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The Uncomplemented Spaces $W(X,Y)$ and $K(X,Y)$

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Abstract. Classical results of Kalton and techniques of Feder are used to study the complementation of the space $W(X,Y)$ of weakly compact operators and the space $K(X,Y)$ of compact operators in the space $L(X,Y)$ of all bounded linear maps from X to Y.

M. Feder [7] showed that if X is an infinite dimensional Banach space and $c_0 \hookrightarrow Y$, then the space $K(X,Y)$ of all compact linear transformations (*i.e.,* compact operators) is not complemented in the space $L(X,Y)$ of all operators from X to Y. Emmanuele [5] and John [8] generalized this result and showed that if $c_0 \hookrightarrow K(X,Y)$, then $K(X,Y)$ is not complemented in $L(X,Y)$. The reader may consult [5–9] for a guide to the extensive literature dealing with this problem.

G. Emmanuele studied the space $W(X,Y)$ of all weakly compact operators [4]. Although Emmanuele noted that the presence of a copy of c_0 in $W(X,Y)$ does not preclude the complementation of $W(X,Y)$ in $L(X,Y)$, he did show that if Y contains a *complemented* copy of c_0 and (x_n^*) is a w^*-null sequence in X^* which is not weakly null, then $W(X,Y)$ is not complemented in $L(X,Y)$. Bator and Lewis [1, Theorem 4], removed the assumption that c_0 is complemented in Y and, in the process, strengthened Theorems 2 and 3 of [4].

Emmanuele [4] also showed that if ℓ_1 is *complemented* in Y and there exists a non-weakly compact operator $U\colon X \to \ell_1$, then $W(X,Y)$ is not complemented in $L(X,Y)$. Of course, if there is a non-weakly compact operator $U\colon X \to \ell_1$, then ℓ_1 is complemented in X [9, Proposition 2]. Emmanuele's result was generalized in [1], where it was demonstrated that if Y is any non-reflexive space and ℓ_1 is complemented in X, then $W(X,Y)$ is not complemented in $L(X,Y)$. In this note, unconditional basic sequences and techniques of Kalton [9] and Feder [7] are used to extend results in [1,4,5,8].

Throughout this note, X and Y denote real Banach spaces. Notation is consistent with that used in Diestel [2].

Theorem 1 (i) *If (y_n) is an unconditional and seminormalized basic sequence in Y and $U\colon X \to [y_n] \subseteq Y$ is any operator such that $\{U^*(y_n^*) : n \in \mathbf{N}\}$ is not relatively weakly compact, then $W(X,Y)$ is not complemented in $L(X,Y)$.*

(ii) *If (y_n) is an unconditional and seminormalized basic sequence in Y and $U\colon X \to [y_n] \subseteq Y$ is any operator such that $\{U^*(y_n^*) : n \in \mathbf{N}\}$ is not relatively compact, then $K(X,Y)$ is not complemented in $L(X,Y)$.*

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Proof The proofs of (i) and (ii) are essentially the same, *i.e.,* replace the phrase "(non) relatively weakly compact" with the phrase "(non) relatively norm compact." We provide the details for (i) and leave (ii) to the reader.

Suppose that $U, (y_n), X$, and Y are as in the hypothesis. Let

$$D = \{U^*(y_n^*) : n \in \mathbf{N}\},$$

and, without loss of generality, suppose that no subsequence from D converges weakly to a point in X^*.

Let X_0 be a separable subspace of X such that $[D]|_{X_0}$ is an isometry, and let R denote this restriction map. Let $J\colon [y_n] \to \ell_\infty$ be a linear isometry, and let $A\colon Y \to \ell_\infty$ be a norm-preserving extension of J. Define $T\colon \ell_\infty \to L(X,Y)$ by

$$T(b)(x) = \sum_n b_n y_n^*(U(x)) y_n.$$

Now suppose that $W(X,Y)$ is complemented in $L(X,Y)$, and let

$$P\colon L(X,Y) \to W(X,Y)$$

be a projection. Let (e_n) denote the canonical unit vector basis of c_0 and notice that $RAPT(e_n) = RAT(e_n)$ for each n. An application of [9, Proposition 5] produces an infinite subset K of $\mathbf{N}$ such that $RAPT(\phi) = RAT(\phi)$ for all $\phi \in \ell_\infty(K)$. However this is a contradiction since $T(\chi_K)_{|X_0}$ is not weakly compact $(T(\chi_K)^*(y_m^*) = U^*(y_m^*)$ for $m \in K)$, $A|_{[y_n]}$ is an isometry, and $RAPT(\chi_K)$ is weakly compact. ■

Remark. The identity operator on ℓ_1 shows that $\{U^*(y_n^*) : n \in \mathbf{N}\}$ may well be relatively weakly compact while U is a non-weakly compact operator.

Corollary 2 (i) *If $c_0 \hookrightarrow Y$ and there is a w^*-null sequence (x_n^*) in X^* which is not weakly null, then $W(X,Y)$ is not complemented in $L(X,Y)$. In fact, if $c_0 \hookrightarrow Y$ and there is any non-weakly compact operator $U\colon X \to c_0$, then $W(X,Y)$ is not complemented in $L(X,Y)$.*

(ii) *If $c_0 \hookrightarrow Y$ and X is infinite dimensional, then $K(X,Y)$ is not complemented in $L(X,Y)$.*

Compare [4, Theorems 2, 3]; [1, Theorem 4]; [7, Corollary 4].

Proof (i) For the first conclusion, let (y_n) be a copy in Y of (e_n), and define $U\colon X \to [y_n] \subseteq Y$ by $U(x) = \sum x_n^*(x) y_n$. Apply Theorem 1(i).

