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BRUNN–MINKOWSKI TYPE INEQUALITIES FOR L_p MOMENT BODIES*

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Abstract. About 15 years ago, Lutwak and Zhang (E. Lutwak and G. Zhang, Blaschke–Santalo inequalities, *J. Differ. Geom.* **47** (1997), 1–16) introduced the notion of L_p moment bodies and established important volume inequalities for them, which were recently generalized by Haberl and Schuster (C. Haberl and E. Schuster, General L_p affine isoperimetric inequalities, *J. Differ. Geom.* **83** (2009), 1–26). In this paper, we establish new Brunn–Minkowski-type inequalities with respect to Blaschke L_p harmonic addition for the quermassintegrals and dual quermassintegrals of L_p moment bodies.

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1. Introduction and main results. Centroid bodies were defined and investigated by Petty [**27**]. They have proven to be a remarkably powerful tool in establishing a number of fundamental affine isoperimetric inequalities due to Petty [**27–29**] (see also [**18**, **19**]). Projection bodies were introduced by Minkowski at the turn of the previous century and have since become a central notion in convex geometry (see e.g. [**8**] and the references therein). All centroid and projection bodies are zonoids. However, the centroid operator and the projection operator are quite different. For example, the centroid operator commutes with linear transformations, while the projection operator does not; the projection operator is translation invariant, but the centroid operator is not.

While projection bodies and the projection operator have been the objects of intensive investigations during more than 50 years, centroid bodies (volume-normalized moment bodies) and their L_p extensions by Lutwak and Zhang [**24**] have attracted increased attention only in the last two decades (see e.g. [**4–6**, **8–15**, **19–25**, **33**, **35**, **36**, **38**]). The aim of this paper is to obtain the Brunn–Minkowski-type inequalities with respect to Blaschke L_p harmonic combinations for quermassintegrals and dual quermassintegrals of L_p moment bodies (see e.g. [**1**, **2**, **26**, **31**, **32**] for related results).

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interior and the set of convex bodies centred at the origin, we write $\mathcal{K}_o^n$ and $\mathcal{K}_s^n$ respectively. The unit ball in $\mathbb{R}^n$ and its surface will be denoted by B and S^{n-1} respectively. The volume and surface area of the convex body K will be denoted by $V(K)$ and $S(K)$ respectively.

If $K \in \mathcal{K}^n$, then its support function $h_K = h(K, \cdot)$ is defined by $h(K, x) = \max\{x \cdot y : y \in K\}$, $x \in \mathbb{R}^n \setminus \{0\}$. Let K be a star body in Euclidean space $\mathbb{R}^n$, that is a compact set

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which is star-shaped with respect to the origin and has a continuous radial function $\rho_K(u) = \rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}$, $u \in S^{n-1}$. Let $\mathcal{S}_o^n$ be the set of star bodies in $\mathbb{R}^n$ containing the origin in their interiors. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If K is an arbitrary non-empty subset of $\mathbb{R}^n$, then the set $K^* = \{x : x \cdot y \leq 1, y \in K\}$ is called the polar set of K. According to the definitions of polar body, support function and radial function, it follows that for $K \in \mathcal{K}_o^n$,

$$h(K^*, u) = \frac{1}{\rho(K, u)}, \quad \rho(K^*, u) = \frac{1}{h(K, u)}. \tag{1.1}$$

Lutwak and Zhang [**24**] introduced the notion of L_p moment bodies: For each compact star-shaped (about the origin) L in $\mathbb{R}^n$ and any real $p \geq 1$, the L_p moment body M_pL of L is defined by

$$h(M_pL, u)^p = c_{n,p} \int_{S^{n-1}} |u \cdot v|^p \rho(L, v)^{n+p} dS(v). \quad u \in S^{n-1}, \tag{1.2}$$

where

$$c_{n,p} = \frac{\Gamma(\frac{n+p}{2})}{\pi^{(n-1)/2}\Gamma(\frac{1+p}{2})}. \tag{1.3}$$

We note that the constant $c_{n,p}$ in the definition of L_p moment body is chosen so that for the unit ball B we have $M_pB = B$. We use M_p^*L to denote the polar of M_pL. From definition (1.2), it is easy to see that for any $L \in \mathcal{S}_o^n$, the L_p moment body $M_pL \in \mathcal{K}_o^n$ is well defined.

