# GEODETIC CONSTANTS AND THE MOTION OF THE MOON (1)

**By A. H. COOK,**
M. A., Ph. D.,
Standards Division, National Physical Laboratory, Teddington,
Middlesex, England.

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RÉSUMÉ. — L'auteur discute l'ensemble des constantes qui permettent de décrire le champ de gravitation externe de la Terre : les paramètres décrivant la forme et deux facteurs d'échelle. L'auteur signale quelques problèmes qui se posent pour effectuer la liaison entre le champ extérieur déduit du mouvement des satellites et les observations à la surface de la Terre. On ne peut calculer les paramètres fondamentaux à partir des observations, sans connaître d'autres constantes astronomiques, et l'auteur montre comment on pourrait construire une solution à partir de données d'origines différentes. Le résultat permet de faire quelques suggestions sur la façon dont devrait se faire le choix des nouvelles valeurs conventionnelles de ces paramètres.

ABSTRACT. — The author discusses the set of constants required to describe the external gravitational field of the Earth : a set of form factors and two scale factors. Some problems in relating the external field as derived from the motions of satellites to the quantities observed at the surface are mentioned. The fondamental parameters cannot be derived from the data without a knowledge of other astronomical constants, and it is shown how a general adjustment of data of different origins can be made. The results enable suggestions to be made about the way in which new conventional values of these parameters should be chosen.

ZUSAMMENFASSUNG. — Verf. untersucht das System von Konstanten, das zur Beschreibung des äusseren Gravitationsfeldes der Erde erforderlich ist : ein Satz von Formfaktoren sowie zwei Skalenfaktoren. Einige Probleme werden erwähnt, die entstehen, wenn man das aus den Satellitenbewegungen abgeleitete äussere Feld zu den an der Erdoberfläche beobachteten Grössen in Beziehung setzen will. Die fundamentalen Parameter können aus den Beobachtungen nicht abgeleitet werden, ohne dass andere astronomische Konstanten bekannt sind; es wird gezeigt, wie eine allgemeine Ausgleichung der Angaben verschiedener

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(1) Extensive revisions incorporate new data presented at the Symposium.

Herkunft durchgeführt werden kann. Die Ergebnisse ermöglichen Hinweise auf das Verfahren, nach dem neue konventionelle Werte dieser Parameter ausgewählt werden sollten.

Резюме. — Автор изучает совокупность постоянных определяющих внешнее гравитационное поле Земли : три параметра определяющие форму Земли и масштаб. Автор обращает внимание на некоторые проблемы возникающие при установлении связи между внешнем полем определенным по движению спутников и наблюдениями на земной поверхности. Невозможно вычислить основные параметры из наблюдений без знания об астрономических постоянных; автор показывает как можно построить полное решение употребляя различные данные. Результаты этого исследования дают некоторые указания о том как следует выбирать условные значения этих параметров.

**1. Introduction.** — In the years since conventional values of the size and shape of the Earth and of the values of gravity at its surface were last established by the *International Ellipsoid and International Gravity Formula*, a very great deal of new information has become available and a number of critical studies have been made. It is clear that as a result in particular of satellite observations and of radar observations of the Moon and of Venus, as well as of new programs of geodetic survey, much more accurate values of the fundamental constants can be derived than were available at the time of the establishment of the *International Ellipsoid and Formula*. It is therefore natural to ask whether new constants should not now be adopted by international agreement. In view of the number of different disciplines concerned : metrology, astronomy, geodesy and satellite studies, to mention only those most directly involved, such a revision may be somewhat lengthy. At the same time a clear distinction should be kept between conventional values adopted as an aid to the statement of observed data into which category the *International Gravity Formula* for example falls, and consistent values expressing the best results of observations. The latter must be derived before the former can usefully be agreed and the best set of quantities to adopt as a conventional set must depend on an analysis of the data.

The choice of the fundamental constants of geodesy and of the Earth-Moon system can be made in a variety of ways, the actual set chosen depending on what measurements can be made most accurately. It seems that the time has come to abandon the set used by de Sitter and Brouwer [1] and to adopt one more in accordance with the fact that the low order harmonics of the gravity field of the Earth can be obtained with the highest accuracy directly from satellite observations rather than indirectly from the constant of precession and that the actual distances of the Sun and the Moon can be obtained more accurately by direct radar measurements than indirectly from the parallaxes.

2. **The external potential of the Earth. Fundamental parameters.** — The constants and data to be discussed in this paper are those connected with the external gravitational potential of the Earth. Other astronomical quantities are brought in only so far as they are needed in the determination of the parameters of the Earth's field.

Since the potential satisfies Laplace's equation throughout the entire region outside the Earth, it may be expressed as a unique, convergent series of spherical harmonics outside a sphere with its centre at the centre of mass of the Earth and radius such that it just encloses all the matter of the Earth. These statements will be put more critically below; meanwhile, take a system of spherical polar co-ordinates with origin at the centre of mass of the Earth, with $r$ as the radius vector, $\theta$ as the geocentric co-latitude and $\lambda$ as the longitude. Then the potential V will be written as

$$V = \frac{GM}{l}\,\frac{l}{r}\left[1 - \sum_{n=2} J_n \left(\frac{l}{r}\right)^n P_n(\cos\theta) + \sum\sum \left(\frac{l}{r}\right)^n A_{nm} P_n^m(\cos\theta)\cos m\lambda \right.$$
$$\left. + \sum\sum \left(\frac{l}{r}\right)^n B_{nm} P_n^m(\cos\theta)\sin m\lambda\right].$$

The notation of the coefficients $J_n$, $A_{nm}$, $B_{nm}$ conforms to that recommended by Commission 7 of the International Astronomical Union.

This expression contains three types of parameter :

— form factors, $J_n$, $A_{nm}$, $B_{nm}$, dimensionless parameters representing the variation of the potential with orientation;

— a scale factor for length, $l$;

— a scale factor for mass, GM.

It has been usual to take as the scale factor for length the Earth's equatorial radius R, as is the case in the *International Ellipsoid*, for instance, although it has often been considered that in critical discussions it is more satisfactory to use the mean radius $R_m$. The equatorial radius will be taken here since use of the mean radius is apt to lead to confusion. The form factors are not independent of the scale factor because it is the products $l^n J_n \ldots$ which are invariant, and the numerical values must correspond to the value of $l$.

Of the form factors, only $J_2$ is taken as a fundamental constant. It is true that all the observable factors are needed for an exact description of the external potential as it is observed to control artificial satellites

https://doi.org/10.1017/S0074180900104802 Published online by Cambridge University Press

and to a much less extent, for a description of the surface values of gravity, but the terms beyond $J_2$ have no detectable effect on the motion of the Moon, nor do uncertainties in them affect the residuals of radar observations, of arc-lengths or of occultations, although in the two latter sets of data reasonably good values of the higher coefficients should be used in the calculations.

The scale factor GM needs some discussion. If the distance of the Moon and $\mu$, the ratio of her mass to that of the Earth are taken as further fundamental constants, GM may be calculated; and in a consistent set of fundamental constants the Moon's distance and GM should not both appear as independent quantities. Further, the factor GM may be expressed in terms of the mean value of gravity over the surface of the Earth, $g_m$, and this quantity also may be calculated if the lunar parameters and the size of the Earth are given. It is not satisfactory to take $g_m$ as an independent parameter because the calculation is not straightforward and free of doubt, involving as it does the effect of the topography that lies above sea level and the inadequacy of the present sampling of surface gravity. In consequence, it seems best to treat $g_m$ as an observed quantity with an error to be determined and not as a fundamental parameter. The choice then lies between GM and the lunar quantities and since very concordant values of GM have been derived in a variety of ways (Kaula, p. 26). GM is taken as the basic parameter and the Moon's distance and $g_m$ are derived from it.

Lastly, the distance of the Sun is involved in the determination of the mass of the Moon from the lunar inequality and is therefore included among the fundamental parameters but it is so well determined by radar that it can be treated as exact for present purposes.

The lunar and solar distances have been taken as direct distances and not as parallaxes which involve the size of the Earth. Now that these quantities can be determined by radar there seems to be every reason for abandoning parallaxes and adopting the direct distances themselves with as will be seen, a considerable simplification in the relations involving them.

The precise definitions of the lunar and solar distances are :

the solar distance $a_{\odot}$ is the astronomical unit;

the lunar distance $a_{☾}$ is the semi-major axis of the variation orbit of the Hill-Brown theory, that is, it includes the principal effect of the solar perturbation of the Moon's orbit and is related to the product GM by the formula

$$GM(1+\mu) = n'^2 a_{☾}^3 (1+\nu_4)^3,$$

where

$$\nu_4 = 9.07681 \times 10^{-4}.$$

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To summarise, the fundamental parameters adopted in this paper are :

the Earth's equatorial radius, R;

the form factor $J_2$, $= \dfrac{C - A}{MR^2}$;

the product GM;

the ratio $\mu$ of the mass of the Moon to that of the Earth;

the astronomical unit, $a_{\odot}$.

The radius of the Moon is involved in radar measurements of the Moon's distance; it is expressed as a fraction $\beta$ of the Moon's distance.

3. **Potential and gravity on the surface of the Earth.** — It is an advantage of the parameters chosen above that certain difficulties that arise in the theory of the potential and of gravity on the actual surface of the Earth do not pervade the whole subject but that their effects can be restricted to the actual data that they perturb, practically speaking only surface gravity values.

