AN EXTENSION OF THE KEGEL–WIELANDT THEOREM TO LOCALLY FINITE GROUPS

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1. Introduction. A famous theorem of Kegel and Wielandt states that every finite group which is the product of two nilpotent subgroups is soluble (see [**1**], Theorem 2.4.3). On the other hand, it is an open question whether an arbitrary group factorized by two nilpotent subgroups satisfies some solubility condition, and only a few partial results are known on this subject. In particular, Kegel [**6**] obtained an affirmative answer in the case of linear groups, and in the same article he also proved that every locally finite group which is the product of two locally nilpotent FC-subgroups is locally soluble. Recall that a group G is said to be an *FC-group* if every element of G has only finitely many conjugates. Moreover, Kazarin [**5**] showed that if the locally finite group $G = AB$ is factorized by an abelian subgroup A and a locally nilpotent subgroup B, then G is locally soluble. The aim of this article is to prove the following extension of the Kegel–Wielandt theorem to locally finite products of hypercentral groups.

THEOREM. *Let the locally finite group $G = AB$ be the product of two hypercentral subgroups A and B, at least one of which is an FC-group, and let H be the Hirsch–Plotkin radical of G. Then the factor group G/H is hyperabelian. In particular, G is a radical group.*

Here a group is called *radical* if it has an ascending normal series with locally nilpotent factors, and the *Hirsch–Plotkin radical* of a group G is the largest locally nilpotent normal subgroup of G. Clearly for locally finite groups the property of being radical is much stronger than local solubility. Note also that locally nilpotent FC-groups are hypercentral. We leave as an open question whether in the above theorem it is possible to omit the hypothesis that one of the factors has finite conjugacy classes.

In the next section some consequences of our result will be obtained. In particular, an extension to locally finite groups of Pennington's theorem on finite groups which are the product of two nilpotent subgroups of coprime orders will be proved. Other information on the structure of periodic radical groups factorized by locally nilpotent subgroups can be found in [**4**].

Most of our notation is standard; in particular we refer to [**1**] and [**7**].

2. Proof of the Theorem. Let the group $G = AB$ be the product of two subgroups A and B. A subgroup K of G is said to be *semifactorized* if $K = (A \cap K)(B \cap K)$. The semifactorized subgroup K of $G = AB$ is called *factorized if* $A \cap B \leq K$. In particular, if $G = AB$ and $A \cap B = 1$, every semifactorized subgroup of G is factorized. It is easy to show that a normal subgroup N of $G = AB$ is factorized if and only if $AN \cap BN = N$ (see [**1**], Lemma 1.1.4).

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LEMMA 1. *Let the finite soluble group $G = AB$ be the product of two subgroups A and B of coprime orders. If A is nilpotent, then the subgroup $Z(A)^G \cap A$ is abelian.*

Proof. Let $\pi = \pi(A)$, and consider the factor group $\bar{G} = G/O_{\pi'}(G)$. Clearly $O_\pi(\bar{G})$ is contained in $\bar{A}$, so that $O_\pi(\bar{G})$ is the Fitting subgroup of $\bar{G}$. Then

$$Z(\bar{A}) \leq C_{\bar{G}}(O_\pi(\bar{G})) = Z(O_\pi(\bar{G})),$$

so that also $Z(\bar{A})^{\bar{G}}$ lies in $Z(O_\pi(\bar{G}))$. Put $Z(O_\pi(\bar{G})) = Z/O_{\pi'}(G)$. Thus $Z(A)^G \cap A$ is a π-subgroup of Z, and hence

$$Z(A)^G \cap A \simeq (Z(A)^G \cap A)O_{\pi'}(G)/O_{\pi'}(G) \leq Z/O_{\pi'}(G).$$

Therefore $Z(A)^G \cap A$ is abelian. □

LEMMA 2. *Let the locally finite group $G = AB$ be the product of two hypercentral subgroups A and B with $\pi(A) \cap \pi(B) = \emptyset$. If G has a local system consisting of finite factorized subgroups, then G is hyperabelian.*