Lutwak et al. [**21**] (see also Campi and Gronchi [**3**]) established an L_p version of the classical Busemann–Petty centroid inequality: For $p \geq 1$ and convex bodies K containing the origin in their interiors,

$$V(K)^{n/p-1} V(M_pK) \leq V(B)^{n/p},$$

with equality if and only if K is an ellipsoid centred at the origin.

Suppose $K, L \in \mathcal{S}_o^n$ and $p \geq 1$. Yuan et al. (see [**37**]) introduced the Blaschke L_p harmonic combination of K and L, $K\widehat{+}_pL$. Let $\xi > 0$ be defined by

$$\xi^{1/(n+p)} = \frac{1}{n} \int_{S^{n-1}} [V(K)^{-1}\rho(K, u)^{n+p} + V(L)^{-1}\rho(L, u)^{n+p}]^{n/(n+p)} dS(u).$$

The body $K\widehat{+}_pL \in \mathcal{S}_o^n$ is defined as the body whose radial function is given by

$$\xi^{-1}\rho(K\widehat{+}_pL, \cdot)^{n+p} = V(K)^{-1}\rho(K, \cdot)^{n+p} + V(L)^{-1}\rho(L, \cdot)^{n+p}.$$

Note that, since $\xi = V(K\widehat{+}_pL)$, we have

$$\frac{\rho(K\widehat{+}_pL, \cdot)^{n+p}}{V(K\widehat{+}_pL)} = \frac{\rho(K, \cdot)^{n+p}}{V(K)} + \frac{\rho(L, \cdot)^{n+p}}{V(L)}. \tag{1.4}$$

The main results of this paper are the following Brunn–Minkowski-type inequalities for the quermassintegrals W_i and the dual quermassintegrals $\widetilde{W}_i$ of L_p moment bodies.

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THEOREM 1. *If $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $i = 0, 1, \ldots, n-1$, then*

$$\frac{W_i(M_p(K \widehat{+}_p L))^{\frac{p}{n-i}}}{V(K \widehat{+}_p L)} \geq \frac{W_i(M_p K)^{\frac{p}{n-i}}}{V(K)} + \frac{W_i(M_p L)^{\frac{p}{n-i}}}{V(L)}, \tag{1.5}$$

with equality if and only if K and L are dilates when $p > 1$ or when $p = 1$ and $0 \leq i < n-1$; there is always equality in (1.5) when $p = 1$ and $i = n-1$.

THEOREM 2. *If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, and $i \neq n$, then for $i < n$ and $n < i < n+p$, we have*

$$\frac{\widetilde{W}_i(M_p^*(K \widehat{+}_p L))^{-\frac{p}{n-i}}}{V(K \widehat{+}_p L)} \geq \frac{\widetilde{W}_i(M_p^* K)^{-\frac{p}{n-i}}}{V(K)} + \frac{\widetilde{W}_i(M_p^* L)^{-\frac{p}{n-i}}}{V(L)}; \tag{1.6}$$

for $i > n+p$, we have

$$\frac{\widetilde{W}_i(M_p^*(K \widehat{+}_p L))^{-\frac{p}{n-i}}}{V(K \widehat{+}_p L)} \leq \frac{\widetilde{W}_i(M_p^* K)^{-\frac{p}{n-i}}}{V(K)} + \frac{\widetilde{W}_i(M_p^* L)^{-\frac{p}{n-i}}}{V(L)}, \tag{1.7}$$

with equality in either of the inequalities (1.6) and (1.7) if and only if K and L are dilates.

2. Preliminaries. In this section we collect some basic well-known facts that will be useful in the proofs of our results. For references about the Brunn–Minkowski theory, see [**8**, **30**].

2.1. L_p Mixed quermassintegrals. Let $K \in \mathcal{K}^n$, the quermassintegrals $W_i(K)$, for $i = 0, 1, \ldots, n-1$, of K, are defined by (see [**8**, **17**])

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS_i(K, u), \tag{2.1}$$

where $S_i(K, \cdot)(i = 0, 1, \ldots, n-1)$ is called the area measure of order i of $K \in \mathcal{K}^n$. In particular,

$$W_0(K) = V(K). \tag{2.2}$$

For $p \geq 1$, $K, L \in \mathcal{K}_o^n$ and $\varepsilon > 0$, the Firey L_p combination $K +_p \varepsilon \cdot L \in \mathcal{K}_o^n$ is defined by (see [**7**, **17**])

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p,$$

where '$\cdot$' in '$\varepsilon \cdot L$' denotes the Firey scalar multiplication.

