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Coefficient estimates for starlike functions

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Let $S_k^*(\alpha, \beta)$ denote the class of functions

$$f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$$

analytic in the unit disc $\Delta \equiv \{z : |z| < 1\}$ and satisfying

$$\left|\left(z\frac{f'(z)}{f(z)} - 1\right)\Big/\left\{2\beta\left(z\frac{f'(z)}{f(z)} - \alpha\right) - \left(z\frac{f'(z)}{f(z)} - 1\right)\right\}\right| < 1$$

for some α, β $(0 \le \alpha < 1,\ 0 < \beta \le 1)$ and for all $z \in \Delta$. In the present paper, sharp coefficient estimates for functions in $S_k^*(\alpha, \beta)$ have been obtained. The results thus obtained not only generalize the corresponding results of Thomas H. MacGregor (*Michigan Math. J.* 10 (1963), 277-281), A.V. Boyd (*Proc. Amer. Math. Soc.* 17 (1966), 1016-1018) and others, but also give rise to analogous results for various other subclasses of starlike functions.

1. Introduction

Let $f(z)$ be regular in the unit disc $\Delta \equiv \{z : |z| < 1\}$ and normalized by the conditions $f(0) = 0$, $f'(0) = 1$. The power series representation for such a function is

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n . \tag{1.1}$$

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If $f(z)$ satisfies the condition

$$\text{Re}\left\{z\,\frac{f'(z)}{f(z)}\right\} > 0 \tag{1.2}$$

for all $z \in \Delta$, then it is well known [9, p. 221] that (1.2) is both necessary and sufficient for f to be univalent and starlike with respect to the origin in Δ. For starlike functions the Bieberbach conjecture $|a_n| \le n$ holds for all n, and equality occurs only for the functions $f(z) = z/(1+\varepsilon z)^2$, where $|\varepsilon| = 1$. MacGregor [6] obtained upper bounds for the moduli of the coefficients of a starlike function whose power series representation in Δ is of the form

$$f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n \tag{1.3}$$

Boyd [1] extended MacGregor's result to the class S_α^* of starlike functions of order α $(0 \le \alpha < 1)$; that is, $f \in S_\alpha^*$ if it is univalent and $\text{Re}\left(zf'(z)/f(z)\right) > \alpha$ for $z \in \Delta$. Various other subclasses of starlike functions have been considered and corresponding coefficient estimates obtained for them by Singh [13, 14], Padmanabhan [10], Eenigenburg [2], McCarty [7], Mogra [8], and others.

Recently, the authors [4] introduced the class of starlike functions of order α $(0 \le \alpha < 1)$ and type β $(0 < \beta \le 1)$ and made a preliminary study of its properties. Later, Gupta and Jain [3] obtained certain results for functions of this class when all a_n are negative. These results were the analogues of the corresponding results obtained by Silverman [12] for starlike functions of order α. However, to broaden the scope of applicability of the results obtained, the authors [5] have very recently modified the definition of starlike functions of order α $(0 \le \alpha < 1)$ and type β $(0 < \beta \le 1)$, which is as follows.

DEFINITION. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc Δ. Then f is said to be starlike of order α and type β, if it satisfies the condition

$$\left|\left(z\,\frac{f'(z)}{f(z)} - 1\right) \Big/ \left\{2\beta\left(z\,\frac{f'(z)}{f(z)} - \alpha\right) - \left(z\,\frac{f'(z)}{f(z)} - 1\right)\right\}\right| < 1 \tag{1.4}$$

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for some α, β $(0 \le \alpha < 1, 0 < \beta \le 1)$ and $z \in \Delta$.

The class of starlike functions of order α $(0 \le \alpha < 1)$ and type β $(0 < \beta \le 1)$ we shall denote by $S^*(\alpha, \beta)$.