Dealing first with minor matters, the gravity field in which the Moon and artificial satellites move includes components due to the atmosphere whereas surface gravity values do not. A correction must therefore be made to the observed surface values, in practice to the mean value only. Again, surface values of gravity and potential contain terms due to the rotation of the Earth; the corrections depend on a well known theory and can be calculated very precisely.

The more difficult problem, which it is hoped to discuss in detail elsewhere, concerns the effect of topography on the expansion of the field on and near the surface in terms of spherical harmonies. Strictly speaking, such an expansion is valid only in the region outside a sphere enclosing the whole mass of the Earth and its use inside that sphere, as is implied by the expression of free air anomalies as a series of surface harmonies and the relation of the terms of that series to the coefficients $J_n$, may not be entirely correct.

The principal conclusions are :

the form of the geoid corresponds to the distant field value of $J_2$;

the second harmonic in surface gravity differs from that corresponding to the distant field value of $J_2$ by between 5 and 10 mgal;

the estimate of $g_m$ from surface gravity is not much affected by an error in the coefficient of the second harmonic.

Fortunately the value of $g_m$ found from a surface harmonic analysis is not seriously perturbed by an incorrect value of the coefficient of $P_2$ in surface gravity. Ideally, since the Legendre functions are orthogonal, the estimate of $g_m$ should be independent of the other coefficients, but because the distribution of data over the surface of the Earth is far from complete or uniform, there is some correlation. Nevertheless, the main point here is that the estimated quantity $g_m$ is relatively unaffected by an erroneous estimate of the coefficient of $P_2$ and therefore is a much better quantity to estimate from the observations than is the value of $g$ on the equator.

4. **Observational equations.** — Let $z$ be an observed quantity that is a function of the fundamental parameters

$$z = z(R, J_2, \ldots).$$

Let $z_0$ be an observed value and $z_c$ a value calculated from some adopted set of parameters. Then in general

$$z_0 = z_c + \frac{\delta z}{\delta R}\delta R + \frac{\delta z}{\delta J_2}\delta J_2 + \ldots + \varepsilon,$$

where $\varepsilon$ is an error of observation and $\delta R$, $\delta J_2$, ... are corrections to be made to the adopted values, in general so as to minimise a sum of weighted squares of residuals.

Thus

$$\frac{1}{z_c}\frac{\delta z}{\delta R}\delta R + \frac{1}{z_c}\frac{\delta z}{\delta J_2}\delta J_2 + \ldots = \frac{z_0 - z_c}{z_c} - \varepsilon.$$

Now write

$$\delta R = x_1 R, \qquad \delta(GM) = x_2 GM, \qquad \delta J_2 = x_3 J_2,$$
$$\delta\mu = x_4\mu, \qquad \delta\beta = x_5\beta \qquad \text{and} \qquad \delta a = x_6 a$$

($\beta$ is the Moon's radius).

The value of the A.U. in kilometres is also required but as will be seen, it enters just one observation equation and that the one of least weight, and has only a small influence on other parameters. Since it is now known to a few parts in a million from radar determinations, it will be taken as a constant, although strictly it should be allowed to vary. The effect of so doing is that there is no coupling between geodetic and lunar parameters and other parameters of the solar system.

Now let

$$y_i = \left(\frac{z_0 - z_c}{z_c}\right)_i.$$

The typical observational equation may therefore be written as

$$k_1 x_1 + k_2 x_2 + \ldots = y_i.$$

In the remainder of this section such observational equations will be derived for all the relevant observations. It is evident that they are equations of the type set up by de Sitter and Brouwer [1] and used subsequently by Jeffreys [2], but they differ from the earlier ones in two ways : first, the form of the external potential field is described directly by the coefficient $J_2$ and not indirectly by the precessional constant and an assumption that the Earth is in an hydrostatic state, and secondly, the parallaxes of the Sun and the Moon are replaced by the actual distances. Together, these two changes, which arise directly from recent technical developments, bring about a considerable simplification in the formulation of the equations and in the appreciation of their interrrelations.

The lunar and terrestrial quantities are not dynamically independent but are related by the equation for the motion of the Moon :

$$GM(1+\mu) = n'^2 a_{☾}^3 (1+\nu_1)^3.$$

$u'$ and $\nu_1$ are treated as exact and the following differential relation is then obtained :

$$x_6 = \frac{1}{3}\left(x_2 + \frac{\mu}{1+\mu} x_4\right).$$

This result is used to eliminate $x_6$.

4.1. *Motions of artificial satellites.* — The first order term proportional to $J_2$ in the secular regression of the node of an artifical satellite is

$$\dot{\Omega}_2 = -\frac{3}{2}\left(\frac{GM}{a^3}\right)^{\frac{1}{2}}\left(\frac{R}{p}\right)^2 J_2 \cos i.$$

This expression is the basis of the observational equation for $J_2$ though it is to be understood that in reducing the observations a second order term proportional to $J_2^2$ is included.

$a$ is the semi-major axis of the orbit and $p$ the semi-latus rectum. $i$ is the inclination of the plane of the orbit to the Earth's equator. $a$ is found from the nodal period $T_N$ :

$$a^{\frac{3}{2}} = (GM)^{\frac{1}{2}} T_N \left\{1 - \frac{3}{8} J_2 \left(\frac{R}{p}\right)^2 (7\cos^2 i - 1)\right\}^{-1}.$$

But

$$p = a(1-e^2)$$

and

$$a^{-\frac{7}{2}} = (GM)^{-\frac{7}{6}} T_N^{-\frac{7}{3}} \left\{1 - \frac{3}{8} J_2 \left(\frac{R}{p}\right)^2 (7\cos^2 i - 1)\right\}^{\frac{7}{3}}$$

so that

$$J_2 (GM)^{-\frac{2}{3}} R^2 = -\frac{2}{3} \dot{\Omega}_2 T_N^{\frac{7}{3}} \sec i \left\{1 - \frac{3}{8} J_2 \left(\frac{R}{p}\right)^2 (7\cos^2 i - 1)\right\}^{-\frac{7}{3}}.$$

In the variational form this becomes

$$x_3 - \frac{2}{3} x_2 + 2x_1 = y_1,$$

where

$$y_1 = \frac{O - C}{C} \quad \text{of} \quad \left[ -\frac{2}{3} \dot{\Omega}_2 T_N^{\frac{7}{3}} \sec i \left\{ 1 - \frac{3}{8} J_2 \left( \frac{R}{p} \right)^2 (7 \cos^2 i - 1) \right\}^{-\frac{7}{3}} \right].$$

In practice all the even order J's give rise to secular components of $\dot{\Omega}$ and the analysis of the motions of various satellites has produced a set of $J_2$, $J_4$, ..., calculated with certain assumed values of R and GM. Hence $y_1$ will be found from the product $J_2 (GM)^{-\frac{2}{3}} R^2$ which will in effect be a weighted mean value of

$$-\frac{2}{3} \dot{\Omega}_2 T_N^{\frac{7}{3}} \sec i \left\{ 1 - \frac{3}{8} J_2 \left( \frac{R}{p} \right)^2 (7 \cos^2 i - 1) \right\}^{-\frac{7}{3}}$$

over all the satellites used. It may also contain contributions from the observed secular motions of perigee.

The uncertainty of $y_1$ will be derived from the quoted uncertainty of the solution for $J_2$.

4.2. *The distance of the Moon.* — The geometry of an observation of the distance of the Moon by radar is shown in figure 1. Let the

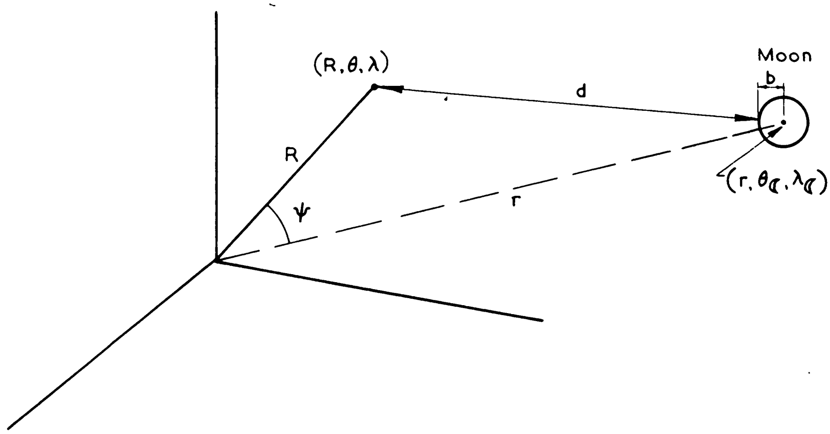


Fig. 1. — Radar measurements of Moon's distance.

co-ordinates of the observatory be $(R_s, \theta, \lambda)$ and let the geocentric co-latitude and longitude of the Moon be $\theta$ and $\lambda$ respectively. Let the observed distance be *d*, and let it be assumed that corrections for periodic terms have been applied so that *d* corresponds to the semi-major axis of the variation orbit. Let the radius of the Moon in the direction

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of the Earth be $b$, let $r$ be the geocentric distance of the Moon and let $\psi$ be the angle between the geocentric directions of the Moon and of the observatory.

Since $R_s$ depends on the equatorial radius and $r$ on the Moon's mean distance, this geometrical relation provides an observation equation relating R, $a_☾$ and $\beta$. The numerical factors depend on the details of the geometry and the work of Yaplee and others [3] leads to the following equation

$$a_☾ - 0.712\,\Delta R - \Delta b = 384\,400.2 \text{ km},$$

where $\Delta R$ is the difference between R and 6 378 163 km and $\Delta b$ is the difference between the radius of the Moon and 1738 km.

