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# ON TRANSLATION PLANES WHICH ADMIT SOLVABLE AUTOTOPISM GROUPS HAVING A LARGE SLOPE ORBIT

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**0. Introduction.** Our main object is to prove the following result.

THEOREM C. *Let* **A** *be an affine translation plane of order* $q^r \geqq q^2$ *such that* $l_\infty$, *the line at infinity, coincides with the translation axis of* **A**. *Suppose* $G$ *is a solvable autotopism group of* **A** *that leaves invariant a set* $\Delta$ *of* $q + 1$ *slopes and acts transitively on* $l_\infty \setminus \Delta$.

*Then the order of* **A** *is* $q^2$.

An autotopism group of any affine plane **A** is a collineation group $G$ that fixes at least two of the affine lines of **A**; if in fact the fixed elements of $G$ form a subplane of **A** we call $G$ a *planar* group. When **A** in the theorem is a Hall plane [**4**, p. 187], or a generalized Hall plane ([**13**]), $G$ can be chosen to be a planar group. But there are also many planes, which satisfy the hypothesis of the theorem, in which it is impossible to choose $G$ to be a planar group; for instance deriving a Walker plane [**19**], or a suitably chosen semifield plane [**9**, Section 4], leads to such examples. However, when $q$ is a prime, then the only known possibilities for **A** in Theorem C are the Hall planes, the derived Walker planes (when $q \equiv -1 \mod 6$) and the recently discovered Cohen-Ganley planes [**1**], which exist whenever $q \equiv \pm 2 \mod 5$.

The main corollary of Theorem C may be described in terms of spreads in projective spaces. Recall that a spread $\Sigma$ in a projective space $\mathbf{P} = PG(2r - 1, q)$ is a collection of $r - 1$ dimension subspaces such that every point of **P** lies in a unique member (or 'component') of $\Sigma$. Also Aut $\Sigma$ is the group of collineations of **P** that permutes the components of $\Sigma$ among themselves. Theorem C yields the following characterization of the spreads associated with the finite Hall planes.

COROLLARY D. *Let* $\Sigma$ *be a spread in* $\mathbf{P} = PG(2r - 1, q)$ *with* $r > 1$. *Suppose* $G$ *is a solvable subgroup of* Aut $\Sigma$ *that fixes individually each member of a set* $\Delta$ *consisting of* $q + 1$ *components of* $\Sigma$. *Then* $G$ *is transitive on* $\Sigma \setminus \Delta$ *only if all the following conditions hold.*

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(i) $\mathbf{P} = PG(3, q)$;
(ii) $\Delta$ *is a regulus*; *and*
(iii) $(\Sigma \setminus \Delta) \cup \Delta'$ *is a regular* (*Desarguesian*) *spread where* $\Delta'$ *is the opposite regulus of* $\Delta$.

*Remark.* If the solvability hypothesis on $G$ is dropped then the only known counterexamples to Corollary D are two well-known spreads in $PG(7, 2)$: the Lorimer-Rahilly spread [**16**] and its transpose the Johnson-Walker spread. If additionally we replace the assumption that $G$ acts trivially on $\Delta$, by the weaker assumption that $G$ fixes $\Delta$ globally, then there arise many infinite families of counterexamples to the corollary (Section 4).

The proof of our main result (Theorem C) follows from a study of certain autotopism groups undertaken in Section 2. Specifically, Section 2 considers an affine translation plane $\mathbf{A}$ of order $q^r > q^2$ which admits an autotopism group $H$ of order $u^\alpha p^\beta$ where
(i) $p$ is the characteristic of $\mathbf{A}$; and
(ii) $u$ is a $p$-primative divisor of $q^{r-1} - 1$.
Our main conclusion is that when $q^r \neq 16$, $H$ is a planar group. This fact allows us to gain further information about $H$ that we require in the proof of Theorem C. A side effect of our analysis has a slight bearing on an old conjecture of Hughes [**4**, p. 178] which asserts that the full autotopism group of a finite semifield plane must be solvable.

COROLLARY 2.8. *Let* $G$ *be the autotopism group of a finite semifield plane of order* $p^r > p^2$, *where* $p$ *is a prime. Suppose that* $u$ *is a primitive divisor of* $p^{r-1} - 1$ *such that* $pu$ *divides* $|G|$.
*Then* $G$ *is a non-solvable group.*

The following result a consequence of Foulser's dimension theorem for subplanes [**3**, Corollary 3.5], is required in the proof of Theorem B.

THEOREM A. *Let* $\mathbf{A}$ *be an affine translation plane of order* $q^r > q$ *and characteristic* $p$. *Suppose* $P$ *is a planar* $p$-*group of* $\mathbf{A}$ *such that its fixed plane* $\mathbf{A}_P$ *has order at least* $q$.
*Then* $|P| \geqq q^{r-1}$ *is only possible when* $q^r = 16$ *or* $\mathbf{A}_P$ *is a Desarguesian Baer subplane of* $\mathbf{A}$.

Special cases of Theorem A follow from [**5**, Section 6] (e.g., when $\mathbf{A}_p$ is a kern plane or $\mathbf{A}_p \cap l_\infty$ is a Desarguesian net). In particular the following known instance of Theorem A ([**2**, Corollary 3]) will be needed in its proof.

