# FREE ACTIONS OF $p$-GROUPS ($p>3$) ON $S^n \times S^n$

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(Received 8 September, 1980)

**1. Introduction and statement of results.** In [3] P. E. Conner showed that no abelian group with rank greater than 2 can act freely on $S^n \times S^n$, the product of two spheres. G. Lewis [6] studied free actions of $p$-groups on $S^n \times S^n$, when $n$ is odd, $n \not\equiv -1(p)$ and $p$ is an odd prime. He showed that any $p$-group which has such an action must be abelian.

This paper studies free actions of non-abelian $p$-groups ($p>3$) on $S^{tp-1} \times S^{tp-1}$. Conner's result shows that it is enough to consider those $p$-groups which contain no elementary abelian subgroup of rank 3, i.e. $p$-groups of rank at most 2. (The rank of a $p$-group is the rank of a maximal elementary abelian subgroup.) For such groups we have the following classification.

THEOREM (Blackburn [5, Satz 12.4]). *Let $G$ be a $p$-group of rank* 2 *with $p>3$. Then $G$ is one of the following isomorphism types:*

(I) *$G$ is metacyclic,*

(II) $G=\langle X, Y, Z : X^p = Y^p = Z^{p^{n-2}} = [X, Z] = [Y, Z] = 1, [Y, X] = Z^{p^{n-3}} \rangle$,

(III) $G=\langle X, Y, Z : X^p = Y^p = Z^{p^{n-2}} = [Y, Z] = 1, [X, Z^{-1}] = Y, [Y, X] = Z^{sp^{n-3}} \rangle$,

*where $n \geq 4$ and $s$ equals* 1, *or some quadratic non-residue* mod $p$.

Using this and some cohomological machinery, we prove the following result.

THEOREM. *Let $G$ be a $p$-group of rank* 2 *where $p$ is a prime number* ($p>3$) *and let $n$ be an odd integer congruent to* $-1$ *modulo some power of $p$. Then $G$ acts freely on $S^n \times S^n$ if and only if $G$ is of type* I *or* III.

The proof is in two parts. In the first, we prove the existence of free actions by defining suitable representations of $G$ which act on $S^n \times S^n$. In the second part, we prove non-existence of free actions by considering certain subgroups of $G$. This is done by using a Gysinoid sequence which holds whenever a finite group acts freely on a manifold with three non-zero homology groups and in particular on $S^n \times S^n$.

The author is greatly indebted to Dr. C. B. Thomas who, as his research supervisor, gave invaluable assistance in the preparation of this work at the Mathematics Department, University College, London. The author also wishes to thank the referee for several helpful suggestions.

**2. Proof of theorem.** We first establish a result on free actions. Let $M$ be an $n$-manifold and $G$ a finite group acting freely on $M$. Suppose that $M$ has precisely three non-zero homology groups in dimensions $0, l, n$ so that $H_0(M) \cong \mathbb{Z}$, $H_l(M) \neq 0$, $H_n(M) \neq 0$. Using the Gysinoid sequence [6], we see that the following sequence is exact:

$$\ldots \to \hat{H}^{i+n-l}(G, H_n(M)) \to \hat{H}^{i-l-1}(G, \mathbb{Z}) \to \hat{H}^i(G, H_l(M)) \to \hat{H}^{i+1+n-l}(G, H_n(M)) \to \ldots$$

If $G$ preserves orientation and $H_i(M)$ is cohomologically trivial then

$$\hat{H}^{n+1}(G,\mathbb{Z}) \cong \hat{H}^0(G,\mathbb{Z}) \cong \mathbb{Z}_{|G|}.$$

Let $M = S^n \times S^n$ ($n$ odd). By Lefschetz's formula $G$ acts trivially on $H_n(M) \cong \mathbb{Z}\times\mathbb{Z}$. Thus $H^0(G,\mathbb{Z}\times\mathbb{Z}) \cong \mathbb{Z}_{|G|}\times\mathbb{Z}_{|G|}$ and the following sequence is exact:

$$\ldots \to H^i(G,\mathbb{Z}\times\mathbb{Z}) \to H^{i+n+1}(G,\mathbb{Z}) \to H^{i-n}(G,\mathbb{Z}) \to H^{i+1}(G,\mathbb{Z}\times\mathbb{Z}) \to \ldots$$

Set $i=0$ and identify $H^{-m}(G,\mathbb{Z})$ with $H^m(G,\mathbb{Z})$. Then the following proposition holds (see [**6**]).