The differential form is

$$x_6 - 0.712 \frac{R}{a_☾} x_1 - x_5 = y_2$$

or eliminating $x_6$;

$$\frac{1}{3}\left(x_2 + \frac{\mu x_4}{1+\mu}\right) - 0.712 \frac{R}{a_☾} x_1 - x_5 = y_2.$$

4.3. *The mass of the Moon.* — Because of the finite mass of the Moon, the Earth moves around the common centre of mass of the Earth and the Moon in an ellipse of semi-major axis $\frac{\mu a_☾}{1+\mu}$ with the period of the Moon's motion. As a consequence the Sun and the planets show an apparent variation in angular position described by the lunar inequality L, equal to

$$\frac{a_☾}{a_\odot} \frac{\mu}{1+\mu}.$$

But, in addition, the velocity of the Earth has a component of the same period with amplitude

$$\frac{n' a_☾ \mu}{1+\mu}$$

and this component was detected in the radio Doppler tracking of the Venus rocket *Mariner II* (Hamilton [4]). If $\nu$ is the radio-frequency and $c$ the velocity of light the variation of Doppler shift will be

$$\nu_D = \frac{\nu}{c} \frac{n' a_☾ \mu}{1+\mu}.$$

If $a_\odot$, $\nu$, $c$ and $n'$ are treated as exact, both methods lead to an observation equation of the same form

$$x_6 + \frac{x_4}{1+\mu} = \frac{\delta L}{L}, \frac{\delta \nu_D}{\nu_D}$$

or, on eliminating $x_6$,

$$\frac{1}{3}x_2 + \left(1 + \frac{1}{3}\mu\right)\frac{4}{1+\mu} = \frac{\delta L}{L}, \quad \frac{\delta \nu_D}{\nu_D} = y_3, \ y_4.$$

4.4. *Mean surface value of gravity.* — The mean value of gravity on the surface, taken with respect to geographical latitude, is

$$g_m = \frac{GM}{R_m^2}\left(1 - \frac{2}{3}m - \ldots\right) \qquad \text{(Appendix 2, p. 57)}$$

but

$$m = \frac{\omega^2 R_m^3}{GM}$$

and

$$R_m = R\left(1 - \frac{1}{3}f - \ldots\right),$$

where $f$ is the polar flattering of the Earth given to first order by

$$f = \frac{3}{2}J_2 + \frac{1}{2}m.$$

Hence

$$g_m = \frac{GM}{R^2}(1 + J_2) - \frac{1}{3}R\omega^2$$

and

$$\frac{\delta g_m}{g_m} = x_2 + \frac{x_3 J_2}{1+J_2} - x_1\left(2 + \frac{1}{3}m\right)$$

that is,

$$x_2 + x_3\frac{J_2}{1+J_2} - x_1\left(2 + \frac{1}{3}m\right) = y_5.$$

4.5. *Measurements of arc lengths.* — Measurements of arc lengths on the surface of the Earth give essentially the mean radius of curvature in the region in which the surveys were made. The distribution of geodetic surveys is now so good and the potential at sea level is so well known from satellite observations that if the geodetic results are reduced with the value of the flattening derived from satellite observations of $J_2$, then the arc length measurements give effectively the mean radius of curvature $R_m\left(1 + 4J_2 + \frac{4}{3}m\right)$ for arcs along a meridian, and $R_m\left(1 + 2J_2 + \frac{2}{3}m\right)$ to first order for arcs along a parallel. Since present geodetic networks cover areas rather than arcs, it is probably correct to consider that they yield the average radius

$$R_m(1 + 3J_2 + m).$$

Since

$$R_m = R\left(1 - \frac{1}{3}f\ldots\right) = R\left(1 - \frac{1}{2}J_2 - \frac{1}{6}m\ldots\right)$$

the corresponding observation equation is

$$x_1\left(1+\frac{5}{2}J_2+\frac{10}{3}m\right)+\frac{5}{2}J_2x_3-\frac{5}{6}mx_2=y_6.$$

In the past, geodetic observations have been reduced to give a value of the flattening as well as of the radius and Jeffreys [2] expressed the results of such observations as a pair of observation equations of independent variance, one for the flattening and one for a combination of the flattening and the radius. It is now considered that the weight of the flattening equation is so small in relation to satellite observations that there is no point in retaining it.

4.6. *The size of the Moon.* — The data giving the size of the Moon are observations of her apparent angular diameter. The observation equation is therefore

$$x_5=y_7.$$

4.7. *Occultations.* — Observations of occultations of stars by the Moon give a relation between the geocentric position of the station from which they are observed and the geocentric position of the Moon at the time of observation. The size of the Moon is also invoved. A variety of results may therefore be derived from the observations, depending on the way in which they are made and the relative accuracies with which the different variables are supposed to be known. Thus if geodetic uncertainties are eliminated, corrections to the Moon's position may be derived, or the distance of the Moon may be found in terms of the radius of the Earth (Fischer [5]). On the other hand, by assuming standard values of the size of the Earth and the distance of the Moon, corrections to the geodetic co-ordinates of the stations of observation may be obtained. The observations of O'Keefe and Anderson [6] are discussed in Appendix 2 where it is shown that corrections to the size of the Earth and the distance of the Moon cannot be found separately but only as the combination $(x_1-x_6)$ and that the size of the Moon cannot be found independently of corrections to the co-ordinates of the Moon.

Occultations therefore give an observation equation

$$x_1-x_6=y_8.$$

that is

$$x_1-\frac{1}{3}\left(x_2+\frac{\mu}{1+\mu}x_4\right)=y_8.$$

4.8. *Direct measurement of* GM. — At the Symposium W. M. Kaula listed a number of estimates of the product GM. It is not clear how most of these are related to other observations used in this paper and for the most part they are ignored but one, from tracking of the space

https://doi.org/10.1017/S0074180900104802 Published online by Cambridge University Press

probe *Ranger* 3, does seem to be a completely independent observation. It is supposed to lead to an equation

$$x_2 = y_3.$$

5. **Data and solutions.** — In this section data corresponding to the observations equations of section 4 are collected and reviewed and a least squares adjustment is made. The data and solutions are not those given in the preliminary version of this paper available at the Symposium because advantage has been taken of the material presented there to obtain an improved solution. The main changes are that the value of the astronomical unit may now be held constant at the radar value, that a Doppler value for the lunar inequality in the Earth's motion is available and that an estimate of GM is available from space probe tracking.

The following values are treated as exact :

| | |
|---|---|
| Velocity of light in vacuo (as adopted by the I. U. G. G.) .................. | 299 792.5 km/s |
| Mean motion of the Moon (de Sitter and Brouwer [1]) .................. | $2.6616995 \times 10^{-6}$ rad.s$^{-1}$ |
| Spin angular velocity of the Earth...... | $7.292115 \times 10^{-5}$ rad.s$^{-1}$ |
| Radar value of the A. U................ | $149.598 \times 10^9$ m |

(*See* papers by Muhleman and Shapiro, p. 153 and 177 in this Symposium).

Data are expressed as differences from values calculated with the following trial values :

| | |
|---|---|
| 1. Equatorial radius of the Earth........ | $R = 6\,378.163$ km |
| 2. » » » ........ | $GM = 3.986034130 \times 10^{14}$ m$^3$.s$^{-2}$ |
| 3. » » » ........ | $J_2 = 1082.60 \times 10^{-3}$ |
| 4. Mass of the Moon.................... | $\mu = 0.01230000$<br>$\mu^{-1} = 81.300813$ |
| 5. Moon's diameter ...................... | $\beta = 4.521 \times 10^{-3}$ (radius $= 1\,738$ km) |

and the following auxiliary values are derived :

| | |
|---|---|
| 6. Distance of the Moon ................ | $a_{☾} = 3.84400000 \times 10^8$ m |
| Flattening of the Earth.............. | $f = 3.35276 \times 10^{-3}$<br>$f^{-1} = 298.262$ |
| Mean radius of the Earth............ | $R_m = 6\,371.033$ km |
| Mean surface gravity................ | $g_m = 979.75156$ cm.s$^{-2}$ |

5.1. *Satellite value of* $J_2$. — The results have recently been reviewed by King-Hele ([7], [8]) from whose papers table I is taken. The main point to notice about this summary is that the scatter of the results obtained by different workers is much greater than the uncertainties estimated by each author. It is highly probable that the signifiance of this fact is that the real uncertainty of the results is not in the errors of the observations but in difficulties in the methods of reduction. There seem to be two such sources of uncertainty in particular. Firstly, some estimates are derived from the motion of the node alone whereas others include data from perigee as well, although it is usually supposed that perigee is less well defined observationally than the node; and secondly, with data from a limited set of satellites, only the first few zonal harmonics can be determined and the higher ones have to be assumed to be zero. No numerical studies have been made of the effects of the first difference between treatments, but Smith [9] has shown that neglect of $J_6$ can cause rather a large change in $J_2$.

TABLE I.