It can be easily seen that $S^*(\alpha, \beta)$ includes the subclasses of starlike functions for different values of the parameters α and β . Hence a study of its various properties leads to a unified study of these subclasses. In the present note, we determine sharp coefficient estimates for the class $S_k^*(\alpha, \beta)$ of starlike functions of order α and type β whose power series representation is of the form (1.3). The results thus obtained not only generalize the corresponding results of MacGregor [*6*], Boyd [*1*], and others, but also give rise to analogous results for the functions of the form (1.3) belonging to the classes introduced and studied by Singh [*13*, *14*], Padmanabhan [*10*], Wright [*15*], McCarty [*7*], and Eenigenburg [*2*].

2. Some coefficient estimates

THEOREM. *Let* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ *be in* $S_k^*(\alpha, \beta)$.

(a) If $\beta(1-\alpha) > k(1-\beta)$ *, let* $M = \left[\frac{\beta(1-\alpha)}{k(1-\beta)}\right]$. *Then*

$$|a_n| \le \frac{k}{(n-1)(m-1)!} \prod_{\mu=0}^{m-1} \left|(2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}\right| \tag{2.1}$$

for $mk + 1 \le n \le (m+1)k$ *,* $m = 1, 2, \ldots, M+1$ *and*

$$|a_n| \le \frac{k}{(n-1)(M+1)!} \prod_{\mu=0}^{M+1} \left|(2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}\right| , \quad n > (M+2)k . \tag{2.2}$$

(b) If $\beta(1-\alpha) \le k(1-\beta)$ *, then*

$$|a_n| \le \frac{2\beta(1-\alpha)}{n-1} \quad \text{for} \quad n \ge k+1 . \tag{2.3}$$

The estimates in (2.1) *are sharp for* $n = mk + 1$ *,* $m = 1, 2, \ldots,$ *while the estimates in* (2.3) *are sharp for all* n .

Proof. We employ the technique used by MacGregor [*6*]. Thus, let $f \in S_k^*(\alpha, \beta)$; then we have

(2.4) $$h(z) = \frac{zf'(z)-f(z)}{2\beta\left(zf'(z)-\alpha f(z)\right)-\left(zf'(z)-f(z)\right)}$$

where h is regular in Δ and satisfies $|h(z)| < 1$ in Δ. Also the power series for $h(z)$ begins with $c_k z^k + c_{k+1} z^{k+1} + \ldots$. Equating coefficients of the same powers on both sides of the equation

$$zf'(z) - f(z) = h(z)\left\{2\beta\left(zf'(z)-\alpha f(z)\right)-\left(zf'(z)-f(z)\right)\right\}$$

or

(2.5) $$\sum_{n=k+1}^{\infty} (n-1)a_n z^n = \left\{c_k z^k + c_{k+1} z^{k+1} + \ldots\right\}\left\{2\beta(1-\alpha)z + \sum_{n=k+1}^{\infty} \left((2\beta-1)n+1-2\alpha\beta\right)a_n z^n\right\},$$

we obtain

(2.6) $$(n-1)a_n = 2\beta(1-\alpha)c_{n-1} \quad \text{for} \quad n = k+1, k+2, \ldots, 2k .$$

Since $|h(z)| < 1$, it follows that $\sum_{n=k}^{\infty} |c_n|^2 \le 1$ and so

(2.7) $$\sum_{n=k}^{2k-1} |c_n|^2 \le 1 .$$

From (2.6) and (2.7), we find that

(2.8) $$\sum_{n=k+1}^{2k} (n-1)^2 |a_n|^2 \le 4\beta^2(1-\alpha)^2 .$$

(2.5) can be rewritten in the form

(2.9) $$\sum_{n=k+1}^{p} (n-1)a_n z^n + \sum_{n=p+1}^{\infty} d_n z^n = h(z)\left\{2\beta(1-\alpha)z + \sum_{n=k+1}^{p-k} \left((2\beta-1)n+1-2\alpha\beta\right)a_n z^n\right\} .$$