Proof. The group G is locally soluble by the theorem of Kegel and Wielandt. Let N be a normal subgroup of G. Since $\pi(A) \cap \pi(B) = \emptyset$, we have $AN \cap BN = N$, so that N is factorized, and hence it also has a local system consisting of finite factorized subgroups. Put $A_1 = Z(A)^G \cap A$, and let a_1, a_2 be elements of A_1. Then there exist finite subgroups E of $Z(A)$ and L of G such that a_1 and a_2 belong to $E^L \cap A$. By hypothesis the finite subgroup $\langle E, L\rangle$ is contained in a finite subgroup K of G such that $K = (A \cap K)(B \cap K)$. Clearly E lies in $Z(A \cap K)$, and a_1, a_2 belong to $Z(A \cap K)^K \cap (A \cap K)$. The subgroup $Z(A \cap K)^K \cap (A \cap K)$ is abelian by Lemma 1, so that $a_1a_2 = a_2a_1$. Therefore the subgroup A_1 is abelian. Similarly $B_1 = Z(B)^G \cap B$ is an abelian subgroup of G. The intersection $Z(A)^G \cap Z(B)^G$ is a factorized subgroup of G, so that

$$\begin{aligned} Z(A)^G \cap Z(B)^G &= (A \cap Z(A)^G \cap Z(B)^G)(B \cap Z(A)^G \cap Z(B)^G) \\ &= (A_1 \cap Z(B)^G)(B_1 \cap Z(A)^G). \end{aligned}$$

Thus $Z(A)^G \cap Z(B)^G$ is the product of two abelian subgroups, and hence it is metabelian by Itô's theorem (see [**1**], Theorem 2.1.1). Assume that the Hirsch–Plotkin radical of G is trivial, and put $X = Z(A)^G$ and $Y = B \cap X$. Then

$$X = (A \cap X)(B \cap X) = A_1Y.$$

It follows from the first part of the proof that $A_1^X \cap Z(Y)^X$ is metabelian. Clearly $A_1^X \cap Z(Y)^X$ is subnormal in G, so that $A_1^X \cap Z(Y)^X = 1$. Then $[A_1, Z(Y)] = 1$, and $Z(Y) \leq Z(X) = 1$, so that $Y = 1$ and $A_1 = X$ is normal in G. Therefore $A_1 = 1$, so that $Z(A) = 1$ and $A = 1$. It follows that $G = B$ is locally nilpotent, and hence $G = 1$. Thus, if G is not trivial, we have that also the Hirsch–Plotkin radical H of G is not trivial. Moreover $H = (A \cap H) \times (B \cap H)$, so that $Z(H)$ is a non-trivial abelian normal subgroup of G. As the hypotheses are inherited by homomorphic images, it follows that G is hyperabelian. □

Let π be a set of primes. A π-subgroup A of a group G is said to be *π-connected* in G if for every π-subgroup B of G such that $AB = BA$, the product AB is also a π-subgroup of G. It is obvious that normal π-subgroups are π-connected and that conjugates of π-connected subgroups are likewise π-connected.

LEMMA 3. *Let G be a periodic locally soluble group, and let π be a set of primes. Then every hyperabelian π-subgroup of G is π-connected.*

Proof. This follows directly from Corollary 2 of [**8**]. □

We will also need the following lemma on π-connected subgroups, a proof of which can be found in [**4**].

LEMMA 4. *Let G be a group, and let π be a set of primes such that $O_\pi(G)=1$. If A is a π-subgroup of G and B is a π-connected subgroup of G such that $AB^g=B^gA$ for every element g of G, then $[A^G,B^G]=1$.*

LEMMA 5. *Let the locally finite group $G=AB$ be the product of two hypercentral subgroups A and B. If G has a local system consisting of finite semifactorized subgroups, then the group G is radical.*

Proof. Since G has a local system consisting of finite semifactorized subgroups, it follows from the theorem of Kegel and Wielandt that G is locally soluble and from Lemma 1.3.2 of [**1**] that for every set π of primes the product $A_\pi B_\pi$ is a Sylow π-subgroup of G and $G=(A_\pi B_\pi)(A_{\pi'}B_{\pi'})$. Thus $A_\pi B_\pi^g=B_\pi^g A_\pi$ for every element g of G. As the subgroup B_π is π-connected in G by Lemma 3, it follows from Lemma 4 that $[A_\pi,B_\pi]$ is contained in $O_\pi(G)$. Similarly we obtain that $[A_{\pi'},B_{\pi'}]$ lies in $O_{\pi'}(G)$. Moreover, it is clear that $O_\pi(G)$ and $O_{\pi'}(G)$ are contained in $A_\pi B_\pi$ and $A_{\pi'}B_{\pi'}$, respectively. Thus the factor group $G/O_\pi(G)O_{\pi'}(G)$ is the product of the hypercentral π-subgroup $A_\pi B_\pi O_{\pi'}(G)/O_\pi(G)O_{\pi'}(G)$ and the hypercentral π'-subgroup $A_{\pi'}B_{\pi'}O_\pi(G)/O_\pi(G)O_{\pi'}(G)$. It is easy to show that this factorized group also has a local system consisting of finite factorized subgroups, so that it follows from Lemma 2 that $G/O_\pi(G)O_{\pi'}(G)$ is hyperabelian. Let H be the Hirsch-Plotkin radical of G. Since

