EMBEDDING $\mathscr{P}$-LIKE COMPACTA IN MANIFOLDS

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1. Introduction. A *compactum* is a compact, metrizable space. A *continuum* is a connected compactum. All *polyhedra* will be finitely triangulable spaces. If α is an open cover of a compactum X, a map of X onto a compactum Y is called an *α-map* provided that the inverse image of each point in Y is contained in some member of α.

If $\mathscr{P}$ is a class of polyhedra, then, following Mardešić and Segal **(10)**, we say a compactum X is *$\mathscr{P}$-like* provided that for each open cover α of X there exists an α-map of X onto some member of $\mathscr{P}$. If the members of $\mathscr{P}$ are connected, each of the following two conditions is necessary and sufficient for X to be $\mathscr{P}$-like: I. Each open cover of X can be refined by an open cover whose nerve is homeomorphic to a member of $\mathscr{P}$; see **(12)**. II. X is homeomorphic to the limit of an inverse sequence of members of $\mathscr{P}$, with bonding maps onto; see **(10)**.

Some previous results on embedding $\mathscr{P}$-like compacta in manifolds are the following. (Let R^n be Euclidean n-space.)

(1) If $\mathscr{P}$ is the class of all polyhedra of dimension $\leqslant n$, a compactum X is $\mathscr{P}$-like if and only if $\dim X \leqslant n$. Then there is the classical result of Menger and Nobeling that every such X can be embedded in R^{2n+1}.

(2) R. H. Bing **(2)** showed that every arc-like continuum embeds in R^2. Also in **(2)** there was given a simple example of a tree-like continuum not embeddable in R^2.

(3) J. R. Isbell **(7)** generalized Bing's result by showing that every limit of an inverse sequence of compact subsets of R^n embeds in R^{2n}.

(4) Bing **(3)** (see also **2**) obtained necessary and sufficient conditions for a circle-like continuum to embed in R^2. Also in **(3)** it was shown that every circle-like continuum embeddable in a 2-manifold is embeddable in R^2.

The use of Čech cohomology (with integer coefficients) provides the following convenient reformulation of part of Bing's results: *A circle-like continuum X can be embedded in R^2 if and only if $H^1(X)$ is zero or infinite cyclic.* Theorem 1 (stated in §3) gives necessary conditions on the n-dimensional Čech cohomology of certain $\mathscr{P}$-like compacta embeddable in $(n+1)$-manifolds, so that one may obtain non-embedding results. Using this theorem, we show that for each $n \geqslant 1$, there exists an n-sphere-like continuum that cannot be embedded in any $(n+1)$-manifold, but can be embedded in R^{n+2}.

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Theorem 2, which depends on a lemma of Isbell **(7)**, provides positive results on embedding inverse limits. We shall show in §3 how Theorems 1 and 2 combine to give different proofs of some of the results of Bing in **(3)**, and we shall point out some other applications of these theorems.

Some of the results of this paper were announced in **(11)** and were part of my dissertation at Yale University, 1963. I am grateful for helpful suggestions from F. E. Browder, W. S. Massey, P. M. Rice, and the late M. K. Fort, Jr.

2. Terminology and notation. We shall use the notation of **(4**, Chapter VIII**)** for inverse and direct limit systems. We deal only with inverse limit *sequences*, which have the positive integers as index set. Thus if (X, f) is an inverse sequence, we have the bonding maps $f_m{}^n : X_n \to X_m (m \leqslant n)$ and the projection maps $f_n : X_\infty \to X_n$. If x is a point of the limit X_∞, we often write x_n for $f_n(x)$.

For each space X and each integral domain D, $H^n(X; D)$ denotes the n-dimensional Čech cohomology D-module of X, and rank $H^n(X; D)$ denotes its rank over D; see **(4**, p. 52**)**. If $\mathscr{P}$ is a class of spaces, we define the *class* of D-modules

$$H^n(\mathscr{P}; D) = \{H^n(P; D) : P \in \mathscr{P}\}.$$

If $\mathscr{M}$ is a class of D-modules, we let

$$\text{rank } \mathscr{M} = \sup\{\text{rank } M : M \in \mathscr{M}\}.$$

We say the class $\mathscr{M}$ is *free*, if each M in $\mathscr{M}$ is free, etc. The integral domains of interest here are the integers, Z, and the integers mod 2, Z_2. We write $H^n(X) = H^n(X; Z)$ and $H^n(\mathscr{P}) = H^n(\mathscr{P}; Z)$.

Let R^n = Euclidean n-space, S^n = standard n-sphere in R^{n+1}, I^n = standard n-cube, T^n = standard n-torus (the n-fold product of S^1). All manifolds will be closed and connected.