*Values of* $J_n$ *determined from satellite orbits.*

| Author and date. | $10^6 J_2$. | $10^6 J_4$. | $10^6 J_6$. | $10^6 J_8$. | $10^6 J_{10}$. | $10^6 J_{12}$. |
|---|---|---|---|---|---|---|
| O'Keefe and others (1959) [10]........... | 1082.5 | −1.7 | – | – | – | – |
| s. d.......... | 0.1 | 0.1 | – | – | – | – |
| Kozai (1961) [11]..... | 1082.19 | −2.1 | – | – | – | – |
| s. d.......... | 0.03 | 0.1 | – | – | – | – |
| Smith (1961) [9]..... | 1083.15 | −1.4 | +0.7 | – | – | – |
| s. d.......... | 0.2 | 0.3 | 0.6 | | | |
| Michielsen (1961) [12] | 1082.7 | −1.7 | +0.7 | 0.1 | – | – |
| s. d.......... | – | – | – | – | – | – |
| Kozai (1962) [13]..... | 1082.48 | −1.84 | +0.39 | −0.02 | – | – |
| s. d.......... | 0.06 | 0.08 | 0.02 | 0.2 | – | – |
| King-Hele, Cook and Rees (1963) [14]..... | 1082.86 | −1.03 | +0.72 | 0.34 | −0.50 | +0.44 |
| s. d........ | 0.1 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 |

Of the results in table I, those of Kozai [13] and of King-Hele, Cook and Rees [14], depend on the most thorough studies. Unfortunately, no reason is known for the rather large discrepancy between them and it seems best to include them both in the observation equations, weighting them according to the authors' estimates of uncertainties.

Then

$$y_1 = (-111 \pm 60) \times 10^{-6} \qquad \text{(Kozai)},$$
$$(+241 \pm 100) \times 10^{-6} \qquad \text{(King-Hele } \textit{and al.}\text{)}$$

5.2. *The distance of the Moon.* — The data are those reported by Yaplee and his collaborators at the Symposium; they have been computed with the trial values of $a$ and $\beta$ used here.

The uncertainty quoted for the centre to centre distance of the Moon from the Earth is 1.1 km but this is mostly the uncertainty in $\beta$ which is taken account of separately in the present adjustment. The uncertainty of the radar measurements themselves is 400 m, mostly due to the uncertainty in the velocity of light.

Then since the result is

$$a_{☾} = 3.844002 \times 10^8 \text{ m}$$

and the trial value is

$$3.8440000 \times 10^8 \text{ m},$$
$$y_2 = (+0.5 \pm 1) \times 10^{-6}.$$

5.3. *The mass of the Moon.* — The lunar inequality, L, has been derived from Spencer-Jones's observations of the close approach of Eros in 1930-1931. There are two principal sources of error other than the observational errors themselves : there may be errors in the star positions with which those of Eros were compared, and the actual orbit of Eros differs from that taken by Spencer-Jones so that residuals of the observations are not random and uncorrelated. Spencer-Jones's data have therefore been re-examined by Jeffreys [15] who used a statistical method designed to deal with this situation, and by Rabe [16] and Delano [17] who each computed improved orbits. Unfortunately the three results are not in good agreement :

| | | |
|---|---|---|
| Jeffreys............ | $6''.4378 \pm 0''.0017$ | |
| Rabe............... | $6\ .4356 \pm 0\ .0028$ | |
| Delano ........... | $6\ .4429 \pm 0\ .0015$ | (mean of two solutions) |
| (Spencer-Jones...... | $6\ .4390 \pm 0\ .0015$) | |

The value adopted will be the mean of all three values

$$6''.4388$$

with a standard deviation of $0''.0020$.

The trial values of the parameters correspond to $L = 6''.4399$ and so

$$y_3 = (-1.70 \pm 3.1) \times 10^{-4}.$$

The Doppler measurements on *Mariner II* reported at the Symposium (Hamilton [4]) give

$$\mu^{-1} = 81.3015 \pm 0.0033.$$

With the trial value $\mu^{-1} = 81.3008$,

$$y_4 = (-9 \pm 40) \times 10^{-6}.$$

5.4. *The mean value of gravity.* — The mean value of gravity over the Earth is derived by adding to the mean value of gravity values in the Potsdam system as found from a harmonic analysis with respect to geographical latitude, a correction for the difference of modern absolute measurements from the Potsdam absolute measurement.

The results of five recent analyses of free-air gravity values are given in table II.

The statistical problem of such analyses arises from the systematic variation of free-air anomalies with height and the correlation of anomalies over very great distances, coupled with poor distribution of observations. Heiskanen [20] and Zhongolovitch [19] have not allowed for the correlation and Jeffreys [18] and Kaula [22] have dealt with it in quite different ways. As with the preceding cases, in which differences are due to the method of treatment of the data and not to the data themselves, an average of the five results will be taken, giving for the mean value of gravity in the Potsdam system :

$$g_m = 979772.3 \pm 2.0 \; mgal.$$

TABLE II.

*Harmonic analyses of free-air gravity (Potsdam standard).*

*(unit : mgal)*

| Author and date. | $g_e$. | $g_m$. | Coefficients of sin² φ. | Coefficients of $P_2(\sin\varphi)$. | Coefficients of $P_4(\sin\varphi)$. |
|---|---|---|---|---|---|
| | 978 000+ | 978 000+ | | | |
| Jeffreys 1943 [18] | – | 772.5 | – | 3 440 | 5.3 |
| s. d. | – | 1.9 | | 5 | – |
| Zhongolovitch 1952 [19] | – | 777.3 (*) | | 3 439 (*) | |
| s. d | – | 1.8 | – | 5 | |
| Heiskanen (1957) [10] | 049.7 | 771.2 | 5290.2 | 3 443.5 | |
| Kaula (1959) [21] | – | 768.0 | | 3 450.8 | 5.6 |
| Uotila (1962) [22] | | 772.3 | | 3 450.5 | |
| Average | – | 772.3 | | – | |
| s. d. of single value | – | 3.3 | | | |
| s. d. of average value | – | 1.5 | | | |

(*) Average of three solutions on different assumptions.

The absolute values of free-air gravity are based on the measurements at Potsdam by Kühnen and Furtwängler [23] and a correction must be applied to bring them into agreement with the most recent absolute determinations. The most reliable results at present available are summarised in table III. The scatter of the values for the correction to the Potsdam values is in part due to the absolute measurements themselves and in part as can be seen from the two sets of American values, to the measurements of the differences between sites of absolute measurements and in the absence of measurements by different methods at the same site it is not possible to separate the contributions from the two sources. The adopted correction is

$$-13.8, \quad \text{s. d. } 0.6 \text{ mgal.}$$

TABLE III.

*Absolute measurements of gravity.*

| Site. | Author and date. | Method. | Difference from Potsdam value. Cook (mgal). | Rose (mgal). |
|---|---|---|---|---|
| Washington (NBS)... | Heyl and Cook (1936) | Reversible pendulum | —16.0 | —16.4 |
| Teddington (NPL).. | Clarke (1939) [24] | Id. | —13.1 | —13.0 |
| Leningrad (VNIIM)... | Ageletskii and Egorov (1956) [25] | Id. | —12.1 | —12.1 |
| Ottawa (NRC)...... | Preston-Thomas and others (1960) [26] | Freely falling bar | —14.0 | —14.7 |
| Sevres (BIPM)..... | Thulin (1961) [27] | Id | —12.7 | —12.6 |
| Princeton.......... | Faller (1963) | Falling interferometer component | —14.5 | —14.7 |
| | Means | Europe | —12.6 | —12.6 |
| | Means | America | —14.8 | —15.3 |
| | Means | Europe and America | —13.7 | —14.0 |
| | Means | s. d. | 1.1 | 1.3 |

The two sets of differences from Potsdam values depend on adjustments of the data reported by A. H. Cook and J. C. Rose to the International Gravity Commission in September 1962.

It is obvious from the relative scatters that the uncertainty in the mean value of gravity in absolute terms is predominantly contributed by the harmonic analysis of free-air gravity.

The mean value of gravity as observed at the surface of the Earth has to be increased by the attraction of the atmosphere, 0.9 mgal, giving finally

$$g_m = 979.7594 \text{ cm/s}^2.$$

The trial value is 979.7516 cm/s² and the observation equation is thus

$$y_5 = (+7.8 \pm 2) \times 10^{-6}.$$

5.5. *Measurements of arc lengths.* — The most recent results of the studies of geodetic triangulation made by Mrs. Fischer are summarised in the World Geodetic System (Kaula, 1963) for which the equatorial radius is 6378.163 ± 0.021 km, assuming the flattening to be that corresponding to the satellite value of $J_2$. Since the flattening is given much more accurately this way than by the geodetic measurements themselves and since the mean radius does not depend, to first order, on the adopted flattening, the result of the arc-length measurements is taken to be just the mean radius. The trial value being 6 378.163 km, the observations give

$$y_6 = (0 \pm 3.5) \times 10^{-6}.$$

5.6. *The size of the Moon.* — The size of the Moon enters the radar measurements of the distance of the Moon and the occultation measurements; in the former it is the radius in the direction of the Earth that is wanted whilst in the latter it is the radius in the plane normal to the direction of the Earth. The trial value used for both is the radius in the plane normal to the direction of the Earth as used in the *American Ephemeris*; apparently it was derived by Newcomb from many measurements of the Moon's apparent angular diameter and it is therefore taken to be proportional to the Moon's distance. No uncertainty is assigned to the value used in the *Ephemeris* and whatever it may be, the occultation observations are subject to additional uncertainty due to the irregular contour of the Moon. As for the radius along the direction of the Earth, almost nothing is known about it for although the corresponding difference of moments of inertia is known, $\frac{B - A}{M a^2} = 0.000072$, the surface of the Moon is not fluid so that the difference of radii cannot be inferred from the moments of inertia.