1. RESULT. *Let* $\mathbf{B}$ *be a Baer subplane of an affine translation plane* $\mathbf{A}$ *of order* $n^2$. *Suppose* $\mathbf{A}$ *admits a collineation group of order* $n$ *that fixes* $\mathbf{B}$ *elementwise. Then* $\mathbf{B}$ *is a Desarguesian plane.*

We shall need to assume that the reader is familiar with translation planes and their connection with spreads and quasifields [**4**, **15**, **16**]. Apart from standard notation we wish to emphasize the following

*Conventions.* (a) Let $G$ be a permutation group of the finite set **A**. Then $G_X$ denotes the elementwise stabilizer of $X \subseteq A$ and $\mathscr{F}(G)$ is the set of fixed points of $G$. But if **A** is an affine plane and $G$ is a planar group we usually write $\mathbf{A}_G$, instead of $\mathscr{F}(G)$, for the fixed plane of $G$.

(b) The integer $q > 1$ is always a power of the prime $p$ and a Sylow $p$-subgroup of any finite group $G$ is called an $S_p$ subgroup of $G$. If $u$ is another prime then a $\{u, p\}$ subgroup of $G$ is required to have order $u^\alpha p^\beta$, where $\alpha$ and $\beta$ are integers.

(c) If **A** is an affine plane then we denote its line at infinity by $l_\infty$ and we call **A** an *affine translation plane* if the group of $l_\infty$ elations is transitive on the affine points of **A**.

*Remark.* In Theorem C we considered planes of order $q^r$. In this theorem, and in fact throughout the paper, there is no need to consider $r$ to be an integer; it is sufficient for $r$ to be a positive rational number (usually $\geqq 2$) such that $q^r$ is an integer.

**1. An upper bound for planar $p$-groups.** The object of this section is to prove Theorem A. We do this using an inductive argument based on the following theorem of Foulser [**3**, Corollary 3.5].

1. Dimension Theorem. *Let* $\Pi$ *be a $p$-group of automorphisms of a finite quasifield $Q$, whose characteristic is $p$. Then* $\dim \mathscr{F}(\Pi) | \dim Q$, *where dimensions are given relative to the prime field in $Q$.*

We shall carry out our induction on the '$\rho$-triples' defined below.

2. *Definition.* Let $q$ be a power of the prime $p$ and suppose $r > 1$. Then $(q, r, p)$ is a *$\rho$-triple* if there exists $(Q, F, \Pi)$ satisfying the following conditions:

(i) $Q$ is a quasifield of order $q^r$ and $F$ is a subquasifield of order $q$; and

(ii) $\Pi$ is a nontrivial $p$-group in $(\mathrm{Aut}\, Q)_F$ such that $\mathscr{F}(\Pi) = F$ and $|\Pi| \geqq q^{r-1}$.

(Note. Foulser's dimension theorem mentioned above forces $r$ to be an integer.)

3. Proposition. *The only $\rho$-triples are those of type* $(q, 2, p)$ *or* $(2, 4, 2)$.

*Proof of Proposition* 3. We proceed by induction on $r$. Let $r = R$ $(> 2)$ be the smallest integer associated with a counterexample to the proposition. This means that there exists a $\rho$-triple $(q, R, p) \neq (2, 4, 2)$. Let $(Q, F, \Pi)$ be chosen to satisfy conditions 2(i) and (ii), relative to $(q, R, p)$.

Also choose $V$ to be a $GF(p)$ subspace of $(Q, +)$ satisfying the following requirements:

(1) $|V| = pq$;

(2) $V \supset F$; and

(3) $V$ is left invariant by the group $\Pi$. (The existence of $V$ follows from the fact that the number of subspaces of $(Q, +)$ that satisfy (1) and (2) is relatively prime to $|\Pi|$.)

We now break up our argument into a series of lemmas and the notation introduced in each lemma is in force until Proposition 3 is proved.

4. LEMMA. *Let* $\Pi_1$ *be the kernel of the restriction map* $\alpha:\Pi \rightarrow \Pi|V$. *Also let* $F_1 = \mathscr{F}(\Pi_1)$. *Then*

(a) $|\Pi_1| \geqq q^{R-2} > 1$;

(b) $Q \supset F_1 \supset F$ (*and the subspace* $F_1$ *is also a subquasifield of* $Q$);

(c) $\Pi$ *leaves* $F_1$ *invariant.*

*Proof.* Since $\alpha(\Pi)$ is semiregular on $V \setminus F$ we have

$$|\Pi_1| = |\Pi| \ / \ |\alpha(\Pi)\,| \geqq |\Pi|/q.$$

Now (a) follows because our hypothesis states $|\Pi| \geqq q^{R-1}$ and $R > 2$. Part (b) is immediate and (c) is valid because $\Pi_1$ is normal in $\Pi$.

5. LEMMA. *There exist integers* $r$, $t$ (*both* $>1$) *such that*

(a) $R = rt$; (b) $|Q| = |F_1|^r$; *and*

(c) $|F_1| = |F|^t = q^t$.