3. Statements of theorems. Some applications.

THEOREM 1. *Let $\mathscr{P}$ be a class of polyhedra and let X be a $\mathscr{P}$-like compactum embedded in an $(n + 1)$-manifold N. If d = rank $H^n(\mathscr{P}; Z_2)$ is finite, then the following statements hold.* (i) *$H^n(X)$ is a finitely generated group of rank no more than rank $H^n(\mathscr{P})$.* (ii) *If $H^n(\mathscr{P})$ is free, then $H^n(X)$ is free.* (iii) *$N - X$ has at most $d + 1$ components.*

Remark 1. The condition $d < \infty$ of course holds when the class $\mathscr{P}$ is finite.

Remark 2. Without the assumption that X is embedded in an $(n + 1)$-manifold, $H^n(X)$ need not be finitely generated or free, even when $\mathscr{P}$ consists of a single polyhedron K with $H^n(K)$ free. The simplest example occurs when K is a circle and X is the dyadic solenoid. Then $H^1(X)$ is isomorphic to the dyadic rationals.

COROLLARY 1. *Let M be an orientable, triangulable n-manifold, and let X be an M-like continuum embedded in an $(n+1)$-manifold N. Then* (i) $H^n(X) = 0$ *or* $H^n(X) \approx Z$; (ii) *$N - X$ has at most two components.*

The proofs are given in §4.

For each finite-dimensional, separable, metrizable space X, let the *embedding number*, emb X, be the least integer p such that X can be embedded in R^p. The following corollary of Theorem 1 (more directly, of Corollary 1) implies that for each n there exists an S^n-like continuum Y with emb $Y = n + 2$.

COROLLARY 2. *Let X be any circle-like continuum that cannot be embedded in the plane. Then for each $n \geqslant 1$, the $(n-1)$-fold suspension $S^{n-1}(X)$ is an S^n-like continuum that cannot be embedded in any $(n+1)$-manifold but can be embedded in R^{n+2}.*

The proof is given in §5.

Definition 1. Let (E, d) be a locally compact metric space and let $f:X \to Y$ be a map where X and Y are compact subsets of E. Then *f can be approximated by embeddings in E* provided that for each $\epsilon > 0$ there exist an open set U with $X \subset U \subset E$ and a 1–1 map $\mu: U \to E$ such that $d(\mu(x), f(x)) < \epsilon$ for all x in X.

THEOREM 2. *Let (E, d) be a locally compact metric space and let (X, f) be an inverse limit sequence of compact subsets of E such that each bonding map $f_n^{n+1}: X_{n+1} \to X_n$ can be approximated by embeddings in E. Then the limit X_∞ can be embedded in E.*

The proof, given in §6, uses a lemma of J. R. Isbell **(7)**. It has common ground with the proof of Theorem 1 of **(7)**.

Note that in Definition 1 if $E = R^2$ and $X = Y = S^1$, the map $f:S^1 \to S^1$ can be approximated by embeddings if and only if for each $\epsilon > 0$ there exists an embedding μ: $S^1 \to R^2$ such that $d(\mu, f) < \epsilon$. This follows from the Schoenflies theorem.

PROPOSITION 1. *A map $f:S^1 \to S^1 \subset R^2$ can be approximated by embeddings if and only if* $|\deg f| \leqslant 1$.

The proof is given in §7. The proof of the part "$\deg f = 1$ implies f can be approximated" is related to the discussion of **(3**, p. 119**)**.

Let us now show how Corollary 1, Theorem 2, and Proposition 1 can be used to obtain some of the results of Bing. The main result is the following reformulated characterization of planar circle-like continua. **(**See Theorems 3 and 4 of **3** and Theorem 4 of **2.)**

PROPOSITION 2. *A circle-like continuum X can be embedded in the plane if and only if $H^1(X) = 0$ or $H^1(X) \approx Z$.*

Proof. Taking $M = S^1$, $N = S^2$ in Corollary 1, we obtain the necessity of the condition. The sufficiency follows from the fact that every circle-like continuum is an inverse limit of circles, from the continuity theorem for Čech cohomology, from Proposition 1, and from Theorem 2.

Remark 1. When the circle-like continuum X is non-planar, $H^1(X)$ is isomorphic to the group of P-adic rationals, for some prime sequence P.

Remark 2. A circle-like continuum X is arc-like if and only if $H^1(X) = 0$.

Two other results in **(3)** are (1) *each circle-like continuum that can be embedded in a* 2*-manifold can be embedded in the plane*, and (2) *the complement of a circle-like continuum in an orientable, connected* 2*-manifold has at most two components.* (1) follows immediately from Corollary 1 and Proposition 2; and (2) (without the assumption that the 2-manifold be orientable) follows from Corollary 1. *Note*: We have required that our manifolds be closed and connected; however, the general case can be easily reduced to this.

Using Theorem 2, R. Bennett **(1)** has shown that *the product of n arc-like continua can be embedded in* R^{n+1}. One has the following more general proposition, which also generalizes Theorem 1 of Isbell **(7)**:

Proposition 3. *The product of n spaces, each of which is the limit of an inverse sequence of compact subsets of* R^k, *can be embedded in* $R^{k(n+1)}$.

