SIMPLE GROUPS STABILIZING POLYNOMIALS

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Abstract

We study the problem of determining, for a polynomial function f on a vector space V, the linear transformations g of V such that $f \circ g = f$. When f is invariant under a simple algebraic group G acting irreducibly on V, we note that the subgroup of $\mathrm{GL}(V)$ stabilizing f often has identity component G, and we give applications realizing various groups, including the largest exceptional group E_8, as automorphism groups of polynomials and algebras. We show that, starting with a simple group G and an irreducible representation V, one can almost always find an f whose stabilizer has identity component G, and that no such f exists in the short list of excluded cases. This relies on our core technical result, the enumeration of inclusions $G < H \leqslant \mathrm{SL}(V)$ such that V/H has the same dimension as V/G. The main results of this paper are new even in the special case where k is the complex numbers.

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1. Introduction

The following problem appears in a variety of contexts: given a polynomial f in n variables, determine the linear transformations g such that $f \circ g = f$. For the f studied here it is obvious that such a g must be invertible, so the answer will be a subgroup of GL_n. Frobenius's 1897 paper [**Frob**] and Dieudonne's 1943 paper [**Dieu**] are both aimed at solving the special case where f is the

determinant. Solutions to this problems appear in many places in algebra as well as in geometric complexity theory; see for example [**MuS**].

This problem is typically solved using arguments that are special to the particular polynomial f being studied. Here we show that a single result gives the answer for a large class of f. Roughly speaking, if f is defined on a vector space V and is invariant under the action of a simple algebraic group G that acts irreducibly on V, we show that 'typically' the stabilizer $O(f)$ of f has identity component G. With this in hand, it is not hard to determine the full group $O(f)$.

This can be viewed as a sort of reverse invariant theory. Suppose an algebraic group G acts on a vector space V and you pick a G-invariant polynomial f on V. The stabilizer $O(f)$ contains G but *a priori* might be bigger. It is known, for example, that when G is a semisimple adjoint complex Lie group and $V = \mathrm{Lie}(G)$, then G is the identity component of $\bigcap_{f \in \mathbb{C}[V]^G} O(f)$; see [**Dix**]. In contrast, we show in Section 6 below that; for simple G apart from SO_5, not only is no intersection necessary, but a single homogeneous generator f of $\mathbb{C}[V]^G$ will do. As further illustrative examples, we show (1) that E_8 is the (identity component of) the automorphism group of an octic form in 248 variables and of a 3875-dimensional algebra (see Sections 3, 5, and 7); and (2) that, up to isogeny and excluding fields of small characteristic, every simple group is the identity component of the stabilizer of a *cubic* form (see Section 8). This latter example shows that the degree of a homogeneous f need not give any information about the identity component of $O(f)$.

The core idea in this paper is that there cannot be many closed connected overgroups H such that $G < H \leqslant \mathrm{SL}(V)$, and that furthermore there are extremely few such H such that V/H and V/G have the same dimension, i.e., such that $k(V)^G$ is an algebraic extension of $k(V)^H$. (In contrast, for finite G and H, every inclusion $G < H < \mathrm{SL}(V)$ leads to a proper algebraic extension $k(V)^G \supsetneq k(V)^H$ by the fundamental theorem of Galois theory.) Indeed, there are so few such H that we can enumerate them in Theorem 13.1. Because of this, we can show that, for most pairs (G, V), there is a polynomial f whose stabilizer has identity component G, and that in the excluded cases there is no such f.

We work with both groups (in the naive sense) as well as affine group schemes over an arbitrary field k. Our Theorems 3.1, 5.4, 6.6, 8.1, 8.2, 13.1, and 15.1 are new already in the case where k is the complex numbers, and Theorem 10.1 is an analog for k an algebraically closed field of prime characteristic that was previously known only in characteristic zero.

Notation and conventions. An *algebraic group scheme* over a field k is an affine group scheme of finite type over k as defined in [**SGA3**]. An *algebraic group* is an algebraic group scheme that is smooth. For an algebraic group scheme G over k and an extension k' of k, we put $G(k')$ for the group of

k'-points of G, that is, the k-algebra homomorphisms $k[G] \to k'$; it is a (concrete) group. If G is smooth and (a) k is algebraically closed or (b) k is infinite and G is reductive, then $G(k)$ is Zariski-dense in G, and sometimes in these cases we will conflate G and $G(k)$, as is commonly done.

For a finite-dimensional vector space V over k, we write $k[V]$ for the ring of polynomial functions on V with coefficients in k, that is, for the symmetric algebra on the dual space of V. The *(naive) stabilizer* in $\mathrm{GL}(V)$ of an $f \in k[V]$ is the (concrete) subgroup $\{g \in \mathrm{GL}(V) \mid f \circ g = f \text{ in } k[V]\}$. The *scheme-theoretic stabilizer* of f is the sub-group-scheme of $\mathrm{GL}(V)$ centralizing f in the sense of [**ConGP**, A.1.9]. In 'most' cases, such as if char k is zero or larger than some bound depending on V and f, the scheme-theoretic stabilizer will be smooth; if additionally (a) or (b) from the previous paragraph holds, then the two notions of stabilizer coincide.