*Proof.* Let $\Pi_2$ be the subgroup of $(\text{Aut } F_1)_F$ induced by $\Pi$ on the $\Pi$ invariant quasifield $F_1$, defined in Lemma 4. Thus by definition 2(ii), $\mathscr{F}(\Pi_2) = F$ and so the dimension theorem (Result 1) yields (c). Part (b) follows if the dimension theorem is applied to $\Pi_1$, since $\mathscr{F}(\Pi_1) = F_1$. Now (a) follows from (b) and (c).

6. LEMMA. $F_1$ *is a Baer extension of* $F$.

*Proof.* Otherwise Lemma 5(c) shows that $t \geqq 3$ and so Lemmas 4(a) and 5 imply:

(i) $\Pi_1 \geqq q^{R-2} = q^{t(r-(2/t))} > |F_1|^{r-1}$.

Hence $(|F_1|, r, p)$ qualifies as a $\rho$-triple relative to $(Q, F_1, \Pi_1)$, because of Lemma 5(b). To avoid contradicting our inductive hypothesis we therefore must have $r = 2$, since $|F_1| \neq 2$. Now $Q$ is a Baer extension of $F_1$ and the semiregularity of $\Pi_1$ on $Q \setminus F_1$ shows that $|\Pi_1| \leqq |F_1|$. But this contradicts statement (i) because $r = 2$. So the lemma is valid.

7. LEMMA. $|Q| = |F_1|^2 = |F|^4$.

*Proof.* By Lemmas 5 and 6, $|Q| = |F_1|^{R/2}$ where $R/2$ is an integer. But Lemmas 4(a) and 6 show that

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$$|\Pi_1| \geqq q^{R-2} = |F_1|^{(R/2-1)}.$$

Thus $(Q, F_1, \Pi_1)$ gives a $\rho$-triple $(|F_1|, R/2, p)$, contrary to our inductive hypothesis unless $R/2 = 2$ or $(|F_1|, R/2, p) = (2, 4, 2)$. But $|F_1| > 2$ and so $R = 4$, as stated in the lemma.

8. COROLLARY. *$|\Pi_1| = q^2$ and the restriction map $\Pi \rightarrow \Pi|F_1$ induces a group of order $q$ on $F_1$.*

*Proof.* By Lemmas 4(a) and 7 we have $|\Pi_1| \geqq q^2$. But Lemma 7 also implies that $Q$ is a Baer extension of $F_1$ and so we can only have $|\Pi_1| = q^2$. By Lemma 4 $\Pi_1$ is the kernel of the restriction map $\Pi \rightarrow \Pi|F_1$ and so the corollary follows.

Result 0.1 when applied to Corollary 8 shows that $F_1 = GF(q^2)$. Hence the restriction $\Pi|F_1$ has order $\leqq 2$ and so Corollary 8 forces $q = 2$. Now Lemma 7 gives $(q, R, p) = (2, 4, 2)$, contrary to our inductive hypothesis. Hence Proposition 3 has been proved.

We now deduce the main result of this section using Proposition 3.

THEOREM A. *Let the quasifield $Q$ have order $q^r$ where the rational number $r > 1$ and let $p$ be the characteristic of $Q$. Suppose $\Pi$ is a $p$-group in* Aut $Q$ *such that*

(i) $|\Pi| \geqq q^{r-1}$; *and*

(ii) $|\mathscr{F}(\Pi)| \geqq q$.

*Then either $Q$ is a Baer extension of $GF(q)$ or $\mathscr{F}(\Pi) = GF(2)$ and $|Q| = 16$.*

*Proof.* We may choose a rational number $m \geqq 1$ such that $|\mathscr{F}(\Pi)| = q^m$. So by Foulser's dimension theorem (Result 1) there is a positive integer $R$ such that $mR = r$. Hence

(i) $|\Pi| \geqq q^{r-1} = q^{m(R-1/m)} \geqq |\mathscr{F}(\Pi)|^{R-1}$.

Thus we have a $\rho$-triple $(|\mathscr{F}(\Pi)|, R, p)$ and so Proposition 3 gives $|Q| = 16$, $|\mathscr{F}(\Pi)| = 2$ or $R = 2$. In the latter event $\mathscr{F}(\Pi)$ is a Baer quasifield and now the semiregularity of $\Pi$ on $Q \setminus \mathscr{F}(\Pi)$ shows $|\Pi| \leqq |\mathscr{F}(\Pi)|$. Thus the relations (i) collapse into equalities and so $|\mathscr{F}(\Pi)| = q$. Result 0.1 shows that $\mathscr{F}(\Pi)$ is a field and so Theorem A is proved.

**2. Autotopism $\{u, p\}$ groups.** Throughout this section **A** is an affine translation plane of order $q^r > 16$ and characteristic $p$. In addition we shall always assume that $q^{r-1} - 1$ possesses a primitive divisor $u$; thus $u$ is a prime divisor of $q^{r-1} - 1$ but not of $p^s - 1$ whenever $q^{r-1} > p^s \geqq p$. Our object is to study autotopism $\{u, p\}$ subgroups of **A**, because of the relevance of such groups to Theorem C. Such a group $H$ need not be planar if $q^r = q^2$; for instance a Hall plane of order $q^2$ contains a nonplanar autotopism group of order $pu$ (generated by a Baer $p$-element and a kern homology of order $u$). However, if $q^r > q^2$ then we shall show that $H$ must be a planar group and, when $q^{r-1} \nmid |H|$, we also have $H \subseteq \Gamma L(1, q^{r-1})$; in fact considerable information about $H$ can be gained even when $q^{r-1} \| H|$, but we shall not analyse this situation as it is not relevant to our main objectives. We summarize most of our conclusions in the following result.