Proposition 1. *Suppose $n$ is odd and $G$ acts freely on $S^n \times S^n$, preserving orientation. Then the sequence*

$$0 \to H^n(G,\mathbb{Z}) \to H^{n+1}(G,\mathbb{Z}) \to \mathbb{Z}_{|G|}\times\mathbb{Z}_{|G|} \to H^{n+1}(G,\mathbb{Z}) \to H^n(G,\mathbb{Z}) \to 0$$

*is exact.*

We now begin the proof of our theorem. By the theorem of Blackburn stated earlier, we have only to consider groups $G$ of types I, II and III. Suppose firstly that $G$ is of type I. Then $G$ is a metacyclic $p$-group and we may let

$$G = \langle A, B \colon A^{p^a} = 1, B^{p^b} = A^{p^c}, B^{-1}AB = A^k;\ c \geq 0,\ k^{p^b} \equiv 1(p^a),\ p^c(k-1) \equiv 0(p^a)\rangle.$$

The group $G$ can be given by the following extension:

$$1 \to \mathbb{Z}_{p^a}\langle A\rangle \to G \xrightarrow{\pi} \mathbb{Z}_{p^b}\langle \bar{B}\rangle \to 1.$$

Let $\alpha\colon A \to e^{2\pi i/p^a} = \rho$ and $\pi^!\beta\colon B \to e^{2\pi i/p^b} = \zeta$, $A \to 1$ be 1-dimensional representations of the subgroup $\mathbb{Z}_{p^a}\langle A\rangle$ and the group $G$ respectively. The induced representation $i_!\alpha$ and the direct sum $p^b(\pi^!\beta)$ are both $p^b$-dimensional representations of $G$. Then $i_!\alpha \oplus p^b(\pi^!\beta)$ induces an action of the group $G$ on the product of two spheres $S^{2p^b-1}\times S^{2p^b-1}$. Also, $1\otimes 1, B\otimes 1, \ldots, B^{p^b-1}\otimes 1$ forms a basis for the induced module associated with the induced representation $i_!\alpha$. Then by [**4**, p. 75] we have:

$$i_!\alpha(B) = \begin{bmatrix} 0 & & 0 & \rho^{p^c} \\ 1 & \cdot & & 0 \\ & \ddots & \ddots & \vdots \\ & & 0 & \vdots \\ 0 & & 1 & 0 \end{bmatrix},$$

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$i_!\alpha(A)=\rho I$, $p^b(\pi^!\beta)(A)=I$ and $p^b(\pi^!\beta)(B)=\zeta I$ (where $I$ denotes the identity matrix). The characteristic values $\lambda$ of $i_!\alpha(B)$ are given as follows:

$$0=\begin{vmatrix} -\lambda & & & 0 & \rho^{p^c} \\ 1 & \cdot & & & 0 \\ & \cdot & \cdot & & \vdots \\ & & \cdot & -\lambda & 0 \\ 0 & & & 1 & -\lambda \end{vmatrix}$$

$$=-\lambda\begin{vmatrix} -\lambda & & & & 0 \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & -\lambda & \\ 0 & & & 1 & -\lambda \end{vmatrix}+\rho^{p^c}\begin{vmatrix} 1 & -\lambda & & & 0 \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & 1 & -\lambda \\ 0 & & & & 1 \end{vmatrix}$$

$$=-\lambda^{p^b}+\rho^{p^c}.$$

This implies that $\lambda\neq 1$.