The rational irreducible representations (the *irreps*) of a simple algebraic group G are denoted $L(\lambda)$, where λ is a dominant weight for G. (We only consider rational representations in this paper.) Each λ can be written uniquely as a sum of fundamental dominant weights $\lambda_1, \ldots, \lambda_r$ of G, which we number as in [**Bou**, Ch. VI, Plates I–IX]. If k has prime characteristic p, the *restricted* representations are those $L(\sum c_i \lambda_i)$ such that $0 \leqslant c_i < p$ for all i, and every irreducible representation can be expressed uniquely as a tensor product of Frobenius twists of restricted ones. If k has characteristic zero, then every irrep is restricted, by definition.

2. Reminders on group actions

Suppose that G is a connected algebraic subgroup of $\mathrm{GL}(V)$. For each $v \in V$, the dimension of the G-orbit Gv and the stabilizer G_v of v are related by the equation $\dim Gv + \dim G_v = \dim G$, as follows by applying the fiber dimension theorem [**EGA4**, Section 13.3] to the map $G \to V$ defined via $g \mapsto gv$.

We define $k[V]^G$ to be the subring of $k[V]$ consisting of elements f that are sent to $f \otimes 1$ under the comodule map $k[V] \to k[V] \otimes k[G]$. If G is reductive and k is infinite, then $G(k)$ is dense in G, and

$$k[V]^G = \{f \in k[V] \mid f \circ g = f \text{ in } k[V] \text{ for all } g \in G(k)\},$$

that is, the collection of $f \in k[V]$ whose naive stabilizers contain $G(k)$.

Put k_{alg} for an algebraic closure of k. As in [**Ro**, Theorem 2], [**Ses**], or [**PoV**, Section 2], there is a nonempty and G-invariant open subset U of $V \otimes k_{\mathrm{alg}}$ such that

$$\mathrm{trdeg}_{/k} k(V)^G = \dim V - \dim Gv \quad \text{for } v \in U.$$

On the other hand, if G is semisimple, an easy argument as in [**PoV**, Theorem 3.3] shows that $k(V)^G$ is the fraction field of $k[V]^G$, and we have

$$\dim k[V]^G = \dim V - \dim G + \dim G_v \quad \text{for } v \in U; \qquad (2.1)$$

that is, the Krull dimension of $k[V]^G$ equals the codimension of a generic orbit in V.

We remark that the orbits in U are orbits of maximal dimension in $V \otimes k_{\text{alg}}$, as can be seen by applying upper semicontinuity as in [**SGA3**, Proposition VI.4.1(i)] to the collection of stabilizers, which form a group scheme over V. Furthermore, if char $k = 0$, the conjugacy class of the stabilizer of $u \in U$ does not depend on the choice of u, as follows from the Luna stratification [**PoV**, Theorem 7.2]. Regardless of the characteristic, we know the following.

LEMMA 2.2. *Let G be a connected algebraic group acting on an irreducible variety V over an algebraically closed field k such that $\dim G_v = 0$ for some $v \in V$. Then there exists a nonempty open subvariety U of V such that $|G_v(k)|$ is finite and constant for $v \in U$.*

Proof. Take Y to be the closure of the image of the map $f : G \times V \to V \times V$ with $f(g, v) = (v, gv)$. Note that the fiber over (v, gv) has k-points $\{(h, v) \mid hv = gv\}$, and so has cardinality equal to $|G_v(k)|$. The hypothesis that some G_v is zero dimensional gives the same conclusion for generic $v \in V$, whereby the map $f : G \times V \to \overline{\text{im } f}$ is generically finite. It follows (by, for example, [**Sp**, Theorem 5.1.6(iii)]) that there is a nonempty open subvariety Y of $\overline{\text{im } f}$ such that all fibers have the same size, equal to the separable degree of the finite extension of function fields $k(G \times V)/k(Y)$. The projection of Y into the first copy of V contains an open subvariety of V (because the image of the morphism projects onto the first copy of V); hence the claim follows. $\qquad\square$

Comparing invariants under G and $\mathrm{Lie}(G)$. For $f \in k[V]$, we can adjoin an indeterminate t to k and expand, for $v, v' \in V$,

$$f(v + tv') = f(v) + t f_1(v, v') + (\text{terms of higher degree in } t),$$

where $f_1(v, v')$ is the directional derivative of f at v in the direction v'. We say that f is *Lie invariant* under $X \in \mathrm{End}(V)$ if $f_1(v, Xv) = 0$ for all $v \in V$.

Example 2.3. If f is homogeneous, then $f_1(v, v) = (\deg f) f(v)$ for all $v \in V$, because it is true for every monomial. Thus f is Lie invariant under the scalar matrices if and only if char k divides $\deg f$.

