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# DECOMPOSITION OF THE MULTIVARIATE BETA DISTRIBUTION WITH APPLICATIONS

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**Summary.** Let $L$ be a positive definite symmetric matrix having a noncentral multivariate beta density of an arbitrary rank, see, e.g. Hayakawa ([2, p. 12, Equation 38]). Then an explicit procedure is given for decomposing the density of $L$ in terms of densities of independent beta variates.

1. **Introduction and decomposition of** $L$. Let $A$ and $B$ be two $p \times p$ positive definite symmetric random matrices having the densities

$$g(A) = K \exp\{-\tfrac{1}{2} \operatorname{tr} A\}|A|^{(N-q-p-1)/2} \tag{1}$$

$$g(B) = K \exp\{-\tfrac{1}{2} \operatorname{tr} B\}|B|^{(q-p-1)/2} \, {}_0F_1[\tfrac{1}{2}q, \tfrac{1}{2}\Omega B] \tag{2}$$

where $K$ is used as a generic symbol for normalizing constants and the hypergeometric series of the matrix argument $\Omega B$ is defined by Constantine ([1, p. 1276]). Then Hayakawa defines a certain correlation matrix $R$ by the relations

$$B = G^{1/2}(I-R)G^{1/2}, \qquad G = A+B. \tag{3}$$

Let $Q$ be a $p \times p$ arbitrary orthogonal matrix, then the density of the random matrix $L = Q(I-R)Q'$ when $\Omega$ has rank two is given by Kabe [3], and by Hayakawa ([2, p. 12, Equation 38]) when $\Omega$ is of a general rank. However, Hayakawa's claim that the densities of $L$ and $(I-R)$ are the same is in error. The density of the matrix $R$ is not so far available in the literature. The density of the matrix $L$ is

$$g(L) = K\,|L|^{(N-q-p-1)/2}|I-L|^{(q-p-1)/2} \, {}_1F_1[\tfrac{1}{2}N, \tfrac{1}{2}q, \tfrac{1}{2}\Omega L]. \tag{4}$$

In case $\Omega$ has rank $s \le p$, then following Radcliffe [6], the density of $L$ may be written as

$$g(L) = K\,|L|^{(N-q-p-1)/2}|I-L|^{(q-p-1)/2}\,\Phi(L_s), \tag{5}$$

where $\Phi(L_s)$ is a certain function of the elements of $L_s$ only, $L_s$ being obtained from $L$ by omitting its last $(p-s)$ rows and columns. Now we write the density (5) as

$$\begin{aligned} g(L_1, L_2, L_s) = {} & K|L_1 - L_2 L_s^{-1} L_2'|^{(N-q-p-1)/2}|L_s|^{(N-q-p-1)/2} \\ & \times |I - L_1 - L_2(I-L_s)^{-1}L_2'|^{(q-p-1)/2}|I-L_s|^{(q-p-1)/2}\Phi(L_s), \end{aligned} \tag{6}$$

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where

$$L = \begin{pmatrix} L_s & L_2' \\ L_2 & L_1 \end{pmatrix}. \tag{7}$$

Setting $L_1 = D + L_2 L_s^{-1} L_2'$, $L_2 = V((I-L_s)L_s)^{1/2}$, we find the density of random variates $D$, $V$, and $L_s$ to be

$$\begin{aligned} g(D, V, I_s) &= K\,|D|^{(N-q-p-1)/2}|I - VV' - D|^{(q-p-1)/2} \\ &\quad \times |L_s|^{(N-q-s-1)/2}|I-L_s|^{(q-s-1)/2}\Phi(L_s). \end{aligned} \tag{8}$$

Again setting $D = (I-VV')^{1/2}R(I-VV')^{1/2}$, we get

$$\begin{aligned} g(R, V, L_s) &= K\,|R|^{(N-q-p-1)/2}|I-VV'|^{(N-p-s-1)/2} \\ &\quad \times |I-R|^{(q-p-1)/2}\psi(L_s), \end{aligned} \tag{9}$$

where

$$\psi(L_s) = K\,|L_s|^{(N-q-s-1)/2}|I-L_s|^{(q-s-1)/2}\Phi(L_s). \tag{10}$$

Introducing $Z = (I-D)^{-1/2}V$ in (8) the density

$$\begin{aligned} g(D, Z, L_s) &= K\,|D|^{(N-q-p-1)/2}|I-D|^{(q-(p-s)-1)/2} \\ &\quad \times |I-ZZ'|^{(q-p-1)/2}\psi(L_s) \end{aligned} \tag{11}$$

is obtained. Obviously, the densities of $D$, $Z$, and $L_s$ are independent. Now the decomposition of the central multivariate beta distribution, given by Khatri and Pillai ([4, p. 1512, §2]), may be stated as follows:

In case $L = (l_{ij})$, then the central part of the multivariate beta density (4) may be decomposed in terms of $p$ beta variates $z_1, z_2, \ldots, z_p$, and $(p-1)$ $Y_i$ vector variates having the joint density

$$\begin{aligned} &g(z_1, z_2, \ldots, z_p, Y_1, \ldots, Y_{p-1}) \\ &\qquad = K \prod_{i=1}^{p} z_i^{(N-q-p+i-2)/2}(1-z_i)^{(q-2)/2} \prod_{i=1}^{p-1}(1-Y_i'Y_i)^{(q-p+i-2)/2}. \end{aligned} \tag{12}$$

Here $L_{ii}$ is obtained from $L$ by omitting its first $i$ rows and $i$ columns and

$$\begin{cases} l_{ii} = z_i + l_{(i)}' L_{ii}^{-1} l_{(i)}, \quad i = 1, \ldots, p-1; \quad l_{pp} = z_p \\ l_{(i)} = (1-z_i)^{1/2}((I-L_{ii})L_{ii})^{1/2}Y_i, \quad i = 1, \ldots, p-1. \\ l_{(i)} = (l_{i,i+1} l_{i,i+2}, \ldots, l_{ip}). \end{cases} \tag{13}$$

We note that $z_1, z_2, \ldots, z_p = |L|$. Further, we note that all independent factors of $|L|$ are expressible in terms of $z$'s. $p$ $z$'s and $p(p-1)/2$ elements of $Y_i$'s account for $p(p+1)/2$ elements of $L$.