Then $A^{p^c}$ acts freely on the first sphere via $i_!\alpha$. Also, $B$ acts freely on the second sphere via $p^b(\pi^!\beta)$. By taking direct sums one can extend this argument to all dimensions. Thus $G$ acts freely on $S^{2t-1}\times S^{2t-1}$, $t\equiv 0(p^b)$.

Suppose next that $G$ is of type III. Let

$$G_1=\langle A, B, C\colon A^{p^{n-2}}=B^p=[B,C]=1, C^p=A^{sp}, [A,C]=B, [A,B]=A^{p^{n-3}}\rangle.$$

(See [**2**, p. 145].) Then we may define an isomorphism from $G_1$ onto $G$ as follows. If $s=1$ define the isomorphism by $A\leftrightarrow XZ$, $B\leftrightarrow Y^{-1}$ and $C\leftrightarrow Z$. Then

$$A^p\leftrightarrow X^pZ^pY^{1+2+\ldots+(p-1)}Z^{p^{n-3}(1+2+\ldots+(p-1))}=Z^p\leftrightarrow C^p.$$

Therefore $A^{p^{n-2}}=B^p=1$ and $A^p=C^p$. $\quad YZ=ZY$ implies that $BC=CB$.

$$AC\leftrightarrow XZZ=ZXY^{-1}Z=ZXZY^{-1}\leftrightarrow CAB.$$
$$AB\leftrightarrow XZY^{-1}=XY^{-1}=Y^{-1}XZ^{p^{n-3}}Z=Y^{-1}XZZ^{p^{n-3}}\leftrightarrow BA^{1+p^{n-3}}.$$

If $s$ is a quadratic non-residue mod $p$ the isomorphism can be defined similarly by $A\leftrightarrow XZ$, $B\leftrightarrow Y^{-1}$ and $C\leftrightarrow Z^s$.

Let $H=\langle B,C\rangle$. Then $H$ is a normal abelian subgroup of index $p$ in $G_1$. Let $\beta: B\to e^{2\pi i/p}=\eta$, $C\to 1$ and $\gamma: C\to e^{2\pi i/p^{n-2}}=\xi$, $B\to 1$ be two 1-dimensional representations of $H$. Then $i_!\beta\oplus i_!\gamma$ induces an action of the group $G_1$ on the product of two spheres $S^{2p-1}\times S^{2p-1}$. $1\otimes 1, A\otimes 1,\ldots, A^{p-1}\otimes 1$ forms a basis for the induced modules

https://doi.org/10.1017/S0017089500004857 Published online by Cambridge University Press

associated with the induced representations $i_!\beta$ and $i_!\gamma$, and:

$$i_!\beta(A)=\begin{bmatrix} 0 & & 0 & 1 \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots\ 0 & \vdots \\ 0 & & 1 & 0 \end{bmatrix}, \quad i_!\beta(B)=\begin{bmatrix} \eta & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \eta \end{bmatrix},$$

$$i_!\beta(C)=\begin{bmatrix} 1 & & & & 0 \\ & \eta^{-1} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & \eta^{-p+1} \end{bmatrix}, \quad i_!\gamma(A)=\begin{bmatrix} 0 & & 0 & \xi^p \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots\ 0 & \vdots \\ 0 & & 1 & 0 \end{bmatrix},$$

$$i_!\gamma(B)=\begin{bmatrix} 1 & & & & 0 \\ & \eta^{-1} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & \eta^{-p+1} \end{bmatrix}, \quad i_!\gamma(C)=\begin{bmatrix} \xi & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \xi \end{bmatrix},$$

As before, no characteristic value of $i_!\gamma(A)$ can equal 1.