THEOREM B. *Let $H$ denote an autotopism group of the plane* **A** *and assume that $|H| = u^\alpha p^\beta$, with $\alpha\beta \neq 0$. Also let $P$ and $U$ denote (resp.) $S_p$ and $S_u$ subgroups of $H$. Then the following statements are valid provided that $q^r > q^2$.*

(a) *$H$ is a planar group.*

(b) *Suppose $U \supseteq V \neq 1$. Then $\mathbf{A}_U = \mathbf{A}_V$ and both planes have order $q$. Moreover $U$ is cyclic such that $C_H(V) = U$.*

(c) *The following conditions are pairwise equivalent.*

(i) *$\mathbf{A}_U \cap l_\infty$ is $H$ invariant;*

(ii) *$H$ contains a non-trivial normal $u$-group;*

(iii) *$U \lhd H$;*

(iv) *$H \subseteq \Gamma L(1, q^{r-1})$;*

(v) *$q^{r-1} \nmid |H|$;*

(vi) *$P$ fixes some points of $l_\infty \setminus (l_\infty \cap \mathbf{A}_U)$.*

*Remarks.* (i) As far as Theorem C is concerned, the only bit of part (c) that we require is (i) $\Rightarrow$ (vi); however the proof of this fact involves proving most of the other implications of part (c).

(ii) Applications of Theorem B to planes with shears are considered at the end of this section.

The main tool in the proof of Theorem B is the following lemma on vector spaces.

1. LEMMA. *Let* **V** *be an elementary abelian group of order $q^r \geqq q^2$ and suppose $U$ is any non-trivial $u$-group in* Aut (**V**, +). *Then the following statements are valid.*

(a) $|\mathscr{F}(U)| = q$.

(b) *$U$ is semiregular on $\mathbf{V} \setminus \mathscr{F}(U)$.*

(c) *$U$ is cyclic.*

(d) *if $r > 2$ then $\mathbf{V} = \mathscr{F}(U) \oplus C_U$ where $C_U$ is the only $U$ submodule of* **V** *disjoint from $\mathscr{F}(U)$.*

(e) *If $r > 2$ and* **W** *is a $U$-submodule of* **V** *then either $\mathbf{W} \subseteq \mathscr{F}(U)$ or $|\mathbf{W}| \geqq q^{r-1}$.*

*Proof.* By Maschke's theorem [**17**, Theorem 15.1] $\mathbf{V} = \mathscr{F}(U) \oplus C$ where $C$ is some $U$ module. As $U$ is fixed point free on $C$ we get

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$$u \Bigg| \left(\frac{q^r}{p^m} - 1\right) \quad \text{where } p^m = |\mathscr{F}(U)|.$$

Since $u$ is also a primitive divisor of $q^{r-1} - 1$ the condition above implies that $q \geqq p^m$ and also that

$$u \Bigg| \left(\frac{q^r}{p^m} - q^{r-1}\right).$$

Now we contradict the primitivity of $u$ unless $q = p^m$. Hence part (a) is valid and part (b) follows immediately by applying part (a) to the cyclic subgroups of $U$. In particular $U$ is faithful and semiregular on $C$. Hence $U$ is a Frobenius complement [**16**, Lemma 4.2]. Since $u$ is also odd (being a primitive divisor), the Frobenius complement $U$ is cyclic [**17**, Theorem 18.1 (4) ] and part (c) is verified. To prove part (d) we assume $r > 2$. The existence of the $U$-module $C_U$ is again guaranteed by Maschke's theorem and so to prove the uniqueness assume there are two distinct $U$-modules $C_1$ and $C_2$ such that

$$\mathbf{V} = \mathscr{F}(U) \oplus C_i \quad \text{for } i = 1, 2.$$

Now $C_1 \cap C_2 \neq 0$ because then $q^r \geqq q^{2(r-1)}$, contradicting the condition $r > 2$. But since $U$ is fixed point free on the non-zero points of $C_1 \cap C_2$ we now find

$$u \Big| \left( |C_1 \cap C_2| - 1\right)$$

and so by the primitivity of $u$ we have

$$|C_1 \cap C_2| \geqq q^{r-1}.$$

But each $C_i$ has order $q^{r-1}$ and so we contradict the assumption that $C_1 \neq C_2$. Hence (d) is valid. Finally, to verify (e), assume that a $U$-module $\mathbf{W} \not\subseteq \mathscr{F}(U)$. Now Maschke's theorem shows that $\mathbf{W}$ contains a subspace $\mathbf{X}$ ($\neq 0$) on which $U$ is fixed point free. So by the primitivity of $u$ we again have $|\mathbf{X}| \geqq q^{r-1}$. Hence the lemma is valid.

It is more convenient to apply Lemma 1 to spreads of order $q^r$, rather than directly to the translation plane **A**. So the next few lemmas are concerned with a spread theoretic version of Theorem B.