The proof requires a slight modification of that of Bennett's. (See also the proof of Isbell's Theorem 1.) It suffices to construct, for each sequence of n maps $f_i: X_i \to R^k$, where X_i is a compact subset of R^k $(1 \leqslant i \leqslant n)$, a 1–1 map μ of the product $(R^k)^{n+1}$ into itself, whose restriction to $(\prod_{i=1}^n X_i) \times 0$ is close to the map $(\prod_{i=1}^n f_i) \times 0$. For this, extend each f_i to a map $F_i: R^k \to R^k$ and, for small $\epsilon > 0$, let

$$\mu(x_1, \ldots, x_{n+1}) = (F_1(x_1) + \epsilon x_2, \ldots, F_n(x_n) + \epsilon x_{n+1}, \epsilon x_1).$$

Added in proof. L. Fearnley obtained essentially the same result independently in Amer. J. Math., *88* (1966), 347–356.

L. Fearnley **(5)** has announced the following two results: (1) *The product of n circle-like continua can be embedded in* R^{n+2}. (2) *The product of n planar circle-like continua can be embedded in* R^{n+1}. Fearnley also stated in **(5)** that these are best possible results. J. R. Isbell obtained independently a proof of (1) using Theorem 2, which he communicated to the author. Also, using Theorem 2 and Propositions 1 and 2, one may similarly prove (2). That (1) is the best possible result follows from Corollary 1 (or, more directly, from the Alexander–Pontryagin duality theorem); in fact, it can be seen that *if X is the product of n circle-like continua, no one of which is arc-like, but some one of which is non-planar, then X cannot be embedded in any* $(n + 1)$*-manifold.*

Fearnley has informed the author that he has generalized (1) to (1)′ *the product of n* S^k*-like continua embeds in* $R^{k(n+1)+1}$. By extending Isbell's proof of (1), one can establish the following more general statement:

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PROPOSITION 4. *Let M^k be a (closed, connected) differentiable k-manifold that can be differentiably embedded in R^{k+1}. Then the product of n M^k-like continua can be embedded in $R^{k+1} \times (M^k)^n$.*

(Here $(M^k)^n$ is the n-fold product of M^k.) It is easy to see that $R^1 \times (S^k)^n$ embeds in R^{kn+1} so that $R^{k+1} \times (S^k)^n$ embeds in $R^{k(n+1)+1}$. Thus Proposition 4 implies Fearnley's result (1)′. It also implies that the product of n T^k-like continua embeds in $R^{k(n+1)+1}$.

Products of n circle-like continua are special cases of T^n-like continua. I have some reasons to believe that if X is a T^n-like continuum and $H^1(X)$ is finitely generated, then Theorem 2 can be used to show that emb $X \leqslant 2n$. Lemma 2, given in the next section, implies that for such an X, $H^1(X)$ is a free abelian group with rank between 0 and n. (It is easy to give examples to show that all such values of the rank can occur.) The case $H^1(X) = 0$ is taken care of by

THEOREM 3. *If X is a T^n-like continuum and $H^1(X) = 0$, then X can be represented as the limit of an inverse sequence of n-cells. Hence by a theorem of J. R. Isbell* **(7)**, *X can be embedded in R^{2n}.*

The proof is given in §8. The first part of the theorem generalizes the fact that every circle-like continuum X with $H^1(X) = 0$ is also arc-like.

4. Proof of Theorem 1 and Corollary 1.

The following lemma (for the case of groups) was used in **(12)**. Let D be an integral domain.

LEMMA 1. *Let (G, ϕ) be a direct system of D-modules with limit G^∞. Then* (i) rank $G^\infty \leqslant$ rank G. (ii) *If G is torsion-free, then G^∞ is torsion-free.*

Here, of course, if M is the index set for the system, rank $G = \sup\{\text{rank } G^m: m \in M\}$; and to say G is torsion-free means each G^m is torsion-free. The proof of the lemma is a straightforward application of the definitions.

LEMMA 2. *Let $\mathscr{P}$ be a class of polyhedra and let X be a $\mathscr{P}$-like compactum. Then for each n,* (i) rank $H^n(X; D) \leqslant$ rank $H^n(\mathscr{P}; D)$. (ii) *If $H^n(\mathscr{P}; D)$ is torsion-free, then $H^n(X; D)$ is torsion-free.*

Proof. This follows from Lemma 1, Remark 3 of **(10**, p. 154**)**, and the continuity theorem for Čech cohomology **(4**, p. 261**)**. Alternatively, one may use Lemma 1, the fact that X is weakly $\mathscr{P}$-like **(12)**, and the definition of Čech cohomology.

