RADIUS OF CONVEXITY OF PARTIAL SUMS OF A CERTAIN POWER SERIES

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Let

$$f(z) = z + a_2 z^2 + \cdots, \tag{1}$$

be regular in the unit disc $E = \{z \,|\, |z| < 1\}$. G. Szegö [5] and Y. Miki [3] proved that if $f(z)$, given by (1), is univalent (starlike with respect to the origin; convex; close-to-convex in E) then any one of the partial sums

$$s_n(z) = z + \sum_{k=2}^{n} a_k z^k, \qquad n = 2, 3 \cdots, \tag{2}$$

is also univalent (starlike with respect to the origin; convex; close-to-convex) in $|z| < \frac{1}{4}$ and that the constant $\frac{1}{4}$ cannot be replaced by a larger one.

MacGregor in [2] considered the class R of functions of the form (1) that are regular in E and satisfy the condition that for z in E, $\operatorname{Re} f'(z) > 0$. It follows from the Noshiro-Warschawski theorem [4; 6] that functions of the class R are univalent in E. MacGregor showed that each partial sum of the form (2) of functions of the class R is univalent in $|z| < \frac{1}{2}$ and that each function of the class R maps $|z| < (\sqrt{2}-1)$ onto a convex domain; and the numbers $\frac{1}{2}$ and $(\sqrt{2}-1)$ are the best possible ones. In the present short note we consider the radius of convexity of partial sums of functions belonging to the class R. We establish:

THEOREM. *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in R$, then any one of the partial sums*

$$s_n(z) = z + \sum_{k=2}^{n} a_k z^k, \qquad n = 2, 3, \cdots,$$

is convex in $|z| < \frac{1}{4}$. The number $\frac{1}{4}$ cannot be replaced by a greater one.

We shall make use of the following estimates in the proof of the theorem.

If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in R$, then

$$|a_k| \leqq \frac{2}{k}, \qquad k = 2, 3, \cdots, \qquad ; [2] \tag{3}$$

$$(4)\qquad |f'(z)| \geqq \frac{1-r}{1+r}, \qquad |z| = r, \qquad 0 \leqq r < 1 \qquad ; [2]$$

$$(5)\qquad \left|\frac{zf''(z)}{f'(z)}\right| \leqq \frac{2r}{1-r^2} \qquad [1, \text{ Lemma } 3.1].$$

PROOF. Let

$$f(z) = s_n(z)+\sigma_n(z),$$

where

$$\sigma_n(z) = \sum_{k=n+1}^{\infty} a_k z^k.$$

Making use of estimate (3), we see that

$$(6)\qquad |\sigma_n'(z)| \leqq 2\sum_{k=n+1}^{\infty} r^{k-1} = \frac{2r^n}{1-r},$$

$$(7)\qquad |z\sigma_n''(z)| \leqq 2\sum_{k=n+1}^{\infty} (k-1)r^{k-1} = \frac{2nr^n}{(1-r)} + \frac{2r^{n+1}}{(1-r)^2}.$$

Now we have

$$1+z\frac{s_n''(z)}{s_n'(z)} = 1+\frac{z\{f''(z)-\sigma_n''(z)\}}{f'(z)-\sigma_n'(z)}$$

$$= 1+\frac{zf''(z)}{f'(z)} + \frac{\left\{\frac{zf''(z)}{f'(z)}\sigma_n'(z)-z\sigma_n''(z)\right\}}{f'(z)-\sigma_n'(z)}.$$

It is well-known that $s_n(z)$ will be convex if Re $[1+\{zs_n''(z)/s_n'(z)\}] > 0$. Making use of estimates (4), (5), (6) and (7) we conclude that Re $[1+\{z\mathscr{S}_n''(z)/\mathscr{S}_n'(z)\}] > 0$ provided

$$1-\frac{2r}{1-r^2}-\frac{\left[\left(\frac{2r}{1-r^2}\right)\frac{2r^n}{1-r}+2\left\{\frac{nr^n}{1-r}+\frac{r^{n+1}}{(1-r)^2}\right\}\right]}{\frac{1-r}{1+r}-\frac{2r^n}{1-r}} > 0,$$

or

$$(8)\qquad \frac{1-2r-r^2}{1-r^2}-\frac{\frac{2r^n}{(1+r)(1-r)^2}\{3r+r^2+n(1-r^2)\}}{\frac{1-r}{1+r}-\frac{2r^n}{1-r}} > 0.$$

If we take $r = \frac{1}{4}$, then on the left-hand side of (8) we obtain

$$\frac{7}{15} - \frac{\dfrac{2}{4^{n-1}}\,\dfrac{(13+15n)}{45}}{\dfrac{3}{5} - \dfrac{2}{3\times 4^{n-1}}}$$

which is greater than zero for $n \geqq 3$. Therefore, we conclude that

$$\text{(9)} \qquad \operatorname{Re}\left\{1+\frac{zs_n''(z)}{s_n'(z)}\right\} > 0 \qquad (n \geqq 3)$$

for $|z| = \frac{1}{4}$. From the maximum principle for harmonic functions it then follows that (9) holds for $|z| \leqq \frac{1}{4}$. Next we consider the case $n = 2$. We have

$$s_2(z) = z + a_2 z^2,$$

and hence

$$1+\frac{zs_2''(z)}{s_2'(z)} = 1+\frac{2a_2 z}{1+2a_2 z}.$$

Thus

$$\operatorname{Re}\left\{1+\frac{zs_n''(z)}{s_2'(z)}\right\} \geqq 1-\frac{2|a_2||z|}{1-2|a_n||z|}$$
$$\geqq 1-\frac{2|z|}{1-2|z|}.$$

We therefore see that $\operatorname{Re}[1+\{zs_2''(z)/s_2'(z)\}] > 0$ provided $|z| < \frac{1}{4}$.

To show that the constant $\frac{1}{4}$ cannot be replaced by a larger one, we consider the function $f_0(z)$ defined as

$$f_0(z) = 2\log(1+z)-z,$$

which belongs to R. If we denote by $s_{2,0}(z)$ the sum of the first 2 terms of the expansion of $f_0(z)$, we find that

$$1+\frac{zs_{2,0}''(z)}{s_{2,0}'(z)} = \frac{1-4z}{1-2z}.$$

which shows that $\operatorname{Re}[1+\{zs_{2,0}''(z)/s_{2,0}'(z)\}] = 0$ when $z = \frac{1}{4}$. This completes the proof of the theorem.

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