Since (2.9) has the form $F(z) = h(z)G(z)$, where $|h(z)| < 1$, it follows that

(2.10) $$\frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\phi})|^2 d\phi \le \frac{1}{2\pi}\int_0^{2\pi} |G(re^{i\phi})|^2 d\phi$$

for each r $(0 < r < 1)$. Expressing (2.10) in terms of the coefficients in (2.9), we get

$$(2.11) \quad \sum_{n=k+1}^{p} (n-1)^2|a_n|^2r^{2n} + \sum_{n=p+1}^{\infty} |d_n|^2r^{2n} \le 4\beta^2(1-\alpha)^2r^2 + \sum_{n=k+1}^{p-k} \big((2\beta-1)n+1-2\alpha\beta\big)^2|a_n|^2r^{2n} .$$

In particular, (2.11) implies

$$(2.12) \quad \sum_{n=k+1}^{p} (n-1)^2|a_n|^2r^{2n} \le 4\beta^2(1-\alpha)^2r^2 + \sum_{n=k+1}^{p-k} \big((2\beta-1)n+1-2\alpha\beta\big)^2|a_n|^2r^{2n} .$$

Letting $r \to 1$ in (2.12), we conclude that

$$(2.13) \quad \sum_{n=k+1}^{p} (n-1)^2|a_n|^2 \le 4\beta^2(1-\alpha)^2 + \sum_{n=k+1}^{p-k} \big((2\beta-1)n+1-2\alpha\beta\big)^2|a_n|^2 .$$

This inequality is equivalent to

$$(2.14) \quad \sum_{n=p-k+1}^{p} (n-1)^2|a_n|^2 \le 4\beta^2(1-\alpha)^2 + \sum_{n=k+1}^{p-k} \left\{\big((2\beta-1)n+1-2\alpha\beta\big)^2-(n-1)^2\right\}|a_n|^2 .$$

Now two cases arise.

(a) If $\beta(1-\alpha) > k(1-\beta)$, then by an inductive argument we will establish the inequalities

$$(2.15a) \quad \sum_{n=mk+1}^{(m+1)k} (n-1)^2|a_n|^2 \le \left\{\frac{k}{(m-1)!}\prod_{\mu=0}^{m-1}\left[(2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}\right]\right\}^2 ,$$

$$(2.15b) \quad \sum_{n=mk+1}^{(m+1)k} \left\{\big((2\beta-1)n+1-2\alpha\beta\big)^2-(n-1)^2\right\}|a_n|^2 \le \left\{\frac{1}{m!}\prod_{\mu=0}^{m-1}\left[(2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}\right]\right\}^2\left\{\big((2\beta-1)mk+2\beta(1-\alpha)\big)^2-m^2k^2\right\} ,$$

for $m = 1, 2, \ldots, M+1$; $M = \left[\frac{\beta(1-\alpha)}{k(1-\beta)}\right]$ where $[p]$ denotes the greatest

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integer not greater than p .

For $m = 1$, (2.15a) gives

$$\sum_{n=k+1}^{2k} (n-1)^2|a_n|^2 \le 4\beta^2(1-\alpha)^2$$

which is the same as (2.8). Thus (2.15a) is valid for $m = 1$. We can prove (2.15b) for $m = 1$ by using (2.8) as follows:

$$\sum_{n=k+1}^{2k} \{((2\beta-1)n+1-2\alpha\beta)^2-(n-1)^2\}|a_n|^2$$

$$\le \frac{\{(2\beta-1)k+2\beta(1-\alpha)\}^2-k^2}{k^2} \sum_{n=k+1}^{2k} (n-1)^2|a_n|^2$$

$$\le \frac{4\beta^2(1-\alpha)^2}{k^2} \{((2\beta-1)k+2\beta(1-\alpha))^2-k^2\} .$$