Now let $\mathscr{P}$, X, N, and $d < \infty$ be as in the statement of Theorem 1. The case $X = N$ is trivial; in the following we assume that X is a proper subset of N. Let U be the open set $N - X$, and write $U = \bigcup_{\lambda \in \Lambda} U_\lambda$, as the union of its connected components (there being exactly one index λ for each component).

Since N is locally connected, each U_λ is open. Hence by **(4**, p. 294**)**, we have for each coefficient domain D a direct sum decomposition

$$(4.1) \qquad H^{n+1}(N, X; D) \approx \textstyle\sum_{\lambda \in \Lambda} H^{n+1}(N, N - U_\lambda; D).$$

Consider the following part of the mod 2 exact sequence of the pair (N, X):

$$H^n(X; Z_2) \xrightarrow{\delta} H^{n+1}(N, X; Z_2) \to H^{n+1}(N; Z_2) \to 0.$$

($H^{n+1}(X; Z_2) = 0$ since X is a proper subset of N). Exactness and the isomorphism $H^{n+1}(N; Z_2) \approx Z_2$ show that

$$\operatorname{rank} H^{n+1}(N, X; Z_2) = \operatorname{rank} \delta H^n(X; Z_2) + 1 \leqslant \operatorname{rank} H^n(X; Z_2) + 1.$$

But by Lemma 2(i), with $D = Z_2$, rank $H^n(X; Z_2) \leqslant d$; hence

$$(4.2) \qquad \operatorname{rank} H^{n+1}(N, X; Z_2) \leqslant d + 1.$$

Now each pair $(N, N - U_\lambda)$ is a relative $(n + 1)$-manifold in the sense of **(4**, p. 311**)**, with $N - (N - U_\lambda) = U_\lambda$ being connected. Hence by Theorem 6.8, parts (iv) and (v), of **(4**, p. 314**)**, we see that

$$(4.3) \qquad H^{n+1}(N, N - U_\lambda; Z_2) \approx Z_2,$$

$$(4.4) \qquad H^{n+1}(N, N - U_\lambda) \approx Z \text{ or } Z_2.$$

From (4.1) with $D = Z_2$, (4.3), and (4.2), we see that $N - X$ has at most $d + 1$ components. This establishes part (iii) of the theorem. The importance of this for the rest of the proof is that we have shown that $N - X$ has only a finite number of components. From this, from (4.1) with $D = Z$, and from (4.4), we see that

$$(4.5) \qquad H^{n+1}(N, X) \text{ is a finitely generated group.}$$

Now exactness of the sequence

$$H^n(N) \to H^n(X) \to H^{n+1}(N, X),$$

the fact that $H^n(N)$ is finitely generated (since N is a compact ANR), and (4.5) show us that $H^n(X)$ is finitely generated. Lemma 2 (i) with $D = Z$ implies that rank $H^n(X) \leqslant$ rank $H^n(\mathscr{P})$. This completes part (i) of the theorem. If $H^n(\mathscr{P})$ is free, then Lemma 2 (ii) says that $H^n(X)$ is torsion-free. But every finitely generated, torsion-free (abelian) group is free. This completes the proof of Theorem 4.

For Corollary 1, we take $\mathscr{P} = \{M\}$. The proof is immediate from the isomorphisms $H^n(M; Z_2) \approx Z_2$ and $H^n(M) \approx Z$.

5. Proof of Corollary 2.

Definition 2. For each space X let $S(X)$ denote the suspension of X, and let $v_X : X \times I \to S(X)$ be the quotient map. We let $S^n(X)$ denote the n-fold suspension of X ($S^0(X) = X$, $S^n(X) = S(S^{n-1}(X))$). If $\mathscr{P}$ is a class of spaces, we define the *suspension* $S(\mathscr{P})$ to be the class $\{S(X):X \in \mathscr{P}\}$.

LEMMA 3. *Let $\mathscr{P}$ be a class of polyhedra, and let X be a $\mathscr{P}$-like compactum. Then $S(X)$ is an $S(\mathscr{P})$-like continuum.*

Proof. Let α be an open cover of $S(X)$. Clearly there exists an open cover β of X such that for each U in β and each t in $I = [0, 1]$, there is a V in α with $v_X(U \times \{t\}) \subset V$. Choose a β-map f of X onto a polyhedron K in $\mathscr{P}$. Let $g:S(X) \to S(K)$ be the *suspension* of f, defined by $gv_X(x, t) = v_K(f(x), t)$. For each point $z = v_K(y, t)$ of $S(K)$, clearly $g^{-1}(z) = v_X(f^{-1}(y) \times \{t\})$. By choice of β, therefore g is an α-map. This completes the proof.