For $g\in G_1$ we may write $g$ in the form $g=C^iB^jA^k$ where $0\leq i<p^{n-2}$ and $0\leq j, k<p$. The actions of the group $G_1$ on the first and second spheres are given respectively by:

$$A^k(x_1,\ldots,x_p)=(x_{p-k+1},\ldots,x_k,\ldots,x_{p-k}),$$
$$B^j(x_1,\ldots,x_p)=(\eta^jx_1,\ldots,\eta^jx_p)$$
$$C^i(x_1,\ldots,x_p)=(x_1,\eta^{-i}x_2,\ldots,\eta^{(-p+1)i}x_p);$$

and

$$A^k(x_1,\ldots,x_p)=(\xi^{kp}x_{p-k+1},\xi^{(k-1)p}x_{p-k+2},\ldots,\xi^px_k,\ldots,x_{p-k}),$$
$$B^j(x_1,\ldots,x_p)=(x_1,\eta^{-j}x_2,\ldots,\eta^{(-p+1)j}x_p),$$
$$C^i(x_1,\ldots,x_p)=(\xi^ix_1,\ldots,\xi^ix_p).$$

So any element $g\in G_1$ which acts freely on the product of two spheres $S^{2p-1}\times S^{2p-1}$ must be equal to the identity. By taking direct sums one may obtain $G_1$, and hence $G$, acting freely on $S^n\times S^n$, $n\equiv -1(p)$.

Finally suppose that $G$ is of type II. Then $G$ contains the subgroup $\langle X, Y, Z^{p^{n-3}}\rangle$, which is a non-abelain group of order $p^3$ and exponent $p$. Therefore the proof of the theorem will be complete once we have established the following lemma.

LEMMA 2. *Let $N$ be a non-abelian group of order $p^3$ and exponent $p$, where $p$ is an odd prime. Then $N$ cannot act freely, preserving orientation, on $S^n \times S^n$, the product of two spheres, when $n \equiv -1(2p)$.*

*Proof of lemma.* Suppose $N$ acts freely, preserving orientation, on $S^n \times S^n$, $n \equiv -1(2p)$. By Proposition 1 the following sequence is exact:

$$0 \to H^n(N, \mathbb{Z}) \to H^{n+1}(N, \mathbb{Z}) \to \mathbb{Z}_{p^3} \times \mathbb{Z}_{p^3} \to H^{n+1}(G, \mathbb{Z}) \to H^n(G, \mathbb{Z}) \to 0,$$

where $2p \mid n+1$. Then the exponent of $H^{n+1}(N, \mathbb{Z})$ is $p^2$ [**7**, Corollary 6.27]. Thus the sequence $\mathbb{Z}^i_{p^2} \times \mathbb{Z}^j_p \to \mathbb{Z}_{p^3} \times \mathbb{Z}_{p^3} \to \mathbb{Z}^{i'}_{p^2} \times \mathbb{Z}^{j'}_p$ is exact for some integers $i, j, i', j'$. By [**7**, Theorem 6.26] $i = i' = 1$ and $j = j' = p+1$, which is a contradiction. Thus $N$ cannot act freely, preserving orientation, on $S^n \times S^n$.

The proof of the theorem is therefore complete.

REMARK. We have constructed free actions of metacyclic $p$-groups on $S^{2t-1} \times S^{2t-1}$, $t \equiv 0(p^b)$, $b \geq 1$. These groups cannot act freely on $S^{2p-1} \times S^{2p-1}$ unless $b = 1$. This can be proved by finding out the integral cohomology ring of metacyclic $p$-groups (see [**1**]).

## REFERENCES

**1.** K. Alzubaidy, Metacyclic $p$-groups and Chern classes, *Illinois J. Math.* (to appear).

**2.** W. Burnside, *Theory of groups of finite order*, 2nd edition (reprinted Dover, 1955).

**3.** P. E. Conner, On the action of a finite group on $S^n \times S^n$, *Ann. of Math.* (2) **66** (1957), 586–588.

**4.** C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras* (Inter-Science, 1962).

**5.** B. Huppert, *Endliche Gruppen I* (Springer-Verlag, 1967).

**6.** G. Lewis, Free actions on $S^n \times S^n$, *Trans. Amer. Math. Soc.* **132** (1968), 531–540.

**7.** G. Lewis, The integral cohomology rings of groups of order $p^3$, *Trans. Amer. Math. Soc.* **132** (1968), 501–529.

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