Associated with the Firey L_p combination, Lutwak [**16**] defined the L_p mixed quermassintegrals as follows: For $K, L \in \mathcal{K}_o^n$, $i = 0, 1, \ldots, n-1$ and real $p \geq 1$, the L_p mixed quermassintegrals $W_{p,i}(K, L)$, $i = 0, 1, \ldots, n-1$, are defined by

$$\frac{n-i}{p} W_{p,i}(K, L) = \lim_{\varepsilon \to 0^+} \frac{W_i(K +_p \varepsilon \cdot L) - W_i(K)}{\varepsilon}. \tag{2.3}$$

For $p = 1$, $W_{1,i}(K, L)$ are just the classical mixed quermassintegrals $W_i(K, L)$ (see [**16**]).

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In [16], Lutwak showed that for $K, L \in \mathcal{K}_o^n$, $p \geq 1$, $i = 0, 1, \ldots, n-1$, there exists a positive Borel measure $S_{p,i}(K, \cdot)$ on S^{n-1} such that the L_p mixed quermassintegral $W_{p,i}(K, L)$ has the following integral representation:

$$W_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) dS_{p,i}(K, u). \tag{2.4}$$

It turns out that the measures $S_{p,i}(K, \cdot)$, $i = 0, 1, \ldots, n-1$, on S^{n-1} are absolutely continuous with respect to $S_i(K, \cdot)$. From (2.4), it follows immediately that for each $K \in \mathcal{K}_o^n$ and $p \geq 1$,

$$W_{p,i}(K, K) = W_i(K). \tag{2.5}$$

We shall require a basic inequality for the L_p mixed quermassintegral $W_{p,i}(K, L)$, which may be stated as follows: for $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $0 \leq i < n$, (see [16])

$$W_{p,i}(K, L)^{n-i} \geq W_i(K)^{n-i-p} W_i(L)^p, \tag{2.6}$$

with equality for $p = 1$ if and only if K and L are homothetic, and for $p > 1$ if and only if K and L are dilates.

2.2. L_p Dual mixed quermassintegrals. For $K \in \mathcal{S}^n$ and arbitrary real i, the dual mixed quermassintegrals $\widetilde{W}_i(K)$ are defined by (see [8, 30])

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) dS(u), \tag{2.7}$$

where $dS(u)$ denotes the area element at $u \in S^{n-1}$. From (2.7), it follows immediately that

$$\widetilde{W}_0(K) = V(K). \tag{2.8}$$

For $K, L \in \mathcal{S}_o^n$, $\varepsilon > 0$ and $p \geq 1$, the L_p harmonic radial combination $K +_{-p} \varepsilon \cdot L \in \mathcal{S}_o^n$ is defined as the star body whose radial function is given by (see [17])

$$\rho(K +_{-p} \varepsilon \cdot L, \cdot)^{-p} = \rho(K, \cdot)^{-p} + \varepsilon \rho(L, \cdot)^{-p}.$$

Associated with the L_p harmonic radial combination, Wang and Leng [34] defined the L_p dual mixed quermassintegrals as follows.

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and any real $i \neq n$, the L_p dual mixed quermassintegral, $\widetilde{W}_{-p,i}(K, L)$ of K and L was defined in [34] by

$$\frac{n-i}{-p} \widetilde{W}_{-p,i}(K, L) = \lim_{\varepsilon \to 0^+} \frac{\widetilde{W}_i(K +_{-p} \varepsilon \cdot L) - \widetilde{W}_i(K)}{\varepsilon}. \tag{2.9}$$

Taking $i = 0$ in (2.9) and using (2.8), we see

$$\widetilde{W}_{-p,0}(K, L) = \widetilde{V}_{-p}(K, L). \tag{2.10}$$

The above definition and the polar coordinate formula for dual mixed quermassintegrals yield the following integral representation of L_p dual mixed quermassintegrals $\widetilde{W}_{-p,i}(K, L)$: for $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and any real $i \neq n$,

$$\widetilde{W}_{-p,i}(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p-i}(u)\rho_L^{-p}(u)dS(u). \tag{2.11}$$

By (2.10) and (2.11), we have

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p}(u)\rho_L^{-p}(u)dS(u). \tag{2.12}$$