Now let X be a circle-like continuum that cannot be embedded in the plane, for example, the dyadic solenoid. Thus by Proposition 2, $H^1(X)$ is neither 0 nor isomorphic to Z. If $n \geqslant 1$, then by Lemma 3 and induction, $S^{n-1}(X)$ is S^n-like. By iteration of the suspension isomorphism, $H^n(S^{n-1}(X)) \approx H^1(X)$. Hence, by Corollary 1, $S^{n-1}(X)$ cannot be embedded in any $(n + 1)$-manifold. Since X is 1-dimensional, X can be embedded in R^3. Therefore, $S^{n-1}(X)$ can be embedded in $R^{3+(n-1)} = R^{n+2}$. This completes the proof of Corollary 2.

Remark. Lemma 3 can be replaced by an argument using inverse limits and Theorem 1* of **(10)**.

6. Proof of Theorem 2.

Definition 3. If (X, d) and (X', d') are compact metric spaces and $f:X \to X'$ is a map, then for each $\epsilon > 0$, let

$$M(f, \epsilon) = \sup\{\delta : x, y \in X \text{ and } d(x, y) < \delta \text{ imply } d'(f(x), f(y)) < \epsilon\}.$$

Note that $M(f, \epsilon) > 0$ since f is uniformly continuous.

The following lemma is a slight modification of a lemma of J. R. Isbell. See **(7**, Lemma 2**)** or **(8**, p. 73**)** for an improved version.

LEMMA 4 (Isbell). *Let (X, f) be an inverse limit sequence of compact metric spaces, X_n having metric d_n. Then there exists a sequence (ϵ_n) of positive numbers for which the following is true. Suppose (E, d) is a complete metric space and suppose that for each n there is an embedding $h_n:X_n \to E$ satisfying the condition*

(6.1) *If $m < n$, then $d(h_m f_m{}^n, h_n) < \delta_m$, where*

$$\delta_m = \min\{M(h_m{}^{-1}, \epsilon_m)/3, 1/m\}.$$

Then the sequence of maps $h_n f_n:X_\infty \to E$ converges to an embedding $X_\infty \to E$.

Now let (E, d) and (X, f) be as in the statement of Theorem 2. The metric used on X_n is the restriction of d. Let the sequence (ϵ_n) be chosen for (X, f) as in Lemma 4. We shall construct recursively sequences (h_n), (U_n), (H_n), and (δ_n) such that for each $n \geqslant 1$, the following conditions A_n, B_n, C_n, and D_n hold, and for each $n \geqslant 2$, the condition E_n holds. (A_n) $h_n \colon X_n \to E$ is an embedding. (B_n) U_n is an open set such that $X_n \subset U_n \subset E$. (C_n) $H_n \colon U \to E$ is a 1–1 and uniformly continuous map extending h_n. (D_n) $\delta_n = \min\{M(h_n^{-1}, \epsilon_n)/3, 1/n\}$. (E_n) $d(h_n, h_{n-1} f_{n-1}^n) < \min\{2^{m-1}\delta_m : 1 \leqslant m < n\}$. It should be remarked that condition D_n is merely the definition of δ_n.

For the first step we let $h_1 \colon X_1 \to E$ be the inclusion map; we let $U_1 = E$, and $H_1 \colon U_1 \to E$ is the identity map. Obviously, conditions A_1, B_1, and C_1 hold (and D_1 defines δ_1). Let $n \geqslant 2$ and suppse we have constructed h_k, U_k, H_k, δ_k for $1 \leqslant k < n$ so that A_k, B_k, C_k, D_k hold for $1 \leqslant k < n$ and E_k holds for $2 \leqslant k < n$. Since the map $f_{n-1}^n \colon X_n \to X_{n-1}$ can be approximated by embeddings in E, there exists a sequence (μ^i) of 1–1 maps $U^i \to E$, where U^i is open and $X_n \subset U^i \subset E$, such that the sequence $(\mu^i | X_n)$ converges uniformly to f_{n-1}^n. Since X_{n-1} is compact, some ϵ-neighbourhood of X_{n-1} is contained in U_{n-1}. Then for i large enough $\mu^i(X_n)$ will lie in U_{n-1}. We can assume that this holds for all i. Thus the composition $H_{n-1} \mu^i$ is defined for all i. Since H_{n-1} is uniformly continuous on its domain, the sequence $(H_{n-1} \mu^i | X_n)$ converges uniformly to $H_{n-1} f_{n-1}^n$. Therefore, there exists an i such that for all $x \in X_n$:

$$d(H_{n-1} \mu^i(x), H_{n-1} f_{n-1}^n(x)) < \min\{2^{m-n}\delta_m : 1 \leqslant m < n\}. \tag{6.2}$$

Since X_n is compact and E is locally compact, there exists an open set U_n, such that $X_n \subset U_n \subset U^i$ and the map $H_n = H_{n-1} \mu^i | U_n$ is uniformly continuous. This defines U_n and H_n. Since H_{n-1} and μ^i are 1–1, so is H_n. Of course we let $h_n \colon X_n \to E$ be defined as the restriction $H_n | X_n$. Now h_n is an embedding since it is 1–1 and its domain is compact. Thus, conditions A_n, B_n, C_n hold. Condition D_n defines δ_n. Finally, inequality (6.2) implies condition E_n, since when $x \in X_n$, $H_{n-1} \mu^i(x) = h_n(x)$ and $H_{n-1} f_{n-1}^n(x) = h_{n-1} f_{n-1}^n(x)$.