The radius for both occultation and radar measurements has therefore been taken to be the *Ephemeris* value and a standard deviation of 1 km has been arbitrarily assigned. Then

$$x_5 = 0, \quad \text{s. d. } 3 \times 10^{-6}.$$

5.7. *Occultations.* — The data of O'Keefe and Anderson [6] as re-computed in Appendix 2, lead to the equation

$$x_1 - x_4 = 3.2 \times 10^{-6} \quad \text{s. d. } 15.3 \times 10^{-6}.$$

5.8. *Values of* GM. — At the Symposium, W. M. Kaula presented a summary of recent estimates of GM (*see* p. 26). For the most part the values indicate the consistency of observations already used separately in the present adjustment but satellite and space probe results import new material, all of very high precision. Kaula has made two estimates from close satellite orbits but the length scale depends on the value of R in the *World Geodetic System*. The only independent value is that from *Ranger* 3 :

$$GM = 3.986016 \pm 0.00005 \times 10^{14} \, m^3 . s^{-2}.$$

With the trial value of 3.986034,

$$y_9 = (-4.5 \pm 1.2) \times 10^{-6}.$$

The satellite values are :

From *Echo I* :

$$3.986037 \pm 0.000012;$$

From *Echo I* and rocket *Vanguard, Explorer* 9 :

$$3.986028 \pm 0.000008.$$

5.9. *Adjustment and discussion.* — The uncertainties of the observations vary greatly and taken at their face value, the range of weights is $10^5$ to 1. In these circumstances the uncertainty of one weight relative to another must be considerable and two points require investigation : how far the standard deviations given above represent independent normally distributed errors, and what effect changes in weighting may have. It has already been seen that some of the " standard deviations " quoted do not correspond to normally distributed errors but indicate the effects of different treatments of systematic errors and so it is clear that a system of weight inversely proportional to the squares of these " standard deviations " does not truly represent the reliability of the data. In the nature of the material, it is difficult to be more precise and so it is important to see the effect of radically different weighting. One solution (A) uses weights inversely proportional to the squares of the standard deviations, the other (B), which undoubtedly undervalues the more precise material, uses weight inversely proportional to the standard deviations themselves. Table V (*see* Appendix 1, p. 54) summarises the data, the weights and the residuals.

The normal equations, the solutions, variances, standard deviations and quantities of independent variance are in tables VI and VII (*see* Appendix 1, p. 56) contains the correlation coefficients between the estimated values of the parameters.

Rather surprisingly, the values of $\chi^2$, that is $\sum \frac{(O-C)^2}{(s.d.)^2}$, are the same for the two solutions. The chance of $\chi^2$ reaching 20.6 by chance on 5 degrees of freedom is 0.1 % so that some observations are inconsistent. $\chi^2$ could be reduced by 9 in solution A and 7 in solution B by taking the mean value of the two estimates of $J_2$ without in any way altering the solution*; $\chi^2$ would then be 11 or 14 respectively on 4 degrees of freedom and the former (solution A) would be just about acceptable. There remains however some inconsistency, either in $g_m$ or GM, apparently and it is probably the former which is the more in error although the

* Later experience shows that this would be entirely justifiable.

quoted uncertainty of the *Ranger* value of GM may also be too small. The other data seem entirely satisfactory.

The discrepancy between the two estimates of $J_2$ is disappointing and the results of the present calculations do not help to decide between the two values. Evidently one or both of the estimates of standard deviation is too small.

The solutions indicate that the value of $g_m$ adopted here is too large. Kaula's value, 4 mgal less, would greatly improve the solution as would the adoption of the American rather than the European absolute values. The latter fact may arise because a larger area of gravity survey depends on American base stations than on European ones. With these two changes, the residual for $g_m$ in solution A can be reduced to zero, and solution B also would presumably be improved.

It can be seen from table VII that the correlation coefficients between the fundamental parameters are generally very small, the only significant one being between R and GM. Correlation coefficients between $a_☾$ and R and $\beta$ are significant and this indicates that GM is a rather better choice of fundamental parameters than $a_☾$.

It seems that the solutions and correlation coefficients are not critically dependent on the weighting, but that the uncertainties of R, GM and $\beta$ do depend strongly on the weighting. The main problem, therefore, is to decide what uncertainties to adopt and in so doing, the main factor to take into account is that the variance of an observation of unit weight is mainly composed of the contributions from the two estimates of $J_2$ which, as has been mentioned, scarcely affect the solutions. Since the effect of these contributions are larger in solution B, the uncertainties in solution A are probably more correct.

Accordingly, the best values of the parameters are taken to be those in table IV, they are approximately the means of A and B with uncertainties of R, GM and $\beta$ twice as great as in A. Table IV shows also the values of quantities derived from the fundamental parameters. In estimating the uncertainties of the derived quantities, no allowance has been made for correlation, partly because the coefficients are generally negligible and partly because the uncertainties are in any case somewhat larger than given by solution A.

Table IV is calculated from the following differential corrections :

| | | | | |
|---|---|---|---|---|
| $x_1$ .............. | — 3 | p. p. m. | s. d. 2 | p. p. m. |
| $x_2$ .............. | — 2 | | 2 | |
| $x_3$ .............. | 0 | | 80 | |
| $x_4$ .............. | —15 | | 70 | |
| $x_5$ .............. | — 0.6 | | 2 | |
| $x_6$ .............. | — 0.7 | | 0.74 | |

https://doi.org/10.1017/S0074180900104802 Published online by Cambridge University Press

TABLE IV.

*Adopted values of fundamental parameters and principal derived quantities.*

| | Adopted value. | s. d. |
|---|---|---|
| R (km) | 6 378.144 | 0.013 |
| GM ($10^{14}$ m$^3$.s$^{-2}$) | 3.986026 | 0.000008 |
| $J_2$ ($10^{-6}$) | 1 082.60 | 0.08 |
| $\mu$ | 0.01229982 | 0.0000008 |
| $\mu^{-1}$ | 81.3020 | 0.0057 |
| $\beta$ ($10^{-3}$) | 4.5204 | 0.002 |
| *Derived quantities :* | | |
| $a_{☾}$ ($10^5$ km) | 3.843973 | 0.000029 |
| Radius of Moon (km) | 1 737.6 | 0.8 |
| $f$ | 0.00335276 | 0.000000012 |
| $f^{-1}$ | 298.26 | |
| $g_m$ (cm/s$^2$) | 979.7554 | 0.0044 |
| $g_e$ (cm/s$^2$) | 978.0413 | 0.0044 |
| (Mass of Earth and Moon)$^{-1}$ | 328 906.9 | – |
| $H = \dfrac{C - A}{C}$ | $3.2748 \times 10^{-3}$ | Newcomb |
| | $3.2756 \times 10^{-3}$ | Rabe |

*Gravity formula :*

$$g = 978.0413[1 + (5.30228 \times 10^{-3}) \sin^2\varphi - 6.4 \times 10^{-6} \sin^2 2\varphi].$$

*Notes to table* IV :

1° The value of reciprocal mass of the Earth and Moon follows from the relation given by D. Brouwer (paper presented to Symposium N° 21) :

$$\frac{S + M(1+\mu)}{M(1+\mu)} = \frac{F_2}{F_3} \frac{n_0^2 A^3}{n'^2 a_{☾}^3} = 0.0055800307 \frac{A^3}{a^3}.$$

2° H follows from the formula

$$P = H\left(K_1 \frac{\mu}{1+\mu} + K_2\right),$$

with

$$K_1 = 944\,154 \qquad \text{and} \qquad K_2 = 5\,310.$$

Newcomb's value for P, corrected for relativistic effects, is 54″,927 per annum, Rabe's value from the motion of Eros is 54″,938.

The values in table IV agree very well with other estimates. In particular, Kaula gives for the World Geodetic System :

$$R = 6\,378.163 \pm 0.015,$$
$$f^{-1} = 298.24 \pm 0.01,$$
$$g_e = 978.0436 \qquad \text{(Potsdam system)}$$

and he reported at the Symposium :

$$GM = 3.986028 \pm 0.000008 \quad \text{from 4 close satellites.}$$

The adopted values in table IV lead to a value of $\chi^2 = 18$, slightly less than the values for solutions A and B.

6. **Units and Standards.** — In the study of the stars and of the solar system the semi-major axis of the Earth's orbit, the astronomical unit,

https://doi.org/10.1017/S0074180900104802 Published online by Cambridge University Press

is a convenient unit of distance because planetary distances in particular can be found much more accurately in terms of it by taking ratios of mean motions than any of these distances can be determined separately in terms of the unit of length, the metre, of terrestrial science. The situation in the study of the Earth's potential field is quite different.

The ratio of the distance of the Moon to the size of the Earth cannot be found simply from a ratio of mean motions but involves the value of gravity on the surface of the Earth. In absolute value, this is measured in terms of the terrestrial units, the metre and the second; it may be objected that the second is derived from an astronomical mean motion but in practice the short time intervals involved in an absolute measurement of gravity are now obtained by comparison with an atomic frequency standard. The uncertainty of the mean value of gravity was seen to be a few parts in a million, the uncertainty in averaging free-air gravity being the dominant contribution.

The distance of the Moon and the mean radius of the Earth have uncertainties that are each also a few parts in a million. They too are measured in terms of the terrestrial units of length and time. The Moon's distance is measured by radar, that is it is a time interval multiplied by the speed of light and the time interval is derived from a terrestrial frequency standard while the speed of light has been measured in terms of a metre derived from wavelength measurements and of a second derived from a terrestrial frequency standard. Many geodetic measurements, and certainly those that are the most accurate over large networks, are now made in terms of the speed of light, using either visible light or short radio-waves, so that they are referred to the same standards as are the radar measurements. It must be noted that it is assumed that the speed of light in the space between the Earth and the Moon is the same as the speed measured in vacuum in the laboratory but there is no way of checking that assumption and it is only possible to derive astronomical quantities by accepting the assumption, that is working in terms of special relativity.

