [1]. It is shown that a representation of $C^*(\Lambda)$ is injective if and only if its restriction onto \mathcal{M} is

injective. Also, \mathcal{M} is abelian and $\mathfrak{D}'_\Lambda = \mathcal{M}'$. Because of its importance, it is asked in [1] if \mathcal{M} is a MASA and there is a faithful conditional expectation from $C^*(\Lambda)$ onto \mathcal{M} . In other words, it would be nice to know if \mathcal{M} is an abelian core in the terminology of [11]. We will answer those questions affirmatively under our conditions. But a simple lemma first, which holds true for all k -graphs.

LEMMA 6.1. *Let Λ be an arbitrary k -graph, and $\mu, \nu \in \Lambda$. Then $\mu \sim \nu$ if and only if, for any $n \leq d(\mu) \wedge d(\nu)$, there are $w \in \Lambda^n, \mu', \nu' \in \Lambda$ such that $\mu = w\mu', \nu = w\nu'$ and $\mu' \sim \nu'$.*

PROOF. It suffices to show the ‘only if’ part as the ‘if’ part is trivial. Factor $\mu = \mu_1\mu_2$ and $\nu = \nu_1\nu_2$ with $d(\mu_1) = d(\nu_1) = n$. Let $y \in s(\mu)\Lambda^\infty$. Then

$$\mu_1\mu_2y = \mu y = \nu y = \nu_1\nu_2y$$

implies $(\mu_1\mu_2y)(0, n) = (\nu_1\nu_2y)(0, n)$, that is, $\mu_1 = \nu_1 =: w$, as $d(\mu_1) = d(\nu_1)$. Also $\sigma^n(\mu_1\mu_2y) = \sigma^n(\nu_1\nu_2y)$ implying $\mu_2y = \nu_2y$. Letting $\mu' = \mu_2$ and $\nu' = \nu_2$ ends the proof. \square

THEOREM 6.2. *Suppose that Λ is a k -graph such that Λ^0 is a maximal tail and $\Lambda^0_{\text{Per}} = \Lambda^0$. Then we have the following.*

(i) \mathcal{M} is a MASA in $C^*(\Lambda)$. Furthermore,

$$\mathcal{M} = \mathfrak{D}'_\Lambda \cong \mathfrak{D}_\Lambda \otimes C^*(\text{Per } \Lambda).$$

(ii) There is a faithful conditional expectation from $C^*(\Lambda)$ onto \mathcal{M} .

PROOF. (i) If Λ is aperiodic, then it follows from [6, Theorem] and [1, Remark 4.5] that \mathfrak{D}_Λ is a MASA in $C^*(\Lambda)$, which equals \mathcal{M} .

We now assume that Λ is periodic. So $\text{Per } \Lambda$ is a nonzero subgroup of \mathbb{Z}^k . Let $\Gamma := \Lambda/\sim$ and $\mathbf{q} : \mathbb{Z}^k \rightarrow \mathbb{Z}^k/\text{Per } \Lambda$ be the quotient map. Then it follows from Theorem 5.2 that Λ is isomorphic to the pullback $\mathbf{q}^*\Gamma$. Keeping the same notation as in Corollary 5.4, we have the embedding

$$\begin{aligned} \pi : C^*(\Lambda) \cong C^*(\mathbf{q}^*\Gamma) &\hookrightarrow C^*(\Gamma) \otimes C^*(\text{Per } \Lambda), \\ s_\lambda &\mapsto s_{[\lambda]} \otimes V^{d(\lambda) - j \circ d(\lambda)}. \end{aligned}$$

Then one can check that

$$\begin{aligned} \mathfrak{B} := \pi(C^*(\mathbf{q}^*\Gamma)) &= C^*(s_\mu s_\nu^* \otimes V^n : d(\mu) - d(\nu) = q(n - j \circ q(n))) \\ &= C^*(s_\mu s_\nu^* \otimes V^n : d(\mu)_i - d(\nu)_i = [n_i]_{d_i} : 1 \leq i \leq r). \end{aligned}$$