The proof will be complete after we use condition E_n to show that (6.1) in Lemma 4 holds. First, let us observe that if m and j are arbitrary integers with $1 \leqslant m \leqslant j$, then condition E_{j+1} implies

$$d(h_j f_j^{j+1}, h_{j+1}) < 2^{m-j-1}\delta_m. \tag{6.3}$$

Now suppose $1 \leqslant m < n$. Then

$$d(h_m f_m^n, h_n) \leqslant \sum_{j=m}^{n-1} d(h_j f_j^n, h_{j+1} f_{j+1}^n) = \sum_{j=m}^{n-1} d(h_j f_j^{j+1} f_{j+1}^n, h_{j+1} f_{j+1}^n)$$
$$\leqslant \sum_{j=m}^{n-1} d(h_j f_j^{j+1}, h_{j+1}).$$

By (6.3), this is

$$< \sum_{j=m}^{n-1} 2^{m-j-1}\delta_m < \delta_m \sum_{j=m}^{\infty} 2^{m-j-1} = \delta_m.$$

This completes the proof of Theorem 2.

7. Proof of Proposition 1. The proof of the necessity of the condition $|\deg f| \leqslant 1$ can be given in a straightforward way using the Schoenflies theorem. However, let us give the following curious proof. Suppose that $f:S^1 \to S^1$ can be approximated by embeddings in R^2, but $|\deg f| > 1$. By Theorem 2, the limit X of the inverse sequence

$$S^1 \xleftarrow{f} S^1 \xleftarrow{f} \dots$$

can be embedded in R^2. Hence by Corollary 1, $H^1(X) = 0$ or $H^1(X) \approx Z$. However, by the continuity theorem, $H^1(X)$ is isomorphic to the group of d-adic rationals, where $d = \deg f$.

Conversely, suppose $f:S^1 \to S^1$ and $|\deg f| \leqslant 1$. To show that f can be approximated by embeddings in R^2, it clearly suffices to show that the map $f':S^1 \to S^1 \times R^1$ defined by $f'(z) = (f(z), 0)$ can be approximated by embeddings $\mu: S^1 \to S^1 \times R^1$. Let $p:R^1 \to S^1$ be the universal covering map $p(t) = \exp(2\pi it)$. Choose a *lifting* $\tilde{f}:R^1 \to R^1$ of f: a map such that $p\tilde{f} = fp$. For each $t \in R^1, \tilde{f}(t+1) - \tilde{f}(t) = \deg f$.

Take first the case $\deg f = \pm 1$. For each $\epsilon > 0$ define $\mu_\epsilon: S^1 \to S^1 \times R^1$ as follows. If $t \in R^1$,

$$\mu_\epsilon(p(t)) = (f(p(t)), \epsilon(\tilde{f}(t) \mp t)).$$

Since the map $t \to \tilde{f}(t) \mp t$ is periodic (of period 1), μ_ϵ is well-defined and continuous. It is easy to see that μ_ϵ is 1–1, and approaches f' as ϵ approaches 0.

Now take the case $\deg f = 0$. Then $\tilde{f}$ is periodic. Let $a = \min \tilde{f}$, and let $t_0 \in [0, 1]$ be such that $\tilde{f}(t_0) = a$. We may assume the additional condition:

(7.1) $$\text{If } t_0 < t < t_0 + 1, \text{ then } \tilde{f}(t) > a.$$

In other words, it is clear that f can be approximated by maps with liftings having this property. Now choose a positive number η such that

$$\eta|\tilde{f}(t) - a| < \pi/2$$

for all t. For each $\epsilon > 0$ we define $\mu_\epsilon:S^1 \to S^1 \times R^1$ as follows. If $z \in S^1$ let $z = p(t)$, where $t_0 \leqslant t \leqslant t_0 + 1$. Then let

(7.2) $$\mu_\epsilon(z) = (f(z), \epsilon(\eta t(\tilde{f}(t) - a) + \tilde{f}(t))).$$

First, μ_ϵ is well-defined and continuous, since $\tilde{f}(t_0 + 1) = \tilde{f}(t_0) = a$. To show that μ_ϵ is 1–1, suppose $\mu_\epsilon(z) = \mu_\epsilon(z')$. Let $z = p(t)$, $z' = p(t')$, where

(7.3) $$t_0 \leqslant t, t' < t_0 + 1.$$

From (7.2) we see that

(7.4) $$f(z) = f(z'),$$

(7.5) $$\tilde{f}(t) - \tilde{f}(t') = \eta t'(\tilde{f}(t') - a) - \eta t(\tilde{f}(t) - a).$$

Now (7.4) gives $p(\tilde{f}(t)) = p(\tilde{f}(t'))$. Hence, the left-hand side of (7.5) is of the form $2\pi n$, where n is an integer. But by choice of η and the fact that $0 \leqslant t, t' < 2$, the right-hand side of (7.5) has absolute value less than 2π. Thus $n = 0$, and we have $\tilde{f}(t) = \tilde{f}(t')$ and $t(\tilde{f}(t) - a) = t'(\tilde{f}(t') - a)$. Hence (7.1) and (7.3) imply $t = t'$. It is clear that μ_ϵ approaches f' as ϵ approaches 0.