It can be said then that the distance of the Moon and the mean radius of the Earth can each be measured separately by reference to terrestrial standards — the wavelength standard of length and the atomic standard of frequency — with much the same accuracy as their ratio may be found by measurements referred to the same standards. These circumstances are most clearly expressed by giving the distance of the Moon and the mean radius of the Earth in the terrestrial unit of length, the metre.

7. **New conventional values and formulæ.** — If one is tempted to suppose that the present adjustment comes close to providing a basis for a new set of conventional geodetic constants, it is salutary to recall that the present estimated uncertainties are rather similar to

those of de Sitter and Brouwer, who gave a probable error of 2 p. p. m. for a mean value of gravity 10 p. p. m. different from the present one and a probable error of 5 p. p. m. for a value of R 16 p. p. m. different from the present one. However the present results give a rather clear guide to the principles to be followed in choosing new conventional constants.

In the first place the values of $J_2$ and $f$ derived from satellite observations seem to be more accurately known than is necessary for surface geodetic and gravity measurements and conventional values may therefore be based on the satellite values. The radius, it seems is given satisfactorily by the adjustment, and there is thus no difficulty in choosing a reference ellipsoid which will in fact be very close to Mrs. Fischer's *World Datum*. The gravity formula is not so straightforward, and difficulties arise from the choice of an absolute value for reference in physics, from the mean value of surface gravity and from the variations with latitude. It has been seen that on account of the measurements of gravity being made at various heights, the surface harmonics in gravity do not correspond exactly to the solid harmonics in the potential. Since the latter are much more accurately determined than the former, it seems reasonable to take them as the standard and to recognise that the residuals from the formula will contain a term arising from the inadequacy of the harmonic representation. The mean value of gravity should represent the observed mean surface value of gravity averaged over the whole Earth and it has been seen that this is affected by a sampling error. For purposes of physics, the mean value is of no interest but instead correct absolute values at a restricted number of stations are wanted. There may thus be a discrepancy between the absolute value of $g_m$ as derived from other astronomical and geodetic parameters and as found from the average surface value of gravity in absolute terms, this discrepancy being due to sampling errors in forming the mean. Since the surface mean value depends on actual measured absolute values at particular stations and since these values should be conserved for purposes of physics and since it seems that the sampling errors are larger than the experimental errors in absolute determinations, it seems best to take the value of $g_m$ in a gravity formula from the surface values in order to keep the correct absolute values.

**Acknowledgement.** — This paper has been prepared as part of the program of general research of the National Physical Laboratory and is published with the permission of the Director of the Laboratory.

## APPENDIX 1.

### Details of the data, equations and solutions of the constants discussed in section 5.

TABLE V.

*Observation equations.*

| Datum. | Observed-Trial (p. p. m.). | s. d. (p. p. m.). | Solution A. Weight. | Solution A. Calculated-Trial (p. p. m.). | Solution A. Observed-Calculated (p. p. m.). | Solution B. Weight. | Solution B. Calculated-Trial (p. p. m.). | Solution B. Observed-Calculated (p. p. m.). |
|---|---|---|---|---|---|---|---|---|
| $J_2$ King-Hele *et al.* | +241 | 100 | 10 | −19 | +260 | 3 | +28 | + 220 |
| $J_2$ Kozai | −111 | 60 | 25 | | − 92 | 5 | | − 132 |
| Moon's distance | + 0.5 | 1 | $10^5$ | − 0.2 | + 0.7 | 300 | + 0.4 | + 0.1 |
| Lunar inequality: | | | | | | | | |
| Position | −170 | 310 | 1 | − 6.5 | −164 | 1 | −27 | − 143 |
| Velocity | − 9 | 40 | 60 | | − 2.5 | 8 | | + 18 |
| Mean gravity | + 7.8 | 2 | $2.25\times10^4$ | + 2.0 | + 5.8 | 150 | + 7.0 | + 0.8 |
| Geodetic survey | 0 | 3.5 | $10^4$ | − 2.5 | + 2.5 | 100 | − 3.6 | + 3.6 |
| Moon's angular diameter | 0 | 3 | $10^4$ | − 0.75 | + 0.75 | 100 | − 0.5 | + 0.5 |
| Occultations | + 3.2 | 15 | 400 | − 1.5 | + 4.7 | 20 | − 3.7 | + 6.9 |
| GM | − 4.5 | 1.2 | $10^5$ | − 3.0 | − 1.5 | 300 | − 0.4 | − 4.1 |
| Variance of observation of unit weight | | | - | - | 227 220 | - | - | 52 555 |
| $\chi^2$ (5 d. f.) | | | - | - | 20.6 | - | - | 20.6 |

## Table VI.

*Normal equations, Solutions and variances.*

### Solution A.

$$\begin{bmatrix} +192\,983 & -45\,708 & +230 & -6.6 & +1\,200 \\ -45\,708 & +132\,626 & -120 & +161 & -333\,333 \\ +230 & -120 & +35.3 & - & - \\ -6.6 & +161 & - & +61.8 & -410 \\ +1\,200 & -33\,333 & - & -410 & 200\,000 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{array}{r} \text{p. p. m.} \\ -350\,555 \\ -257\,831 \\ -832 \\ -508 \\ -50\,000 \end{array}$$

| | | | Variance. | s. d. |
|---|---|---|---|---|
| R........ | $x_1 = -$ | 2.5 p. p. m. | $1.96 \times 10^{-12}$ | 1.4 p. p. m. |
| GM...... | $x_2$ | $-$ 3.0 | 1.18 | 1.1 |
| $J_2$........ | $x_3$ | $-$18.7 | 6 498 | 81 |
| $\mu$........ | $x_4$ | $-$ 5.5 | 3 745 | 61 |
| $\beta$........ | $x_5$ | $-$ 0.75 | 1.14 | 1.1 |

*Combinations with independent variance.*

| | Variance. |
|---|---|
| $x_2$ ............................................ | $1.96 \times 10^{-12}$ |
| $x_1 - 0.23\, x_2$ ...................................... | 1.18 |
| $x_3 + 6.5\, x_1 - 3.4\, x_2$ .............................. | 6 440 |
| $x_4 - 0.067\, x_1 + 1.5\, x_2$ ............................ | 3 741 |
| $x_5 + 0.006\, x_1 - 0.167\, x_2 - 0.002\, x_4$ ................. | 1.14 |

Derived distance of Moon :

$a_{☾}$: $x_6 = -1.0$ p. p. m., s. d. 0.44 p. p. m.

### Solution B.

$$\begin{bmatrix} +754.54 & -332 & +17.15 & -0.09 & +3.6 \\ -332 & +501 & -12 & +3.4 & -100 \\ +17.15 & -12 & 8 & - & - \\ -0.09 & +3.4 & - & +8.8 & -1.2 \\ +3.6 & -100 & - & -1.2 & 400 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{array}{r} \text{p. p. m.} \\ -194.2 \\ -536 \\ +164 \\ -240 \\ -150 \end{array}$$

| | | | Variance. | s. d. |
|---|---|---|---|---|
| R................. | $x_1 = -$ | 3.7 p. p. m. | $105 \times 10^{-12}$ | 10 p. p. m. |
| GM............... | $x_2$ | $-$ 0.4 | 160.7 | 13 |
| $J_2$ ................. | $x_3$ | +27.8 | 7 066 | 84 |
| $\mu$................. | $x_4$ | $-$27.2 | 5 990 | 77 |
| $\beta$................. | $x_5$ | $-$ 0.5 | 141 | 12 |

*Combinations with independent variance.*

| | Variance. |
|---|---|
| $x_2$ ....................................... | $160.7 \times 10^{-12}$ |
| $x_1 - 0.43\, x_2$ ............................... | 73.2 |
| $x_3 + 2.14\, x_1 - 1.5\, x_2$ ......................... | 6 570 |
| $x_4 - 0.01\, x_1 + 0.35\, x_2$ ........................ | 5 970 |
| $x_5 + 0.009\, x_1 - 0.25\, x_2 - 0.003\, x_4$ ........... | 131.4 |

Derived distance of Moon :

$a_{☾}$: $x_6 = -0.1$ p. p. m., s. d. 4.2 p. p. m.

TABLE VII.

*Correlation coefficients.*

SOLUTION A.

| | R. | GM. | $J_2$. | $\mu$. | $\beta$. | $a_{☾}$. |
|---|---|---|---|---|---|---|
| R | 1 | 0.18 | 0.07 | —0.006 | 0.21 | +0.32 |
| GM | | 1 | —0.03 | —0.04 | —0.15 | +1 |
| $J_2$ | | | 1 | +0.001 | +0.012 | —0.03 |
| $\mu$ | | | | 1 | +0.002 | +0.5 |
| $\beta$ | | | | | 1 | – |
| $a_{☾}$ | | | | | | 1 |

SOLUTION B.

| | R. | GM. | $J_2$. | $\mu$. | $\beta$. | $a_{☾}$. |
|---|---|---|---|---|---|---|
| R | 1 | 0.52 | —0.13 | —0.03 | —0.13 | 0.24 |
| GM | | 1 | 0.09 | —0.056 | —0.26 | 1 |
| $J_2$ | | | 1 | —0.005 | —0.024 | +0.09 |
| $\mu$ | | | | 1 | +0.015 | +0.07 |
| $\beta$ | | | | | 1 | 0.26 |
| $a_{☾}$ | | | | | | 1 |

# APPENDIX 2.

## The form of an ellipsoid of revolution and the gravity field outside it.

The theory of the gravity field outside an ellipsoid of revolution that is an equipotential of the matter inside it is available to quantities of the order of $(J_2)^3$ (Cook [28], Lambert [29]). It is usually given in terms of the equatorial radius vector and the equatorial value of gravity at the surface but formulæ in terms of the mean radius and mean gravity are also available (Cook [30]) and it is those that are required in this paper.