2. LEMMA. *Let $\Gamma$ be a spread of order $q^r > q^2$ admitting an autotopism group $H$ of order $u^\alpha p^\beta$ where $\alpha\beta \neq 0$. Then*

(i) *$H$ is a planar group*;

(ii) *the fixed plane of any nontrivial $u$-subgroup of $H$ has order $q$*;

(iii) *$H$ acts faithfully on $\lambda$, where $\lambda$ denotes one of the components of $\Gamma$ that is left invariant by $H$*: *and*

(iv) *the commutator $[\sigma, \theta] \neq 1$ whenever $\sigma$ and $\theta$ are nontrivial $p$ and $u$ elements (resp.) in $H$.*

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*Proof.* As $H$ is an autotopism group we may assume it leaves invariant at least two distinct components of $\Gamma$, which we shall label $\lambda$ and $\mu$. Let $U \neq 1$ be any $u$-group in $H$ and write

$$L = \lambda \cap \mathscr{F}(U) \quad \text{and} \quad M = \mu \cap \mathscr{F}(U).$$

Now by Lemma 1, $|L| = q = |M|$ and so part (ii) is valid. To prove that $H$ itself is planar we first consider the following situation.

Case A. When $H$ contains a nontrivial normal $u$-subgroup $U$.

Let $\bar{U}$ be an $S_u$ subgroup of $H$ that contains $U$ and let $P$ be any $S_p$ subgroup of $H$. Now by part (ii) the plane $\mathbf{A}_{\bar{U}} = \mathbf{A}_U$ and so $P$ leaves $\mathbf{A}_{\bar{U}}$ invariant. But the $H$ invariant components $\lambda$ and $\mu$ are in $\mathbf{A}_{\bar{U}}$ and so $P$ induces a planar group on $\mathbf{A}_{\bar{U}}$. Hence $H = \langle P, \bar{U} \rangle$ is also a planar group. Now to complete the proof of part (i) it remains to consider the negation of case A.

Case B. $H$ does not contain any normal nontrivial $u$-group.

As $H$ is solvable it now must contain an elementary abelian normal $p$-group $K \neq 1$. Now $K$ is certainly planar and $H$ leaves $\mathbf{A}_K$ invariant and hence also $k = \lambda \cap \mathscr{F}(K)$. Now by Lemma 1(e) either $k \subseteq \mathscr{F}(\bar{U}) \cap \lambda$ or $|k| \geqq q^{r-1}$, whenever $\bar{U}$ is any $S_u$ subgroup of $H$. But since $r > 2$, $|k| \geqq q^{r-1}$ contradicts the Baer condition for $\mathbf{A}_K$ and so

$$k \subseteq \mathscr{F}(\bar{U}) \cap \lambda.$$

Similarly

$$\mu \cap \mathscr{F}(K) \subseteq \mathscr{F}(\bar{U}) \cap \mu$$

and so $\mathbf{A}_K \subseteq \mathbf{A}_{\bar{U}}$. Now any $S_p$ subgroup $P$ normalizes $K$ and so induces a planar group on $\mathbf{A}_K$. Hence $H = \langle \bar{U}, P \rangle$ fixes elementwise a subplane of $\mathbf{A}_U$ and part (i) is proved. Part (iii) is an immediate corollary. Finally, to verify part (iv), assume that $\sigma\theta = \theta\sigma$. Now by Lemma 1(d)

$$\lambda = \mathscr{F}(\theta) \oplus C_\theta$$

where $C_\theta$ is the unique $\theta$ module in $\lambda$ disjoint from $\mathscr{F}(\theta)$. The uniqueness of $C_\theta$ shows that its normalizer $\sigma$ also fixes $C_\theta$ and hence

$$|\mathscr{F}(\sigma) \cap C_\theta| > 1.$$

But now $\theta$ leaves $\mathscr{F}(\sigma) \cap C_\theta$ invariant and so by Lemma 1(e),

$$|\mathscr{F}(\sigma) \cap C_\theta| \geqq q^{r-1},$$

contradicting the Baer condition for the plane $\mathbf{A}_\sigma$. Hence the proof of the lemma is complete.

Until the proof of Theorem B is complete we shall continue assuming the notation and hypothesis of Lemma 2.

LEMMA 3. *If $U$ is an $S_u$ subgroup of $H$ then the following conditions are pairwise equivalent.*

(1) $U \lhd H$.

(2) *$H$ contains a normal $u$-group* $\neq 1$.

(3) $H \subseteq \Gamma L(1, q^{r-1})$.

(4) $q^{r-1} \nmid |H|$.

*Proof.* (1) $\Rightarrow$ (2) is vacuous. To prove (2) $\Rightarrow$ (3) assume that $U_0 \lhd H$ where $U_0$ is a nontrivial $u$-group in $H$. Now by Lemma 1, $\lambda = F_0 \oplus C_0$ where $F_0 = \mathscr{F}(U_0) \cap \lambda$ and $C_0$ is the only nonzero $U_0$ module disjoint from $F_0$. So the planar group $H$ leaves $C_0$ invariant and, because of the Baer condition for subplanes, acts faithfully on $C_0$. Since $U_0$ is a cyclic (Lemma 1(c) ) normal subgroup of $H$ which acts irreducibly on $C_0$ we find [**17**, Proposition 19.8] that $H \subseteq \Gamma L(1, q^{r-1})$. Hence (2) $\Rightarrow$ (3) is valid while (3) $\Rightarrow$ (1) and (4) are easily verified. It is now sufficient to verify that (4) $\Rightarrow$ (2). Assume (2) is false. Since $H$ is solvable it must now contain a normal $p$-group $K \neq 1$ and now Lemma 2(iv) implies that $u|(|K| - 1)$. Hence the primitivity of $u$ implies that $q^{r-1}||K|$, contrary to the hypothesis of condition (4). Hence the lemma is valid.