Now suppose that $U\colon X \to c_0 \subseteq Y$ is not weakly compact. Thus $U^*\colon \ell_1 \to X^*$ is not weakly compact. Since the closed and absolutely convex hull of the canonical unit vector basis (e_n^*) of ℓ_1 contains a non-empty open subset of ℓ_1, it follows that $\{U^*(e_n^*) : n \in \mathbf{N}\}$ is not relatively weakly compact. Apply Theorem 1(i) again.

(ii) The Josefson–Nissenzweig theorem [2] and the copy of c_0 in Y — precisely the argument used by Feder — automatically produce elements which satisfy the hypotheses of (ii) in the theorem. ■

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Theorem 3 (i) *If there is an unconditional basic sequence (x_n) in X such that $[x_n]$ is complemented in X and an operator $T\colon [x_n] \to Y$ such that $\{T(x_n) : n \in \mathbf{N}\}$ is not relatively weakly compact, then $W(X,Y)$ is not complemented in $L(X,Y)$.*

(ii) *If there is an unconditional basic sequence (x_n) in X such that $[x_n]$ is complemented in X and an operator $T\colon [x_n] \to Y$ such that $\{T(x_n) : n \in \mathbf{N}\}$ is not relatively compact, then $K(X,Y)$ is not complemented in $L(X,Y)$.*

Proof As above, details for the proof of (i) will be presented and the proof of (ii) will be left to the reader.

Suppose that $W(X,Y)$ is complemented in $L(X,Y)$. Consequently, $W([x_n],Y)$ is complemented in $L([x_n],Y)$. Now let $(x_{n_i}) = (b_i)$ be a subsequence of (x_n) such that no subsequence of $(T(b_i))$ converges weakly to a point of Y. Let $B = [b_i]$, and note that $W(B,Y)$ is complemented in $L(B,Y)$. Let $P\colon L(B,Y) \to W(B,Y)$ be a projection and let $L = T|_{[b_i]}$. Further, let $J\colon [T(x_n) \colon n \in \mathbf{N}] \to \ell_\infty$ be an isometric embedding, and let $A\colon Y \to \ell_\infty$ be a continuous linear extension of J.

Now define $S\colon \ell_\infty \to L(B,Y)$ by

$$S(\gamma)(b) = \sum \gamma_i b_i^*(b) L(b_i) = L\Big(\sum \gamma_i b_i^*(b) b_i\Big)$$

for $b \in B$ and $\gamma \in \ell_\infty$. Certainly $PS(\gamma)$ is weakly compact for each γ. Also,

$$S(e_i) = b_i^* \otimes L(b_i), \quad \text{and} \quad APS(e_i) = b_i^* \otimes JL(b_i) = AS(e_i)$$

for each i. Appealing to [9, Proposition 5] again, we obtain an infinite subset K of $\mathbf{N}$ such that $APS\chi_K = AS\chi_K$. However, $AS\chi_K(b_i) = JL(b_i)$ for $i \in K$, and $\{JL(b_i) : i \in K\}$ is not relatively weakly compact. ■

Corollary 4 *(See* [4, Theorem 5]; [1, Theorem 3]; [9, Lemma 3].*)*

(i) *If X contains a complemented copy of ℓ_1 and Y is not reflexive, then neither $W(X,Y)$ nor $K(X,Y)$ is complemented in $L(X,Y)$.*

(ii) *If X contains a complemented copy of ℓ_1 and Y is infinite dimensional, then $K(X,Y)$ is not complemented in $L(X,Y)$.*

Proof Every separable subspace of the (non-reflexive) space Y is a quotient of ℓ_1, and an operator $T\colon \ell_1 \to Y$ is (weakly) compact if and only if $\{T(e_n^*) : n \in \mathbf{N}\}$ is relatively (weakly) compact. ■

If (L_n) is a sequence in $K(X,Y)$ and $L\colon X \to Y$ is a non-compact operator such that $\sum_n L_n(x)$ converges unconditionally to $L(x)$ for each $x \in X$, then clearly $(\sum_{i=1}^n L_i)_{n=1}^\infty$ is not Cauchy in $L(X,Y)$. A re-blocking of the sequence (L_n) easily produces a non-null sequence (U_n) of compact operators which converges unconditionally in the strong operator topology to L. Consequently, the next theorem extends the main result in [7] and includes the main result in [5, 8].

Theorem 5 *If (T_n) is a sequence in $K(X,Y)$ such that $\sum_n |\langle T_n(x), y^*\rangle| < \infty$ for each $x \in X$ and $y^* \in Y^*$ and $||T_n|| \not\to 0$, then $K(X,Y)$ is not complemented in $L(X,Y)$.*

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Proof An appeal to Corollaries 2 and 4 above and [9, Theorem 4] shows that we may assume that c_0 does not embed in Y and that ℓ_∞ does not embed in $K(X,Y)$. Therefore, $\sum T_n(x)$ is unconditionally convergent in Y for each $x \in X$. Now let (T_{n_i}) be a subsequence of (T_n) such that if T is the operator defined by $T(x) = \sum_i T_{n_i}(x), x \in X$, then T is not compact. (If one assumes that all subsequences generate a compact operator, then let Σ be the σ-algebra of all subsets of $\mathbf{N}$ and use the Diestel–Faires theorem [3, p. 20] and the unconditional convergence of $\sum T_n$ in the strong operator topology to obtain a copy of ℓ_∞ in $K(X,Y)$.) Consequently, T has an unconditional compact expansion, and [7, Theorem 1] guarantees that $K(X,Y)$ is not complemented in $L(X,Y)$. ■

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