The equation for the meridian in polar co-ordinates (radius vector $r$, co-latitude $\theta$) about the centre, may be written

$$r^2 = \frac{a^2(1-e^2)}{1-\frac{2}{3}e^2(1-P_2(\cos\theta))},$$

where

$$e^2 = \frac{a^2-b^2}{a^2}.$$

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In terms of the flattening $f = \dfrac{a-b}{a}$,

$$\frac{r}{a} = 1 - \frac{1}{3}f - \frac{1}{5}f^2 - \frac{13}{105}f^3 - \left(\frac{2}{3}f + \frac{1}{7}f^2 - \frac{1}{21}f^3\right) P_2(\cos\theta)$$
$$+ \left(\frac{12}{35}f^2 - \frac{72}{77}f^3\right) P_4(\cos\theta) - \frac{40}{231}f^3 P_6(\cos\theta) + \ldots.$$

Hence the mean radius $R_m$, is

$$a\left(1 - \frac{1}{3}f - \frac{1}{5}f^2 - \frac{13}{105}f^3 - \ldots\right).$$

and

$$\frac{R}{R_m} = 1 - \left(\frac{2}{3}f + \frac{23}{63}f^2\right) P_2(\cos\theta) + \frac{12}{35}f^2 P_4(\cos\theta) + \ldots.$$

The relation between $f$ and $J_2$ defined as $\dfrac{C-A}{MR_m^2}$ and $m = \dfrac{\omega^2 R_m^3}{GM}$ is

$$f = \frac{3}{2}J_2 + \frac{1}{2}m - \frac{3}{8}J_2^2 + \frac{3}{56}m^2 + \frac{1}{28}mJ_2,$$

so that

$$\frac{R}{R_m} = 1 - \left(J_2 + \frac{1}{3}m + \frac{4}{7}J_2^2 + \frac{13}{84}mJ_2 - \frac{25}{252}m^2 - \ldots\right) P_2(\cos\theta) + \ldots.$$

The radius of curvature in the meridian is

$$-\frac{\left(r^2 + \dfrac{dr}{d\theta}\right)^{\frac{1}{2}}}{\dfrac{d\varphi}{d\theta}} \qquad (\varphi \text{ is geographic latitude})$$

that is

$$R_m\left\{1 + \frac{8}{3}f - \frac{10}{3}fP_2(\cos\theta) + \ldots\right\}.$$

The mean value is

$$R_m\left\{1 + 4J_2 + \frac{4}{3}m + \ldots\right\}.$$

The radius of curvature in the prime vertical is $\dfrac{r\sin\theta}{\cos\varphi}$ that is

$$R_m\left\{1 + \frac{4}{3}f + \frac{4}{3}fP_2(\cos\theta) + \ldots\right\}$$

the mean value being

$$R_m\left(1 + 2J_2 + \frac{2}{3}m + \ldots\right).$$

In terms of geographical latitude, the sea-level value of gravity is

$$\frac{g}{g_e} = 1 + \left(\frac{5}{2}m - f + \frac{15}{4}m^2 - \frac{17}{14}fm\right)\sin^2\varphi + \frac{1}{8}f(f - 5m)\sin^2 2\varphi.$$

The mean value is

$$\frac{g_m}{g_e} = 1 - \frac{1}{3}f + \frac{5}{6}m + \frac{1}{9}f^2 + \frac{5}{4}m^2 - \frac{121}{126}mf.$$

Since

$$g_e = \frac{GM}{R_m^2}\left[1 + \frac{1}{3}f - \frac{3}{2}m + \frac{2}{45}f^2 - \frac{13}{14}fm\right],$$

$$g_m = \frac{GM}{R_m^2}\left[1 - \frac{2}{3}m + \frac{2}{45}f^2 - \frac{10}{9}fm\right].$$

## APPENDIX 3.

### The occultation observations of O'Keefe and Anderson.

The geometry of occultation observations is shown in figures 2, 3 and 4. They are discussed in terms of projections onto the plane that

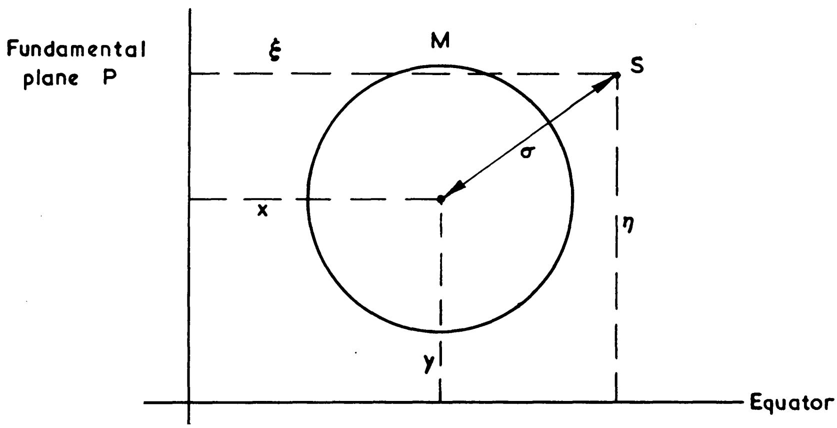


Fig. 2. — Occultations : Coordinates in the fundamental plane.

is normal to the direction of the star and that passes through the centre or the Earth. This plane, P, is known as the fundamental plane. Rectangular axes are taken in P with the origin at the centre of the Earth and the $x$-axis the intersection of the plane P with the plane of the equator. M is the projection of the Moon on P, the co-ordinates of the centre of M being $x$ and $y$. S is the projection of the station from which the occultation is observed and its co-ordinates are $\xi$ and $\eta$.

Let $\sigma$ be the distance of S from the centre of M. Then

$$\sigma^2 = (\xi - x)^2 + (\eta - y)^2.$$

Let $k$ be the radius of M. Knowing the time of the occultation, $\sigma$ may be calculated from the tabulated positions of the Moon and the

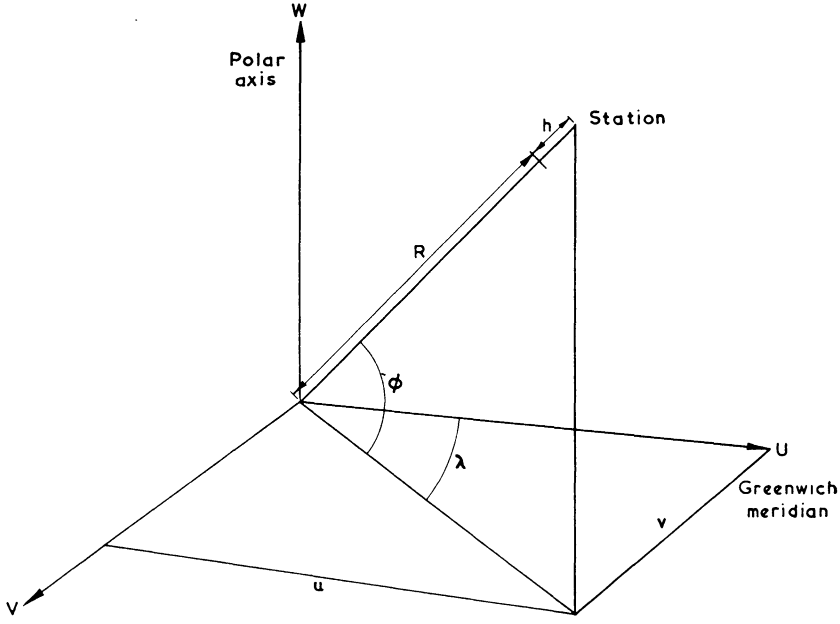


Fig. 3. — Occultations : station coordinates.

star and the distance of the Moon and the co-ordinates of the station and a difference $s$ may be determined :

$$s = \sigma - k.$$

$s$ being found from measured times, is not obtained in kilometers but as a fraction of the distance of the Moon, that is the observed quantity is

$$\frac{s}{d} = \frac{\sigma}{d} - \beta,$$

where $d$ is the distance of the Moon. Let $\alpha_{☾}$ and $\delta_{☾}$ be her right ascension and declination and let $\alpha_*$ and $\delta_*$ be those of the star. Then

$$x = d \sin(\alpha_{☾} - \alpha_*) \cos \delta,$$
$$y = d \{ \sin \delta_{☾} \cos \delta_* - \cos \delta_{☾} \sin \delta_* \cos (\alpha_{☾} - \alpha_*) \}.$$

https://doi.org/10.1017/S0074180900104802 Published online by Cambridge University Press

Let $R_s$ be the radius vector at the station for the adopted ellipsoid of reference, $h$ the height of the station above this ellipsoid and $\varphi$ and $\lambda$ the geocentric latitude and longitude of the station. $h$, $\varphi$ and $\lambda$ are found from calculations of the form of the geoid from gravity anomalies.