We now use Theorem A to extend the list of equivalent conditions given in the previous lemma.

LEMMA 4. *Let $U$ be an $S_u$ subgroup of $H$. Then the following conditions are pairwise equivalent, when* $q^r \neq 16$.

(1) $U \lhd H$.

(2) *The plane $\mathbf{A}_U$ is invariant under $H$.*

(3) *$\mathbf{A}_U \cap l_\infty$ is $H$ invariant.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are immediate, while (3) $\Rightarrow$ (2) follows from the fact that $(\mathbf{A}_U \cap l_\infty) \cup \mathbf{A}_H$ is a generating set for $\mathbf{A}_U$. Finally, to verify (2) $\Rightarrow$ (1), assume (2) and consider the kernel $N$ of the restriction map $H \to H|\mathbf{A}_U$. Since $U$ is also an $S_u$ subgroup of $N$ it is sufficient to check that $U \lhd N$. If $U = N$ we are done, so assume that $up||N|$. Now by Lemma 3, applied to $N$, we have $q^{r-1}||N|$ unless $U \lhd N$. Thus (1) is false only when the elementwise stabilizer of $\mathbf{A}_U$ is divisible by $q^{r-1}$. But $\mathbf{A}_U$ has order $q$ and so Theorem A can be applied. This yields $q^r = q^2$ or 16, contrary to our assumptions. The lemma follows.

It will now be convenient to state the following simple fact on projective planes.

*Remark* 5. Suppose $\mathbf{P}_1$ and $\mathbf{P}_2$ are subplanes of a projective plane $\mathbf{P}$ and that they intersect in a (nondegenerate) subplane $\mathbf{P}_0$. Also let $l$ by any line of $\mathbf{P}_0$. Then

$$\mathbf{P}_1 \cap l \supseteq \mathbf{P}_2 \cap l \Rightarrow \mathbf{P}_1 \supseteq \mathbf{P}_2.$$

*Proof*. Each $\mathbf{P}_i$ is generated by the points of $(\mathbf{P}_i \cap l) \cup \mathbf{P}_0$.

LEMMA 6. *Suppose $U$ is an $S_u$ subgroup and $P$ an $S_p$ subgroup of $H$. Then the following conditions are equivalent.*

(1) *$P$ fixes some point of $l_\infty \setminus (l_\infty \cap \mathbf{A}_U)$.*

(2) *$l_\infty \cap \mathbf{A}_U$ is $H$ invariant.*

*Proof*. Assume if possible that (1) holds while (2) is false. Now appropriate conditions listed in Lemmas 3 and 4 show that $H$ does not contain a normal $u$-subgroup $\neq 1$. So by the solvability of $H$ it contains a nontrivial normal $p$-group $K$. Since $P$ contains $K$, condition (1) shows that $\mathbf{A}_K \neq \mathbf{A}_U$. Hence

$$\mathbf{A}_K \cap \lambda \neq \mathbf{A}_U \cap \lambda,$$

e.g., use Remark 5. But since $K \lhd H$, $\mathbf{A}_K \cap \lambda$ is now seen to be a $U$-submodule of $\lambda$ distinct from $\mathbf{A}_U \cap \lambda$. Now Lemma 1(e) contradicts the Baer condition for the plane $\mathbf{A}_K$. Hence (1) $\Rightarrow$ (2) is valid. To prove the converse assume $l_\infty \cap \mathbf{A}_U$ is $H$-invariant. Now by Lemma 4, $U \lhd H$ and so by Lemma 1(d),

$$\lambda = (\mathbf{A}_U \cap \lambda) \oplus C_U$$

where $C_U$ is $H$ invariant. But now clearly $\mathscr{F}(P) \cap C_U \neq 0$ and so $\mathbf{A}_P \nsubseteq \mathbf{A}_U$. Hence Remark 5 shows that (1) must occur and so the lemma is proved.

It is clear that Lemmas 1 to 6 constitute a proof of Theorem B. When $q = p$ we can sharpen the conclusions of Theorem B by showing that $\mathbf{A}$ satisfies the additional properties listed below: in particular $\mathbf{A}$ must now have odd order.