From definition (2.11), it follows immediately that for each $K \in \mathcal{S}_o^n$, real $i \neq n$ and $p \geq 1$,

$$\widetilde{W}_{-p,i}(K, K) = \widetilde{W}_i(K). \tag{2.13}$$

The L_p Minkowski inequality for L_p dual mixed quermassintegrals states that **[34]** if $K, L \in \mathcal{S}_o^n$, $p \geq 1$, then, for real $i < n$ or $i > n + p$,

$$\widetilde{W}_{-p,i}(K, L)^{n-i} \geq \widetilde{W}_i(K)^{n+p-i}\widetilde{W}_i(L)^{-p}, \tag{2.14}$$

for $n < i < n + p$,

$$\widetilde{W}_{-p,i}(K, L)^{n-i} \leq \widetilde{W}_i(K)^{n+p-i}\widetilde{W}_i(L)^{-p}, \tag{2.15}$$

with equality in either of the inequalities (2.14) and (2.15) if and only if K and L are dilates.

3. Brunn–Minkowski inequalities for L_p moment bodies.

In order to prove Theorems 1 and 2, we need the following lemma.

LEMMA 1. *Let $K \in \mathcal{S}_o^n$, $p \geq 1$, then for $u \in S^{n-1}$,*

$$\frac{h(M_p(K\widehat{+}_pL), u)^p}{V(K\widehat{+}_pL)} = \frac{h(M_pK, u)^p}{V(K)} + \frac{h(M_pL, u)^p}{V(L)}. \tag{3.1}$$

Proof. From (1.4) and (1.2), we have

$$\begin{aligned}\frac{h(M_p(K\widehat{+}_pL), u)^p}{V(K\widehat{+}_pL)} &= c_{n,p}\int_{S^{n-1}} |u\cdot v|^p \frac{\rho(K\widehat{+}_pL, v)^{n+p}}{V(K\widehat{+}_pL)}dS(v)\\ &= c_{n,p}\int_{S^{n-1}} |u\cdot v|^p \left[\frac{\rho(K, v)^{n+p}}{V(K)} + \frac{\rho(L, v)^{n+p}}{V(L)}\right]dS(v)\\ &= c_{n,p}\int_{S^{n-1}} |u\cdot v|^p \frac{\rho(K, v)^{n+p}}{V(K)}dS(v) + c_{n,p}\int_{S^{n-1}} |u\cdot v|^p \frac{\rho(L, v)^{n+p}}{V(L)}dS(v)\\ &= \frac{h(M_pK, u)^p}{V(K)} + \frac{h(M_pL, u)^p}{V(L)}.\end{aligned}$$

□

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Proof of Theorem 1. Let $K, L \in \mathcal{S}_o^n$ and $p \geq 1$. From (2.4), (3.1) and (2.6), we obtain for all $Q \in \mathcal{K}_o^n$,

$$\begin{aligned}
\frac{W_{p,i}(Q, M_p(K\widehat{+}_pL))}{V(K\widehat{+}_pL)} &= \frac{1}{nV(K\widehat{+}_pL)} \int_{S^{n-1}} h^p_{M_p(K\widehat{+}_pL)}(u) dS_{p,i}(Q,u) \\
&= \frac{1}{n} \int_{S^{n-1}} \frac{h^p_{M_p(K\widehat{+}_pL)}(u)}{V(K\widehat{+}_pL)} dS_{p,i}(Q,u) \\
&= \frac{1}{n} \int_{S^{n-1}} \frac{h^p_{M_pK}(u)}{V(K)} dS_{p,i}(Q,u) + \frac{1}{n} \int_{S^{n-1}} \frac{h^p_{M_pL}(u)}{V(L)} dS_{p,i}(Q,u) \\
&= \frac{W_{p,i}(Q, M_pK)}{V(K)} + \frac{W_{p,i}(Q, M_pL)}{V(L)} \\
&\geq W_i(Q)^{\frac{n-p-i}{n-i}} \left[\frac{W_i(M_pK)^{\frac{p}{n-i}}}{V(K)} + \frac{W_i(M_pL)^{\frac{p}{n-i}}}{V(L)} \right].
\end{aligned}$$

Taking $Q = M_p(K\widehat{+}_pL)$ in the above inequality, and using (2.5), we obtain the desired inequality (1.5).

According to the equality condition of inequality (2.6), we know that equality holds in inequality (1.5) if and only if M_pK, M_pL and $M_p(K\widehat{+}_pL)$ are homothetic when $p = 1$ and $0 \leq i < n-1$ or M_pK, M_pL and $M_p(K\widehat{+}_pL)$ are dilates when $p > 1$. There is always equality in (1.5) when $p = 1$ and $i = n-1$.