$$\bigcap_p O_p(G)O_{p'}(G)=H,$$

the group G/H is residually hyperabelian. In particular, if π is a finite set of primes, the Sylow π-subgroup $A_\pi B_\pi$ of G is hyperabelian, so that also $O_\pi(G)$ is hyperabelian. Assume that $H=1$. Then for every finite set π of primes we have $O_\pi(G)=1$, and so also $[A_\pi,B_\pi]=1$. It follows that $[A,B]=1$ and $G=AB$ is locally nilpotent, so that $G=1$. As the hypotheses are inherited by homomorphic images, this argument proves that the group G is radical. □

If G is a locally nilpotent group and π is a set of primes, we will denote by G_π the unique Sylow π-subgroup of G. In [**4**] the following result on Sylow subgroups of locally finite factorized groups was proved.

LEMMA 6. *Let the periodic radical group $G=AB$ be the product of two locally nilpotent subgroups A and B, at least one of which is hyperabelian. Then for every set π of primes the product $A_\pi B_\pi$ is a Sylow π-subgroup of G, and $G=(A_\pi B_\pi)(A_{\pi'}B_{\pi'})$.*

LEMMA 7. *Let the periodic radical group $G=AB$ be the product of two hypercentral subgroups A and B, and let H be the Hirsch–Plotkin radical of G. Then the factor group G/H is hyperabelian.*

Proof. Let p be a prime. If g is an element of G, we have $G = AB^g$, and so the product $A_pB_p^g$ is a Sylow p-subgroup of G by Lemma 6. Moreover, B_p is a p-connected subgroup of G by Lemma 3, and hence it follows from Lemma 4 that $[A_p, B_p]$ is contained in $O_p(G)$. Therefore for every prime p the Sylow p-subgroup A_pB_pH/H of G/H is hypercentral. As G is radical, it follows that G/H is hyperabelian. □

Proof of the Theorem. Suppose that A is an FC-group, and consider the factorized group

$$\langle A, Z(B)\rangle = A(B \cap \langle A, Z(B)\rangle).$$

Let E be a finite normal subgroup of A and L a finite subgroup of $Z(B)$. Then the finite subgroup $\langle E, L\rangle$ is contained in a finite subgroup K of $\langle A, Z(B)\rangle$ such that $K = (A \cap K)(B \cap K)$ (see [**1**], Lemma 1.2.3). Since A has a local system consisting of finite normal subgroups, it follows that the group $\langle A, Z(B)\rangle$ has a local system consisting of finite semifactorized subgroups. Therefore $\langle A, Z(B)\rangle$ is a radical group by Lemma 5. Clearly $Z(B)^G = Z(B)^{BA} = Z(B)^A$ is contained in $\langle A, Z(B)\rangle$, and hence it is a radical normal subgroup of G. If $Z(B)^G = 1$, then $B = 1$ and $G = A$ is hypercentral. As the hypotheses are inherited by homomorphic images, it follows that G is a radical group. Finally, the factor group G/H is hyperabelian by Lemma 7. □

We give now a series of consequences of our Theorem. The first of these is an extension of Kegel's theorem on finite groups triply factorized by nilpotent subgroups (see [**1**], Corollary 2.5.11).

Corollary 8. *Let the locally finite group $G = AB = AC = BC$ be the product of three hypercentral subgroups A, B and C, at least one of which is an FC-group. Then G is locally nilpotent.*

Proof. The group G is radical by the Theorem, and hence it is locally nilpotent by Theorem B of [**4**]. □

Note that in the hypotheses of Corollary 8 the group G need not be hypercentral. In fact, there exists a locally finite triply factorized group $G = AB = AK = BK$, where A, B and K are abelian, K is normal in G and $Z(G) = 1$ (see [**1**], Theorem 6.1.3 or [**8**]).

From our result one can also deduce the following extension of Pennington's theorem on finite products of nilpotent groups of coprime order (see [**1**], Theorem 2.5.3).