Now suppose that (2.15a) and (2.15b) hold for $m = 1, 2, \ldots, q-1$. Using (2.14) with $p = (q+1)k$ and the inductive hypothesis concerning (2.15a), we obtain the inequalities

$$\sum_{n=qk+1}^{(q+1)k} (n-1)^2|a_n|^2$$

$$\le 4\beta^2(1-\alpha)^2 + \sum_{n=k+1}^{qk} \{((2\beta-1)n+1-2\alpha\beta)^2-(n-1)^2\}|a_n|^2$$

$$= 4\beta^2(1-\alpha)^2 + \sum_{m=1}^{q-1} \sum_{n=mk+1}^{(m+1)k} \{((2\beta-1)n+1-2\alpha\beta)^2-(n-1)^2\}|a_n|^2$$

$$\le 4\beta^2(1-\alpha)^2 + \sum_{m=1}^{q-1} \left\{\frac{1}{m!} \prod_{\mu=0}^{m-1} \left[(2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}\right]\right\}^2 \{((2\beta-1)mk+2\beta(1-\alpha))^2-m^2k^2\}$$

$$= \left\{\frac{k}{(q-1)!} \prod_{\mu=0}^{q-1} \left[(2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}\right]\right\}^2 .$$

The last equality can be easily obtained by an inductive argument on q . This last sequence of inequalities implies (2.15a) where $m = q$.

Continuing our argument, we use (2.15a) with $m = q$ to deduce (2.15b) for $m = q$ as follows:

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$$\sum_{n=qk+1}^{(q+1)k} \left\{\left((2\beta-1)n+1-2\alpha\beta\right)^2-(n-1)^2\right\}|a_n|^2$$

$$\leq \frac{\left((2\beta-1)qk+2\beta(1-\alpha)\right)^2-q^2k^2}{q^2k^2} \sum_{n=qk+1}^{(q+1)k} (n-1)^2|a_n|^2$$

$$\leq \frac{\left((2\beta-1)qk+2\beta(1-\alpha)\right)^2-q^2k^2}{q^2k^2} \left\{\frac{k}{(q-1)!} \prod_{\mu=0}^{q-1} \left[(2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}\right]\right\}^2$$

$$= \left\{\frac{1}{q!} \prod_{\mu=0}^{q-1} \left[(2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}\right]\right\}^2 \left\{\left((2\beta-1)qk+2\beta(1-\alpha)\right)^2-q^2k^2\right\} .$$

This completes the proof of (2.15a) and (2.15b). Now (2.1) follows from (2.15a).

To prove (2.2), suppose $n > (M+2)k$. Putting $p = (q+1)k$ in (2.14), we have

$$\sum_{n=qk+1}^{(q+1)k} (n-1)^2|a_n|^2 \leq 4\beta^2(1-\alpha)^2 + \sum_{n=k+1}^{qk} \left\{\left((2\beta-1)n+1-2\alpha\beta\right)^2-(n-1)^2\right\}|a_n|^2 .$$

Hence, for $n > (M+2)k$, we have

(2.16) $(n-1)^2|a_n|^2$

$$\leq 4\beta^2(1-\alpha)^2 + \sum_{n=k+1}^{qk} \left\{\left((2\beta-1)n+1-2\alpha\beta\right)^2-(n-1)^2\right\}|a_n|^2$$

$$= 4\beta^2(1-\alpha)^2 + \sum_{n=k+1}^{(M+2)k} \left\{\left((2\beta-1)n+1-2\alpha\beta\right)^2-(n-1)^2\right\}|a_n|^2$$

$$+ \sum_{n=(M+2)k+1}^{qk} \left\{\left((2\beta-1)n+1-2\alpha\beta\right)^2-(n-1)^2\right\}|a_n|^2$$

$$= 4\beta^2(1-\alpha)^2 + \sum_{m=1}^{M+1} \sum_{n=mk+1}^{(m+1)k} \left\{\left((2\beta-1)n+1-2\alpha\beta\right)^2-(n-1)^2\right\}|a_n|^2$$

$$+ \sum_{m=M+2}^{q-1} \sum_{n=mk+1}^{(m+1)k} \left\{\left((2\beta-1)n+1-2\alpha\beta\right)^2-(n-1)^2\right\}|a_n|^2$$

$$\leq 4\beta^2(1-\alpha)^2 + \sum_{m=1}^{M+1} \sum_{n=mk+1}^{(m+1)k} \left\{\left((2\beta-1)n+1-2\alpha\beta\right)^2-(n-1)^2\right\}|a_n|^2 .$$

Using (2.15a) in (2.16), we obtain

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$$(n-1)^2|a_n|^2 \le \left\{\frac{k}{(M+1)!}\prod_{\mu=0}^{M+1}\left[(2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}\right]\right\}^2 ;$$

that is,

$$|a_n| \le \frac{k}{(n-1)(M+1)!}\prod_{\mu=0}^{M+1}\left[(2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}\right] , \quad n > (M+2)k .$$

This proves (2.2).