Since Γ is aperiodic by Theorem 5.3, the diagonal algebra \mathfrak{D}_Γ of $C^*(\Gamma)$ is a MASA by [6, Theorem]. So $\mathfrak{D}'_\Gamma \cap (C^*(\Gamma) \otimes C^*(\text{Per } \Lambda)) = \mathfrak{D}_\Gamma \otimes C^*(\text{Per } \Lambda)$ is a MASA of $C^*(\Gamma) \otimes C^*(\text{Per } \Lambda)$ [16, 17]. It is now not hard to verify that $\mathfrak{D}'_\Gamma \cap \mathfrak{B}$ is $\mathfrak{D}_\Gamma \otimes C^*(V_1^{d_1}, \dots, V_r^{d_r})$, and it is a MASA of \mathfrak{B} . Moreover, its preimage

$$\begin{aligned} \pi^{-1}(\mathfrak{D}'_\Gamma \cap \mathfrak{B}) &= \pi^{-1}(\mathfrak{D}_\Gamma \otimes C^*(V_1^{d_1}, \dots, V_r^{d_r})) \\ &= \mathfrak{D}'_\Lambda \cap C^*(\Lambda) = \mathfrak{D}_\Lambda C^*(W_h : h \in \text{Per } \Lambda) \\ &\cong \mathfrak{D}_\Lambda \otimes C^*(\text{Per } \Lambda) \end{aligned}$$

by Corollary 3.6. (Recall from Lemma 3.5 that, for any $h \in \text{Per } \Lambda$, W_h is a central unitary in $\mathcal{M}(C^*(\Lambda))$.) Thus $\mathfrak{D}'_\Lambda \cap C^*(\Lambda)$ is a MASA of $C^*(\Lambda)$, which is isomorphic to $\mathfrak{D}_\Lambda \otimes C^*(\text{Per } \Lambda)$.

To finish proving (i), we now show that $\mathcal{M} = \mathfrak{D}'_\Lambda \cap C^*(\Lambda)$, namely

$$\mathcal{M} = \mathfrak{D}_\Lambda C^*(W_h : h \in \text{Per } \Lambda).$$

Obviously, the right-hand side is contained in the left-hand side. To show the other inclusion, we make use of Lemma 6.1. Let $\mu \sim \nu$. Then $(\mu, \nu) = (w\mu', w\nu')$ with $d(w) = d(\mu) \wedge d(\nu)$ and $\mu' \sim \nu'$. Note that $d(\mu') \wedge d(\nu') = 0$. Let $h = d(\mu') - d(\nu')$. It follows from Lemma 3.5 that $s_{([\lambda], d(\mu))} = W_h s_{([\lambda], d(\nu))}$. In particular, this implies $s_{\mu'} = W_h s_{\nu'}$. Hence

$$s_\mu s_\nu^* = W_h s_w s_{w'} s_{\nu'}^* s_w^* = s_{ww'} s_{w\nu'}^* W_h \in \mathfrak{D}_\Lambda C^*(W_h : h \in \text{Per } \Lambda).$$

(ii) This immediately follows from Corollaries 3.6, 5.4 and (i) above, as \mathfrak{D}_Γ is the canonical MASA of the AF-algebra \mathfrak{F}_Γ , the fixed point algebra of the gauge action of $C^*(\Gamma)$. □

The fact that \mathcal{M} is a MASA of $C^*(\Lambda)$ can also follow from the facts that $\mathcal{M} = \mathfrak{D}'_\Lambda$ proved above and that $\mathcal{M}' = \mathfrak{D}'_\Lambda$ shown in [1].

Let us end this paper by remarking on the simplicity of $C^*(\Gamma)$ and the centre of $C^*(\Lambda)$, which generalize [4, Corollary 8.8] and [7, Proposition 6.1].