8. Proof of Theorem 3. Suppose we are given a commutative diagram

$$\begin{tikzcd}
(\tilde{X}, \tilde{x}_0) \arrow[r, "\tilde{f}"] \arrow[d, "p"] & (\tilde{Y}, \tilde{y}_0) \arrow[d, "q"] \\
(X, x_0) \arrow[r, "f"] & (Y, y_0)
\end{tikzcd}$$

where p and q are covering maps and $\tilde{X}$ and $\tilde{Y}$ are simply connected. (It will be assumed that all base spaces of covering maps are 0-connected and locally 0-connected.) If f is given, there is exactly one $\tilde{f}$ making the diagram commutative, by **(6**, p. 257**)**. Let G and H be the covering transformation groups of p and q, respectively. Let $\phi:\pi_1(X, x_0) \to G$ and $\psi:\pi_1(Y, y_0) \to H$ be the standard isomorphisms **(6**, p. 260**)**. For instance, ϕ is defined as follows. Let $a = [\alpha] \in \pi_1(X, x_0)$. Lift the path α to the path $\tilde{\alpha}$ beginning at $\tilde{x}_0$. Then $\phi(a)$ is the unique g in G such that $g(\tilde{x}_0) = \tilde{\alpha}(1)$. Under these assumptions, the verification of the following lemma is straightforward.

LEMMA 5. *For each* $a \in \pi_1(X, x_0)$, $\tilde{f} \circ \phi(a) = (\psi f_*(a)) \circ \tilde{f}$.

COROLLARY. *Suppose f induces the trivial homomorphism on fundamental groups. Then $\tilde{f}$ is periodic in the sense that for each $g \in G$, $\tilde{f}g = \tilde{f}$.*

LEMMA 6. *Let (X, f) and $(\tilde{X}, \tilde{f})$ be inverse limit sequences with each $\tilde{X}_n$ simply connected. Suppose each $f_n^{n+1}: X_{n+1} \to X_n$ induces the trivial homomorphism on fundamental groups. Suppose that for each n we have a covering map $p_n:\tilde{X}_n \to X_n$ such that $f_n p_{n+1} = p_n \tilde{f}_n$. Then the induced map $p_\infty:\tilde{X}_\infty \to X_\infty$ is 1–1 and onto.*

Proof. Let G_n denote the covering transformation group of p_n. To show that p_∞ is 1–1, suppose $\tilde{x}, \tilde{y} \in \tilde{X}_\infty$ and $p_\infty(\tilde{x}) = p_\infty(\tilde{y})$. Then since $p_n(\tilde{x}_n) = p_n(\tilde{y}_n)$, there is $g_n \in G_n$ such that $\tilde{y}_n = g_n(\tilde{x}_n)$, for each n. But

$$\tilde{y}_n = \tilde{f}_n^{n+1}(\tilde{y}_{n+1}) = \tilde{f}_n^{n+1} g_{n+1}(\tilde{x}_{n+1}),$$

which by the preceding corollary equals $\tilde{f}_n^{n+1}(\tilde{x}_{n+1}) = \tilde{x}_n$. Thus $\tilde{y} = \tilde{x}$. To show that p_∞ is onto, consider $x \in X_\infty$. Let $\tilde{x}_n = \tilde{f}_n^{n+1}(p_{n+1}^{-1}(x_{n+1}))$. This is indeed a single point by the preceding corollary. Now for each n,

$$p_n(\tilde{x}_n) = p_n \tilde{f}_n^{n+1} p_{n+1}^{-1}(x_{n+1}) = f_n^{n+1} p_{n+1}\, p_{n+1}^{-1}(x_{n+1}) = f_n^{n+1}(x_{n+1}) = x_n.$$

From this (with n replaced by $n + 1$) we have

$$\tilde{x}_n = \tilde{f}_n^{n+1} p_{n+1}^{-1}(x_{n+1}) = \tilde{f}_n^{n+1}(\tilde{x}_{n+1}).$$

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Thus the point $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots)$ is in $\tilde{X}_\infty$. Then from $p_n(\tilde{x}_n) = x_n$, we have $p_\infty(\tilde{x}) = x$. This completes the proof of the lemma.