(b) If $\beta(1-\alpha) \le k(1-\beta)$, then (2.13) gives

$$\sum_{n=k+1}^{p} (n-1)^2|a_n|^2 \le 4\beta^2(1-\alpha)^2$$

or

$$(n-1)^2|a_n|^2 \le 4\beta^2(1-\alpha)^2 \quad \text{if} \quad n \ge k+1 ;$$

that is,

$$|a_n| \le \frac{2\beta(1-\alpha)}{n-1} \quad \text{if} \quad n \ge k+1 ,$$

which gives (2.3).

The function f , given by

$$z\frac{f'(z)}{f(z)} = \frac{1-(2\alpha\beta-1)z^k}{1-(2\beta-1)z^k} \quad \text{where} \quad \beta(1-\alpha) > k(1-\beta) ,$$

shows that the estimates in (2.1) are sharp for $n = mk + 1$, $m = 1, 2, 3, \ldots$ while the estimates in (2.3) are sharp for the function

$$f(z) = z\exp\{[2\beta(1-\alpha)/(n-1)]z^{n-1}\} ,$$

where $\beta(1-\alpha) \le k(1-\beta)$ and $n \ge k+1$.

Putting $\beta = 1$ in the theorem, we get the following result due to Boyd [1].

COROLLARY 1. *If* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ *is starlike of order* α , $0 \le \alpha < 1$, *then*

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$$|a_n| \le \frac{k}{(n-1)(m-1)!} \prod_{\mu=0}^{m-1} \left(\mu + \frac{2(1-\alpha)}{k}\right)$$

where $mk+1 \le n \le (m+1)k$, $m = 1, 2, \ldots$. *The result is sharp for* $n = mk + 1$, $m = 1, 2, \ldots$ *for the function*

$$f(z) = \frac{z}{\left(1-z^k\right)^{2(1-\alpha)/k}} .$$

The following result, due to MacGregor [6], can be obtained by taking $(\alpha, \beta) = (0, 1)$ in the theorem.

COROLLARY 2. *If* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ *is in* $S_k^*(0, 1)$ *, then*

$$|a_n| \le \frac{k}{(n-1)(m-1)!} \prod_{\mu=0}^{m-1} (\mu+(2/k))$$

where $mk+1 \le n \le (m+1)k$, $m = 1, 2, \ldots$. *The estimates are sharp for* $n = mk + 1$ $(m = 1, 2, \ldots)$, *for*

$$f(z) = \frac{z}{\left(1-z^k\right)^{2/k}} .$$

REMARKS. (i) The coefficient estimates determined in [5] for the starlike functions of order α and type β can be obtained by putting $k = 1$ in the theorem.

(ii) Putting

$$\beta = \tfrac{1}{2} ,$$

$$(\alpha, \beta) = (0, \tfrac{1}{2}) ;$$

$$(\alpha, \beta) = \left(0, (2\delta-1)/2\delta \text{ where } \delta > \tfrac{1}{2}\right) ;$$

$$(\alpha, \beta) = \left((1-\gamma)/(1+\gamma), (1+\gamma)/2\right)$$

where $0 < \gamma \le 1$, and replacing α by $1 - \alpha$ and β by $\frac{1}{2}$ in the theorem, we get respectively the corresponding coefficient estimates for the functions of the form (1.3) belonging to the classes introduced by McCarty [7], Singh [13, 14], Padmanabhan [10], and Eenigenburg [2].

(iii) The results due to Schild [11], Singh [13, 14], Eenigenburg [2], Mogra [8], McCarty [7], can be obtained by taking different values of the parameters α, β $(0 \le \alpha < 1, 0 < \beta \le 1)$ with $k = 1$ in the theorem.

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