Take rectangular co-ordinates, $u$, $v$, w, with the W-axis along the Earth's polar axis, U in the equator in the direction of the meridian of Greenwich and V in the perpendicular direction. Then

$$u = (R_s + h) \cos\varphi \cos\lambda,$$
$$v = (R_s + h) \cos\varphi \sin\lambda,$$
$$w = (R_s + h) \sin\varphi.$$

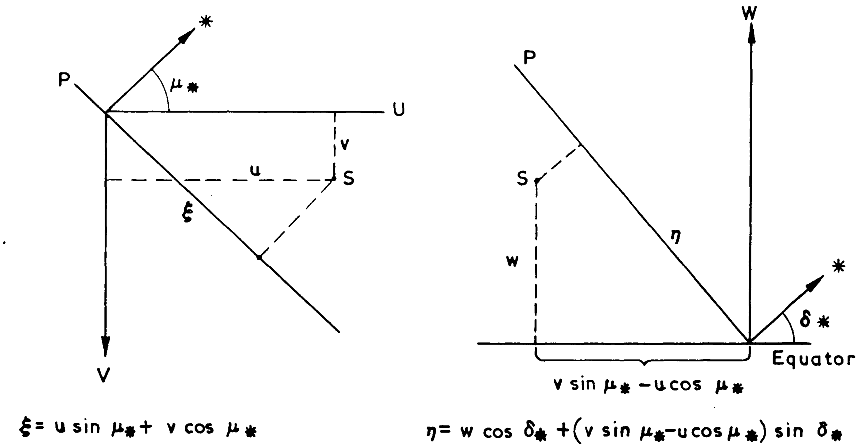


Fig. 4. — Occultations : projection of station on the fundamental plane.

Projecting onto the fundamental plane,

$$\xi = u \sin\mu_* + v \cos\mu_*,$$
$$\eta = w \cos\delta_* + (v \sin\mu_* - u \cos\mu_*) \sin\delta_*,$$

where $\mu_*$ is the Greenwich hour angle of the star.

Thus finally,

$$\xi = (R_s + h) \cos\varphi \sin(\mu_* + \lambda)],$$
$$\eta = (R_s + h) [\sin\varphi \cos\delta_* - \cos\varphi \sin\delta_* \cos(\mu_* + \lambda)].$$

The differential required to compare with the observations is

$$\delta\left(\frac{\sigma}{d}\right).$$

Now

$$\frac{\sigma}{d}\delta\left(\frac{\sigma}{d}\right) = \frac{\xi - x}{d}\left\{\frac{\delta\xi}{d} - \frac{\delta x}{d} - \frac{\xi - x}{d}\frac{\delta d}{d}\right\} + \frac{\eta - y}{d}\left\{\frac{\delta\eta}{d} - \frac{\delta y}{d} - \frac{\eta - y}{d}\frac{\delta d}{d}\right\}.$$

It is permissible to ignore the errors in $\varphi$ and $\lambda$, $\alpha_*$ and $\delta_*$ so that

$$\frac{\delta\xi}{d} = \frac{\xi}{d}\,\frac{\delta(\mathrm{R}_s+h)}{\mathrm{R}_s+h}, \qquad \frac{\delta\eta}{d} = \frac{\eta}{d}\,\frac{\delta(\mathrm{R}_s+h)}{\mathrm{R}_s+h}$$

and again supposing that the error in $h$ may be ignored,

$$\frac{\delta\xi}{d} = \frac{\xi}{d}\,\frac{\delta\mathrm{R}}{\mathrm{R}}, \qquad \frac{\delta\eta}{d} = \frac{\eta}{d}\,\frac{\delta\mathrm{R}}{\mathrm{R}}.$$

In $x$ and $y$ errors in the position of the Moon must be allowed for (it is evident that errors in the positions of the star and the Moon cannot be separated). Thus

$$\begin{aligned}\frac{\delta x}{d} &= \frac{x}{d}\,\frac{\delta d}{d} - d\sin(\alpha_{☾}-\alpha_*)\sin\delta_{☾}\Delta\delta_{☾}\\ &\quad + d\cos(\alpha_{☾}-\alpha_*)\cos\delta_{☾}\,\Delta\alpha_{☾},\\ \frac{\delta y}{d} &= \frac{y}{d}\,\frac{\delta d}{d} + d[\cos\delta_{☾}\cos\delta_* + \sin\delta_{☾}\sin\delta_*\cos(\alpha_{☾}-\alpha_*)]\,\Delta\delta_{☾}\\ &\quad + d\cos\delta_{☾}\sin\delta_*\sin(\alpha_{☾}-\alpha_*)\,\Delta\alpha_{☾}.\end{aligned}$$

Since $d$ may be calculated very accurately as a function of time,

$$\frac{\delta d}{d} = \frac{\delta a_{☾}}{a_{☾}}$$

Hence

$$\delta\left(\frac{\sigma}{a}\right) = \frac{\xi-x}{\sigma}\,\frac{\xi}{a}\left(\frac{\delta\mathrm{R}}{\mathrm{R}} - \frac{\delta a_{☾}}{a_{☾}}\right) + \frac{\eta-y}{\sigma}\,\frac{\eta}{a}\left(\frac{\delta\mathrm{R}}{\mathrm{R}} - \frac{\delta a_{☾}}{a_{☾}}\right)$$

$+$ terms proportional to $\Delta\delta_{☾}$ and $\Delta\alpha_{☾}$ which are almost constant for a given occultation but that vary from one to another because of the variations in $\Delta\alpha_{☾}$ and $\Delta\delta_{☾}$.

The observation equation may therefore be written as

$$\left(\frac{\xi-x}{\sigma}\,\frac{\xi}{a_{☾}} + \frac{\eta-y}{\sigma}\,\frac{\eta}{a_{☾}}\right)(x_1-x_6) - \beta + c_i = \frac{s}{a_{☾}}.$$

It is evident that $x_1$ and $x_6$ cannot be found separately. Likewise, $\beta$ cannot be separated from the constants $c_i$ for the separate occultations. Accordingly a least squares adjustment of occultations will yield an equation for $(x_1-x_6)$.

The results can be put in other forms if additional information is imported. O'Keefe and Anderson used the dynamical relation between the motion of the Moon and the value of gravity at the surface of the Earth to determine in effect the scale of the Earth-Moon system while Mrs. Fischer has supposed that the radius of the Earth is much better known than the distance of the Moon and recomputed the data to give the Moon's distance.

The data for the four occultations observed by O'Keefe and Anderson are given in table VIII. The value of $s_1$ is taken from Mrs. Fischer's [5] revision of the material and is equal to her quantity $\Delta\sigma'$ which is the result of applying corrections for the known form of the geoid to the original data and of using the parameters of the ellipsoid of the *Astro-geodetic World Datum*. The mean radius of the World datum 6 371.04 km is slightly different from the trial value 6 371.06 km used in this Appendix and the value of $a_☾$ used by O'Keefe, 384 403.7 km, is also different from the present trial value; a correction

$$+ \Lambda \left( \frac{0.02}{6\,871} + \frac{3.7}{334\,400} \right)$$

has therefore to be made to $s_1$ giving the quantity $s_2$. The observation equations are then

$$\Lambda\,(x_1 - x_6) + c_i' = s_2,$$

where

$$\Lambda = \frac{1}{a_☾ \sigma} \{ \xi(\xi - x) + \eta(\eta - y) \}.$$

The solutions to the normal equations are

$$\begin{aligned} x_1 - x_6 &= 6.29 \times 10^{-6}, \\ z_1 &= 2\,058.7 \text{ km} \\ z_2 &= 2\,820.3 \\ z_3 &= 1\,914.1 \\ z_4 &= 1\,687.1 \end{aligned}$$

and the residuals are shown in the last column of table VIII. The standard deviation of a single observation is 14.3 m (on 4 degrees of freedom) and the standard deviation of $(x_1 - x_6)$ is $15.3 \times 10^{-6}$.

The corresponding result for the trial values used in the body of the paper is

$$x_1 - x_6 = 3.2 \times 10^{-6}.$$

### Table VIII.

*Occultations, data of O'Keefe and Anderson* (1952).

| | Latitude. | Star (*). | $\xi$ (km). | $\eta$ (km). | $x$ (km). | $y$ (km). | A (km). | $s_1$ (km). | $s_2$ (km). | Residuals (m). |
|---|---|---|---|---|---|---|---|---|---|---|
| Antonito, Col. | 36°57 | 2 540 | 1 077 | 5 752 | — 537 | 5 099 | 5 494 | 2 030.7 | 2 073.9 | — 4.7 |
| Clay Center, Kans. | 39 23 | 2 540 | 2 148 | 5 672 | + 535 | 5 020 | 7 167 | 2 032.9 | 2 089.2 | + 4.6 |
| El Paso, Tex. | 31 41 | 501 | 4 815 | 2 105 | 3 305 | 1 239 | 9 094 | 2 781.9 | 2 853.3 | + 0.1 |
| Sequin, Tex. | 29 37 | 501 | 5 304 | 2 263 | 3 794 | 1 397 | 9 969 | 2 777.7 | 2 856.0 | — 0.3 |
| Lubbock; Tex. | 33 09 | 348 | 5 221 | 2 994 | 3 490 | 2 814 | 9 034 | 1 862.7 | 1 933.7 | —13.1 |
| Alvarado Tex. | 32 20 | 348 | 5 350 | 3 051 | 3 620 | 2 870 | 9 273 | 1 887.9 | 1 960.7 | +13.1 |
| Hughson, Calif. | 37 34 | 996 | 4 488 | 2 300 | 2 800 | 1 877 | 8 549 | 1 664.7 | 1 731.9 | +13.8 |
| Desert Center, Calif. | 34 04 | 996 | 4 975 | 2 278 | 3 287 | 1 854 | 9 364 | 1 631.9 | 1 705.5 | —15.5 |
| Arivoca, Ariz. | 31 09 | 996 | 5 251 | 2 265 | 3 563 | 1 842 | 9 282 | 1 647.0 | 1 724.2 | + 1.6 |

(*) Number in Zodiacal Catalogue.

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