Now let X be as in the statement of Theorem 3. By **(10**, Theorem 1***)** we may assume that X is the limit of an inverse limit sequence of tori:

$$T^n \xleftarrow{f_1^2} T^n \xleftarrow{f_2^3} \ldots \longleftarrow X. \tag{8.1}$$

Since $H^1(X) = 0$, the limit of the induced direct sequence is zero:

$$H^1(T^n) \xrightarrow{(f_1^2)^*} H^1(T^n) \xrightarrow{(f_2^3)^*} \ldots \to 0. \tag{8.2}$$

This does not imply that each $(f_i^{i+1})^*$ is zero, but it is easy to see that, since $H^1(T^n)$ is finitely generated, there is for each i a $j > i$ such that $(f_i^j)^* = 0$. Hence, by cofinality, we may assume that the original sequence (8.1) is such that in (8.2) every homomorphism is zero.

Let $p:R^n \to T^n$ be the standard universal covering:

$$p(t_1, \ldots, t_n) = (\exp(2\pi i t_1), \ldots, \exp(2\pi i t_n)).$$

Lifting the maps (8.1), we obtain the commutative diagram

$$\begin{tikzcd} R^n \arrow[d, "p"] & R^n \arrow[l, "\tilde{f}_1^2"'] \arrow[d, "p"] & \ldots \arrow[l, "\tilde{f}_2^3"'] & \tilde{X} \arrow[l] \arrow[d, "p_\infty"] \\ T^n & T^n \arrow[l, "f_1^2"'] & \ldots \arrow[l, "f_2^3"'] & X \arrow[l] \end{tikzcd} \tag{8.3}$$

where $\tilde{X}$ is defined as the limit of the top row, and p_∞ is the induced map. Now the natural homomorphisms $\rho:\pi_1(T^n) \to H_1(T^n)$ and

$$\alpha:H^1(T^n) \to \mathrm{Hom}(H_1(T^n), Z)$$

are isomorphisms; see **(6**, p. 348; **9**, p. 77**)**. Hence, by the above and the naturality of ρ and α, the homomorphisms $(f_i^{i+1})_* : \pi_1(T^n) \to \pi_1(T^n)$ are trivial. Thus Lemma 6 applies to give that p_∞ in (8.3) is 1–1 and onto.

By the Corollary to Lemma 5, the maps $\tilde{f}_i^{i+1}$ are periodic (invariant under translation by lattice points in R^n). For each positive integer a let I_a be the n-cell

$$I_a = \{t \in R^n : |t_j| \leqslant a, j = 1, \ldots, n\}. \tag{8.4}$$

By the periodicity of $\tilde{f}_i^{i+1}$,

$$\tilde{f}_i^{i+1}(R^n) = \tilde{f}_i^{i+1}(I_a). \tag{8.5}$$

In particular, each set $\tilde{f}_i^{i+1}(R^n)$ is compact. For each i let I_i^n be an n-cell of the form (8.4) such that

$$\tilde{f}_i^{i+1}(R^n) \subset I_i^n. \tag{8.6}$$

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By (8.5) and (8.6), the restricted inverse sequence

$$I_1^n \xleftarrow{\tilde{f}_1^2|I_2^n} I_2^n \xleftarrow{\tilde{f}_2^3|I_3^n} \cdots$$

of n-cells has the same limit, $\tilde{X}$, as the top row of (8.3). Since $\tilde{X}$ is compact, p_∞ is a homeomorphism. Thus X can also be represented as an inverse limit of n-cells. This completes the proof.

References

1. R. Bennett, *Embedding products of chainable continua*, Proc. Amer. Math. Soc., *16* (1965), 1026–1027.
2. R. H. Bing, *Snake-like continua*, Duke Math. J., *18* (1951), 653–663.
3. ——— *Embedding circle-like continua in the plane*, Can. J. Math., *14* (1962), 113–128.
4. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology* (Princeton, 1952).
5. L. Fearnley, *Embeddings of topological products of circularly chainable continua*, Amer. Math. Soc. Not., *12* (1965), 480.
6. P. J. Hilton and S. Wylie, *Homology theory* (Cambridge, 1960).
7. J. R. Isbell, *Embeddings of inverse limits*, Ann. Math., *70* (1959), 73–84.
8. ——— *Uniform spaces*, Mathematical Surveys, 12, Amer. Math. Soc. (Providence, 1964).
9. S. MacLane, *Homology* (Berlin, 1963).
10. S. Mardešić and J. Segal, *ϵ-Mappings onto polyhedra*, Trans. Amer. Math. Soc., *109* (1963), 146–164.
11. M. C. McCord, *On embedding manifold-like continua in manifolds*, Amer. Math. Soc. Not., *12* (1965), 138.
12. ——— *Universal $\mathscr{P}$-like compacta*, Michigan Math. J., *13* (1966), 71–85.

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