7. COROLLARY. *Let $\mathbf{A}$ be an affine translation plane of order $p^r >$* Max $(p^2, 16)$ *and suppose $u$ is a primitive divisor of $p^{r-1} - 1$. Assume $\mathbf{A}$ admits an autotopism group of order $u^\alpha p^\beta$ with $\alpha\beta \geqq 1$. Then all the following statements are valid*:

(i) $p > 2$;

(ii) $p \mid r - 1$; *and*

(iii) $\mathbf{A}$ *does not admit affine elations.*

*Proof*. If $\mathbf{A}$ admits a planar $p$-group of order $p^{r-1}$ then Theorem A leads to a contradiction. Otherwise Theorem B implies that our autotopism group of order $u^\alpha p^\beta$ is in $\Gamma L(1, p^{r-1})$. This is only possible if $p \mid r - 1$ and so (ii) holds. Now $p = 2$ is impossible because $\mathbf{A}$ cannot admit Baer involutions. Hence (i) applies. By [**3**, Theorem 4.2], affine elations and planar $p$-elements cannot coexist in $\mathbf{A}$ unless $p \mid r$, contradicting (ii). So the corollary is valid.

A consequence of Corollary 7 (that we shall not use) is that autotopism $(u, p)$ groups tend to be nonsolvable, especially in planes admitting shears.

8. COROLLARY. *Let $u$ be a primitive divisor of $p^{r-1} - 1$ and assume $p^r >$ Max $(p^2, 16)$. Suppose* **A** *is an affine translation plane admitting an autotopism group $G$ such that $pu||G|$.*

*Then $G$ is nonsolvable if $p = 2$ or* **A** *admits affine elations.*

*Proof.* If $G$ is solvable it has a Hall $(u, p)$ subgroup [**17**, Theorem 11.1] to which we may apply Corollary 7.

**3. Main theorems.** We shall now prove Theorem C and Corollary D. The following simple fact will be used in our proof.

*Remark* 1. Let $T$ be a finite transitive group acting on a set $\Omega$ and let $m$ be any prime divisor of $|\Omega|$. Then the $S_m$ subgroups of $T$ are fixed point free on $\Omega$.

LEMMA 2. *Let* **A** *be an affine translation plane with characteristic $p$ and order $q^r > q^2$. Suppose $G$ is a solvable autotopism group of* **A** *that leaves invariant a set $\Delta$ of $q + 1$ slopes and acts transitively on $l_\infty \setminus \Delta$. Then $q^{r-1} - 1$ cannot have a primitive divisor when $q^r \neq 16$.*

*Proof.* To get a contradiction suppose $u$ is a primitive divisor of $q^{r-1} - 1$. The transitivity of $G$ now implies that $pu$ is a divisor of $|G|$ and so, by the solvability hypothesis, $G$ contains a Hall subgroup $H$ of order $u^\alpha p^\beta$ with $\alpha\beta \geqq 1$ [**17**, Theorem 11.1]. By Theorem B, parts (a) and (b), we find that the $S_u$ subgroups of $H$ are planar groups with fixed planes of order $q$. But now Remark 1 implies that every $S_u$ subgroup of $H$ fixes $\Delta$ identically. Next consider the action of **P**, an $\mathbf{S}_p$ subgroup of $H$, on the line $l_\infty$. This time Remark 1 shows that

$$\mathscr{F}(P) \cap l_\infty \subset \Delta$$

and so condition c(vi) of Theorem B fails. So Theorem B (cf. condition c(i) ) shows that if $U$ is an $S_u$ subgroup of $H$ then $H$ cannot leave $\Delta = \mathbf{A}_U \cap l_\infty$ invariant. This contradicts the invariance of $\Delta$ under $G$ and so the lemma is proved.

We can now complete the proof of our main result using the argument of [**6**, Proposition 3.5].

THEOREM C. *Let* **A** *be an affine translation plane of order $q^r \geqq q^2$. Suppose $G$ is a solvable autotopism group of* **A** *that leaves invariant a set $\Delta$* of *$q + 1$ slopes and acts transitively on $l_\infty \setminus \Delta$. Then* **A** *is a plane of order $q^2$.*

*Proof.* When **A** has order 16 the theorem can be verified by using the techniques of Johnson, Ostrom and Walker (e.g. [**10**] ). (Alternatively, this case can be handled by using the recently completed classification of all translation planes of order 16, due to Riefart and Dempwolff [**18**].) So to get a contradiction we may assume that

$q^r > \text{Max}(16, q^2)$.

Now Lemma 2 shows that $q^{r-1} - 1$ has no primitive divisors and so by Zsigmondy's theorem ( [**20**], [**16**, p. 63] ), $q^r = p^3$ where $p$ is a Mersenne prime. Thus $G$ contains a 2-group $S$ of order $2^{x+1}$ where $p + 1 = 2^x$. Now consider the action of $S$ on one of the sides $l$ of the autotopism triangle associated with $G$. Since $l$ has $p^3 - 1$ nonzero affine points and $2 \| p^3 - 1$, we may conclude that $S$ has an orbit of length at most 2 in this set of $p^3 - 1$ points. Hence $G$ has a 2-subgroup $S_1$ of order $2^x$ which fixes at least three points in the projective closure of **A**. By similarly considering the action of $S_1$ on another side $m$, of the autotopism triangle fixed by $G$, we find that $S_1$ contains a subgroup $S_2$ of order $2^{x-1}$, such that $S_2$ fixes at least three points on $m$. Hence $S_2$ must be trivial as **A** cannot admit Baer involutions. Thus $p + 1 = 3$, contrary to our assumption that $q^r > 16$. The result follows.

It is obvious that the following corollary is equivalent to Corollary D mentioned in the introduction.