Since M_pK, M_pL and $M_p(K\widehat{+}_pL)$ are origin-symmetric, equality holds in (1.5) if and only if they are dilates (except for $p = 1$ and $i = n-1$). Thus, $M_pK = \lambda_1 M_p(K\widehat{+}_pL)$ and $M_pL = \lambda_2 M_p(K\widehat{+}_pL)$ for some $\lambda_1, \ \lambda_2 > 0$. An argument similar to [**31**, p. 227] now yields that K and L are dilates. □

For $i = 0$ we note the following special case of Theorem 1:

Corollary 1. *If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, then*

$$\frac{V(M_p(K\widehat{+}_pL))^{\frac{p}{n}}}{V(K\widehat{+}_pL)} \geq \frac{V(M_pK)^{\frac{p}{n}}}{V(K)} + \frac{V(M_pL)^{\frac{p}{n}}}{V(L)},$$

with equality if and only if K and L are dilates.

Proof of Theorem 2. Let $K, L \in \mathcal{S}_o^n$, $p \geq 1$, then by (2.11), (1.1) and (3.1), we have for all $Q \in \mathcal{K}_o^n$,

$$\begin{aligned}
\frac{\widetilde{W}_{-p,i}(Q, M_p^*(K\widehat{+}_pL))}{V(K\widehat{+}_pL)} &= \frac{1}{nV(K\widehat{+}_pL)} \int_{S^{n-1}} \rho_Q^{n+p-i}(u) \rho^{-p}_{M_p^*(K\widehat{+}_pL)}(u) dS(u) \\
&= \frac{1}{nV(K\widehat{+}_pL)} \int_{S^{n-1}} \rho_Q^{n+p-i}(u) h^p_{M_p(K\widehat{+}_pL)}(u) dS(u) \\
&= \frac{1}{n} \int_{S^{n-1}} \rho_Q^{n+p-i}(u) \frac{h^p_{M_p(K\widehat{+}_pL)}(u)}{V(K\widehat{+}_pL)} dS(u)
\end{aligned}$$

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$$= \frac{1}{n}\int_{S^{n-1}} \rho_Q^{n+p-i}(u)\frac{h^p_{M_pK}(u)}{V(K)}dS(u) + \frac{1}{n}\int_{S^{n-1}} \rho_Q^{n+p-i}(u)\frac{h^p_{M_pL}(u)}{V(L)}dS(u)$$
$$= \frac{\widetilde{W}_{-p,i}(Q, M_p^*K)}{V(K)} + \frac{\widetilde{W}_{-p,i}(Q, M_p^*L)}{V(L)}.$$

If $i < n$ or $n < i < n+p$, then using (2.14) and (2.15) we obtain

$$\frac{\widetilde{W}_{-p,i}(Q, M_p^*(K\widehat{+}_pL))}{V(K\widehat{+}_pL)} \geq \widetilde{W}_i(Q)^{\frac{n+p-i}{n-i}}\left[\frac{\widetilde{W}_i(M_p^*K)^{-\frac{p}{n-i}}}{V(K)} + \frac{\widetilde{W}_i(M_p^*L)^{-\frac{p}{n-i}}}{V(L)}\right]. \tag{3.2}$$

Taking $Q = M_p^*(K\widehat{+}_pL)$ in the above inequality and using (2.13) we get the desired inequality,

$$\frac{\widetilde{W}_i(M_p^*(K\widehat{+}_pL))^{-\frac{p}{n-i}}}{V(K\widehat{+}_pL)} \geq \frac{\widetilde{W}_i(M_p^*K)^{-\frac{p}{n-i}}}{V(K)} + \frac{\widetilde{W}_i(M_p^*L)^{-\frac{p}{n-i}}}{V(L)}.$$

Using now (2.14) and (2.15) the equality conditions follow from arguments similar to the ones in the proof of Theorem 1. The case $i > n+p$ is treated analogously. □

The following corollary is a direct consequence of Theorem 2 when $i = 0$.

Corollary 2. *If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, then*

$$\frac{V(M_p^*(K\widehat{+}_pL))^{-\frac{p}{n}}}{V(K\widehat{+}_pL)} \geq \frac{V(M_p^*K)^{-\frac{p}{n}}}{V(K)} + \frac{V(M_p^*L)^{-\frac{p}{n}}}{V(L)},$$

with equality if and only if K and L are dilates.

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