COROLLARY 6.3. *Keep the same conditions as in Theorem 6.2. Suppose further that Λ^0 is finite and Λ is cofinal. Then $C^*(\Lambda/\sim)$ is simple and the centre of $C^*(\Lambda)$ is $C^*(W_h : h \in \text{Per } \Lambda)$.*

PROOF. Clearly $(\Lambda/\sim)^0$ is finite and so $C^*(\Lambda/\sim)$ is unital. By Proposition 4.9, it is easy to see that Λ/\sim is also cofinal. By Theorem 5.3, Λ/\sim is aperiodic. As [9, Proposition 4.8], one can see that $C^*(\Lambda/\sim)$ is simple. Thus $\mathcal{Z}(C^*(\Lambda)) = \pi^{-1}(\mathcal{Z}(\pi(C^*(\Lambda)))) = \pi^{-1}(I \otimes C^*(V_1^{d_1}, \dots, V_r^{d_r})) = C^*(W_1, \dots, W_r)$ from (3.3). □

Acknowledgements

The author would like to heartily thank Professor Aidan Sims for pointing out a mistake in the proof of Theorem 3.2 in an earlier version, and for providing very valuable suggestions and comments. Also, the author is very grateful for Professor Ken Davidson’s useful comments and encouragement, and for the referee’s very careful reading which improved the presentation of the paper.

References

[1] J. Brown, G. Nagy and S. Reznikoff, ‘A generalized Cuntz–Krieger uniqueness theorem for higher-rank graphs’, *J. Funct. Anal.* **266** (2014), 2590–2609.
 [2] T. M. Carlsen, S. Kang, J. Shotwell and A. Sims, ‘The primitive ideals of the Cuntz–Krieger algebra of a row-finite higher-rank graph with no sources’, *J. Funct. Anal.* **266** (2014), 2570–2589.

- [3] K. R. Davidson and D. Yang, 'Periodicity in rank 2 graph algebras', *Canad. J. Math.* **61** (2009), 1239–1261.
- [4] K. R. Davidson and D. Yang, 'Representations of higher rank graph algebras', *New York J. Math.* **15** (2009), 169–198.
- [5] J. H. Hong and W. Szymański, 'The primitive ideal space of the C^* -algebras of infinite graphs', *J. Math. Soc. Japan* **56** (2004), 45–64.
- [6] A. Hopenwasser, 'The spectral theorem for bimodules in higher rank graph C^* -algebras', *Illinois J. Math.* **49** (2005), 993–1000.
- [7] A. an Huef, M. Laca, I. Raeburn and A. Sims, 'KMS states on the C^* -algebra of a higher-rank graph and periodicity in the path space', *J. Funct. Anal.* **268** (2015), 1840–1875.
- [8] S. Kang and D. Pask, 'Aperiodicity and the primitive ideal space of a row-finite higher rank graph C^* -algebra', *Internat. J. Math.* **25** (2014), 1450022.
- [9] A. Kumjian and D. Pask, 'Higher rank graph C^* -algebras', *New York J. Math.* **6** (2000), 1–20.
- [10] P. Lewin and A. Sims, 'Aperiodicity and cofinality for finitely aligned higher-rank graphs', *Math. Proc. Cambridge Philos. Soc.* **149** (2010), 333–350.
- [11] G. Nagy and S. Reznikoff, 'Abelian core of graph algebras', *J. Lond. Math. Soc. (2)* **85** (2012), 889–908.
- [12] I. Raeburn, *Graph Algebras*, CBMS 103 (American Mathematical Society, Providence, RI, 2005).
- [13] D. I. Robertson and A. Sims, 'Simplicity of C^* -algebras associated to higher-rank graphs', *Bull. Lond. Math. Soc.* **39** (2007), 337–344.
- [14] J. C. Rosales and P. A. García-Sánchez, *Finitely Generated Commutative Monoids* (Nova Publishers, Commack, NY, 1999).
- [15] J. Shotwell, 'Simplicity of finitely aligned k -graph C^* -algebras', *J. Operator Theory* **67** (2012), 335–347.
- [16] S. Wassermann, 'The slice map problem for C^* -algebras', *Proc. Lond. Math. Soc. (3)* **32** (1976), 537–559.
- [17] S. Wassermann, 'Tensor products of maximal abelian subalgebras of C^* -algebras', *Glasg. Math. J.* **50** (2008), 209–216.

DILIAN YANG, Department of Mathematics and Statistics,
University of Windsor, Windsor, Ontario, Canada N9B 3P4
e-mail: dyang@uwindsor.ca