COROLLARY D. *In addition to the hypothesis of Theorem* C *assume that* $GF(q)$ *is in the kern of* **A** *and that* $G|\Delta$ = *identity. Then* **A** *is a Hall plane.*

*Proof.* By Theorem C **A** has order $q^2$. So every $S_p$ subgroup $S$ of $G$ is planar and $\mathbf{A}_S$ is a Baer subplane such that $\Delta$ is its slope set. There are now two cases to consider:

(i) $G$ leaves invariant a Baer subplane $\mathbf{A}_0$ such that $\mathbf{A}_0 \cap l_\infty = \Delta$; or

(ii) there exist distinct $S_p$ subgroups of $G$, say $S$ and $T$, such that $\mathbf{A}_S$ and $\mathbf{A}_T$ are distinct Baer subplanes (containing $\Delta$).

If possibility (i) occurs then the $p$-complement of $G$ leaves $\mathbf{A}_0$ invariant. There is no loss in generality if we allow $G$ to contain a group of kern homologies of order $q - 1$; hence the $p$-complement in $G$ contains a (Hall) subgroup $H$ whose order is divisible by $(q - 1)^2$. So the representation $H \to H|\mathbf{A}_0$ has kernel divisible by $q - 1$. Thus $G_{\mathbf{A}_0}$, the elementwise stabilizer of $\mathbf{A}_0$, is a group of order $q(q - 1)$ and so **A** must be a Hall plane (e.g., [**5**, Theorem 5] ).

Next consider case (ii). Obviously there are now $q + 1$ Baer subplanes across the partial spread associated with $\Delta$ and so **A** is derivable. Also $G$ inherts to a group $\bar{G}$ which contains several shears groups of order $q$. Now the Hering-Ostrom theorem [**15**, Theorem 35.10] contradicts the solvability of $G$ unless $q^2 = 3^2$ and $G = SL(2, 3)$. Hence **A** must be a Hall plane [**15**, Theorem 49.6]. This completes the proof of the corollary.

**4. Concluding remarks.** Bearing in mind the hypothesis of Theorem B, it is natural to consider the classification of all spreads $\Gamma$ which satisfy the following more general conditions.

*Hypothesis* $(H)$. $\Gamma$ is a spread in $PG(2r - 1, q)$ admitting an automorphism group $G$ such that Aut $G$ fixes globally a set $\Delta$ of $q + 1$ components and acts transitively on $\Gamma \setminus \Delta$.

There are now very many known spreads in $PG(3, q)$ that satisfy hypothesis $(H)$; many families of such examples can be constructed by using the procedure of Cohen and Ganley [**1**, Theorem 7.1]. Thus, contrary to our earlier expectations (e.g., see [**7**] or [**8**, problem A]), we now feel that the spreads in $PG(3, q)$, satisfying hypothesis $(H)$, are probably too numerous to classify. So we raise a slightly modified version of our earlier question [**8**, problem A].

*Problem* $(P)$. Classify all spreads $\Gamma$ satisfying hypothesis $(H)$ when $r \geqq 3$.

The case when $r = 3$ always occurs in Desarguesian spreads of order $2^{3s}$ with $G = SL(2, 2^s)$. Moreover, Kantor has recently constructed many families of spreads of even order $q^3$ (with $q = 2^{2m+1}$, $m > 1$) that also satisfy hypothesis $(H)$, with $G = SL(2, q)$ [**11**, Case (3), p. 252]. But when $r = 4$ there are only three known spreads which satisfy hypothesis $(H)$. These are the Lorimer spread of order 16, its transpose and the (unpublished) Denniston-Walker spread in $PG(7, 8)$. (In the last case, $G$ is a normal extension of $\mathbf{Z}_{73}$ by $SL(2, 8)$.) So as a supplement to problem $(P)$ we raise the following

*Question*. Does hypothesis $(H)$ imply $r \leqq 4$ and is $r \geqq 3$ only possible when $q$ is even?

Let us briefly reconsider hypothesis $(H)$ for $r \geqq 2$, when $q$ is prime. We now have far fewer examples. It turns out that if the order of $\Gamma$ exceeds 16 then the only known possibilities for $\Gamma$ are the "Cohen-Ganley systems" in $PG(3, p)$ and the spreads derived from them. Here by a Cohen-Ganley system we mean any spread of order $p^2$ constructed by the procedure described in Cohen et al. [**1**, Theorem 7.1]. At present there are only three known infinite families of C.G. systems, viz. Hall spreads, spreads derived from Walker spreads and the Cohen-Ganley spreads [**1**, Section 6]. Thus the following question related to problem $(P)$ may admit a complete solution, at least modulo the C.G. systems.

*Problem* $(Q)$. Classify all translation planes of order $p^r$ that admit collineation groups with a slope orbit of length $p^r - p$.

We end by noting that further work on the type of problems discussed in this article must take into account the bizzare translation planes of Kantor [**12**], that have order $q^3$ and exist whenever $q$ is a square prime power. Each of these planes admits a collineation group $G$ with a slope orbit of length $q^3 - q$ (cf. Theorem C) and yet they do not satisfy hypothesis ($H$), at least relative to $q$, because the corresponding kern subplanes have order $q^{3/2}$.

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