Corollary 9. *Let the locally finite group $G = AB$ be the product of two nilpotent subgroups A and B, with classes c and d, respectively. If one of the factors A and B is an FC-group, then the subgroup $G^{(c+d)}$ is a locally nilpotent π-group, where $\pi = \pi(A) \cap \pi(B)$. In particular, if $\pi(A) \cap \pi(B) = \varnothing$, the group G is soluble with derived length at most $c + d$.*

Proof. The group G is radical by the Theorem, so that the subgroup $G^{(c+d)}$ is a π-group by Theorem D of [**3**]. Moreover, it follows from Theorem D of [**4**] that $G^{(c+d)}$ is locally nilpotent. □

Corollary 10. *Let the locally finite group $G = AB$ be the product of an abelian subgroup A and a nilpotent subgroup B of class $c > 1$, and let H be the Hirsch–Plotkin* *radical of G. Then the factor group G/H is soluble with derived length at most* $[\log_2(c-1)]+2$.

Proof. The group G is radical by the Theorem. Let V/H be the Hirsch–Plotkin radical of G/H, and put $Z/H=Z(V/H)$. Then A and $Z(B)$ are contained in Z by Theorem C of [**4**]. Thus $G=ZB$ and G/Z is a nilpotent group with class at most $c-1$. It follows that G/Z has derived length at most $[\log_2(c-1)]+1$, and hence G/H is a soluble group with derived length at most $[\log_2(c-1)]+2$. □

Lemma 11. *Let the locally finite group $G=AB$ be the product of two locally nilpotent FC-subgroups A and B. Then G has a local system consisting of finite soluble semifactorized subgroups. Moreover, for every set π of primes the product $A_\pi B_\pi$ is a Sylow π-subgroup of G and $G=(A_\pi B_\pi)(A_{\pi'}B_{\pi'})$.*

Proof. The subgroups A and B have local systems consisting of finite normal subgroups, and hence the statement follows from the theorem of Kegel and Wielandt and from Lemmas 1.2.3 and 1.3.2 of [**1**]. □

Corollary 12. *Let the locally finite group $G=A_1\ldots A_n$ be the product of finitely many pairwise permutable locally nilpotent FC-subgroups $A_1,\ldots,A_n$ such that $\pi(A_i)\cap\pi(A_j)=\varnothing$ if $i\neq j$. Then G is hyperabelian.*

Proof. It follows from Lemma 11 and Lemma 2 that the subgroup A_1A_2 is hyperabelian, so that there exists a non-trivial abelian normal subgroup N of A_1A_2 which either is contained in A_1 or in A_2. Suppose that $N\leq A_2$. Then the normal closure

$$N^G=N^{A_1A_2\ldots A_n}=N^{A_3\ldots A_n}$$

is contained in the subgroup $A_2A_3\ldots A_n$. By induction on n it can be assumed that $A_2A_3\ldots A_n$ is hyperabelian, so that N^G is a hyperabelian normal subgroup of G. Therefore the Hirsch–Plotkin radical of G is not trivial. As the hypotheses are inherited by homomorphic images, the group G is radical, and then it is also clear that G is hyperabelian. □

Let G be a locally finite group. A set $\{G_p \mid p\in\pi(G)\}$ of Sylow subgroups of G, one for each prime p in the set $\pi(G)$, is called a *Sylow basis* of G if $G=\langle G_p \mid p\in\pi(G)\rangle$ and $G_pG_q=G_qG_p$ for all p,q in $\pi(G)$. It seems to be unknown whether a locally finite group having a Sylow basis is locally soluble. As a consequence of our results we prove here the following.

Corollary 13. *Let G be a locally finite group having a Sylow basis. If the Sylow subgoups of G are FC-groups, then G is locally soluble.*

Proof. Let $\{G_p \mid p\in\pi(G)\}$ be a Sylow basis of G. Then for every finite subset π of $\pi(G)$ the subgroup $\langle G_p \mid p\in\pi\rangle$ is hyperabelian by Corollary 12. Therefore G is locally soluble. □

Corollary 14. *Let G be a periodic locally soluble group whose Sylow subgroups are FC-groups. If the set of primes $\pi(G)$ is finite, then G is hyperabelian.*

Proof. In order to prove that G is hyperabelian, it can be assumed that the group G is countable (see [**7**] Part 1, Corollary to Theorem 2.15). Then it follows form a result of Baer that G has a Sylow basis (see [**2**], Lemma 2.1). Therefore G is hyperabelian by Corollary 12. □

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