By using the decomposition (12) it follows from (11) that the $(p-s)\times(p-s)$ matrix $D$ may be expressed in terms of $(p-s)$ independent beta variates and $\frac{1}{2}(p-s)(p-s-1)$ $y_i$ variates $i = 1, 2, \ldots, p-s-1$; $Z$ contains $(p-s)\times s$ variates and $L_s$ has $s(s+1)/2$ variates and this accounts for $p(p+s)/2$ elements of $L$.

2. **Applications.** If $L$ has the central distribution (4), then $|L|$ has the distribution of $z_1 z_2 \ldots z_p$ denoted by $\Lambda(N, p, q)$. Now in Kshirsagar's [5] notations

$$\Lambda_0 = |\Gamma'(B-A)\Gamma|/|\Gamma' B\Gamma| = |L_s|, \tag{14}$$

$\Gamma$ is $s \times p$ arbitrary,

$$\Lambda^* = \Lambda/\Lambda_0 = |L|/|L_s| = |L_1 - L_2 L_s^{-1} L_2'| = |D| = \Lambda'\Lambda'', \tag{15}$$

$$\Lambda' = \frac{|\Gamma' AB^{-1}(B-A)\Gamma|\,|\Gamma' B\Gamma|}{|\Gamma' A\Gamma|\,|\Gamma'(B-A)\Gamma|} = |I-VV'| = |I-V'V|, \Lambda'' = |R|, \tag{16}$$

$$\Lambda^* = \Lambda^5\Lambda^6 = |I-VV'|\,|P(I-VV')^{-1}P'| = |P'P| = |D|. \tag{17}$$

The independence of the distributions of $|R|$ and $|(I-VV')|$ follow from (9). $|I-VV'|$ is $\Lambda(N-s, p-s, s)$, $|R|$ is $\Lambda(N-2s, p-s, q-s)$. The independence of $|I-VV'|$ and $|P'(I-VV')^{-1}P|$ is obvious from (8), $D = PP'$, $P$ is $(p-s)\times(p-s)$, $\Lambda^5$ is $\Lambda(N-s, q-s, p-s)$, and $\Lambda^6$ is $\Lambda(N-q, s, p-s)$.

Incidentally, it may be mentioned that the distribution of the residual criterion $\Lambda_0$ may be obtained explicitly even if $\Gamma$ is not of the type $(I, 0)$ as assumed by Radcliffe [6], and Kshirsagar [5] and it has a noncentral multivariate beta distribution of rank $s$. We may obtain this distribution by using the results given in next section.

3. **Some further results**. Let $\hat{\Sigma}$ be a $p \times p$ positive definite symmetric matrix having a noncentral Wishart distribution with $N$ degrees of freedom (d.f.) and of rank $\leq s$, with population covariance matrix $\Sigma$. Then the noncentral part involves the roots of

$$|\Sigma^{-1}\Omega\Omega'\Sigma^{-1} - \lambda\hat{\Sigma}| = 0. \tag{18}$$

If $B$ is an $s \times p$ arbitrary matrix of rank $s$ $(<p)$ then the matrix $B\hat{\Sigma}B'$ has a noncentral Wishart distribution with $N$ d.f. and of rank $s$ with population covariance matrix $B\Sigma B'$. This noncentral distribution is obtained by changing $p$ to $s$ everywhere and changing $\Sigma$ to $B\Sigma B'$, $\Omega$ to $B\Omega$ and $\hat{\Sigma}$ to $B\hat{\Sigma}B'$, i.e. the noncentral part will involve the roots of

$$|B\Omega\Omega B' - \lambda B\Sigma B' B\hat{\Sigma} B' B\Sigma B'| = 0. \tag{19}$$

Thus in (14) for arbitrary $\Gamma$, the numerator $\Gamma'(B-A)\Gamma$ has a noncentral Wishart distribution of rank $s$ and denominator central Wishart distribution of rank $s$, and hence the distribution of $\Lambda_0$ may be obtained. If $p \times p$ $M$ has a central (or noncentral) multivariate beta distribution

$$g(M) = K\,|M|^{(N-p-1)/2}|G-M|^{(q-p-1)/2} \tag{20}$$

then for arbitrary $\Gamma$ $s \times p$, the matrix $\Gamma' M\Gamma = W$ has the distribution

$$g(W) = K\,|W|^{(N-s-1)/2}|\Gamma' G\Gamma - W|^{(q-s-1)/2} \tag{21}$$

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The noncentral distribution of $\Gamma' M \Gamma$ is derived from the noncentral distribution of $M$ exactly on same lines, as in case of the Wishart distribution. If $M$ has the distribution

$$g(M) = K\,|M|^{(N-p-1)/2}|G+M|^{-(q+N)/2} \tag{22}$$

then $\Gamma' M \Gamma = W$ has the density

$$g(W) = K\,|W|^{(N-s-1)/2}|\Gamma' G \Gamma + W|^{-(q+N)/2}, \tag{23}$$

and the noncentral case follows exactly on same lines as in noncentral Wishart distribution.

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