For an affine group scheme $G \leqslant \mathrm{GL}(V)$, we can view $\mathrm{Lie}(G)$ as the fiber over 1_G of the map $G(k[x]/(x^2)) \to G(k)$ induced by $x \mapsto 0$. From this, the following is obvious. If $f \in k[V]$ is invariant under the group G, then f is Lie invariant under $\mathrm{Lie}(G)$. The converse holds if $\mathrm{char}\, k = 0$ [**J**, Lemma 2], but in characteristic $p \neq 0$ the situation is more complicated. For example, every element of the subalgebra $k[V]^{(p)}$ generated by $\{h^p \mid h \in k[V]\}$ is Lie invariant under $\mathrm{Lie}(G)$. We have the following, which is an application of a result of Skryabin from [**Sk**].

LEMMA 2.4. *Let V be a representation of a semisimple algebraic group H over an algebraically closed field k of prime characteristic p.*

(0) *If $k[V]^H = k$, then the subring of $k[V]$ of elements Lie invariant under $\mathrm{Lie}(H)$ is $k[V]^{(p)}$.*

(1) *If $k[V]^H = k[f]$ for some homogeneous f of degree not divisible by p, then the subring of $k[V]$ of elements Lie invariant under $\mathrm{Lie}(H)$ is $k[V]^{(p)}[f]$, a free $k[V]^{(p)}$ module of rank p.*

Recall that case (0) of the lemma encompasses all representations with $\dim k[V]^H$ equal to zero, and that every representation with $\dim k[V]^H = 1$ has $k[V]^H = k[f]$ for some homogeneous f; see [**Po 80**, Proposition 12] for $k = \mathbb{C}$ and [**BeGL**, 6.1] for the general case.

Proof. For (0), this is a straightforward application of [**Sk**, Theorem 5.5]. For (1), Example 2.3 shows that, for each $v \in V$, $f_1(v, v) = 0$ if and only if $f(v) = 0$. Therefore, the variety Z consisting of those $v \in V$ such that the linear form $f_1(v, -)$ vanishes is contained in the vanishing set Y of f, and in fact is the singular set of Y. Since f is irreducible (because H is connected and has only trivial characters), Z is a proper subvariety, so Z has codimension at least two in V, and Skryabin's result gives the claim. $\qquad\square$

3. The compact real form of E_8

More than 125 years ago, Killing classified the finite-dimensional simple complex Lie algebras by introducing the notion of a root system and then classifying the simple root systems. (This is apart from some small errors, corrected in [**Ca 1894**].) In the paper containing the classification, [**Ki**], he explicitly posed the opposite problem of giving, for each simple root system, a concrete description of a simple Lie algebra with that root system [**Ki**, page 38]. He answered this problem for the root systems of types A_n, B_n, C_n, and D_n

by showing that they come from $\mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, and $\mathfrak{so}_{2n}$, respectively, and these descriptions are now a standard part of the theory as in [**Bou**, Section VIII.13] or [**Sp**, 7.4.7]. Analogous descriptions for types E_6 and E_7 date back 120 years to Cartan's thesis [**Ca 1894**, pages 139–147], and were followed by treatments of G_2 by Engel [**Engel**] and Cartan (without proof, [**Ca 1914**, page 298]), and of F_4 by Chevalley and Schafer [**ChS**], and by a refinement of the E_7 case by Freudenthal [**Fre**]. For E_8, the only such interpretation known is as the automorphism group of its Lie algebra—see Section 16 for a more comprehensive discussion—and we now give another one. Recall that the smallest faithful irreducible representation of a group of type E_8 is its adjoint representation of dimension 248.

The paper [**CeP**] gives an explicit formula for a degree-8 invariant polynomial f on the Lie algebra of the compact real form of E_8 that is not in the span of the fourth power of the Killing form q, obtained by decomposing the representation with respect to the maximal subgroup of type D_8. Alternatively, such an invariant may be found by picking any degree-8 polynomial f_0 and defining f to be the average with respect to Haar measure, $f(v) := \int_G f(gv)\,dg$; the resulting f will be E_8 invariant and almost certainly not in $\mathbb{R}q^4$.

THEOREM 3.1. *The stabilizer in* $\mathrm{GL}_{248}(\mathbb{R})$ *of the octic polynomial f displayed in* [**CeP**, (2.3)] *is generated by the compact form of E_8 and ± 1. The stabilizer of f in* $\mathrm{GL}_{248}(\mathbb{C})$ *is generated by the complex E_8 and the eighth roots of unity.*

Proof. As the compact real E_8 and the eighth roots of unity stabilize f by construction, it suffices to verify that nothing else stabilizes f, for which it suffices to consider the complex case. Put S for the identity component of the stabilizer of f in GL_{248}. Because the representation is irreducible, it follows that S is reductive, and hence semisimple because f is not constant, and hence simple because the representation is tensor indecomposable for E_8. If S properly contains E_8, then it has classical type, and its smallest nontrivial representation has dimension 248 and is not symplectic; that is, S is SL_{248} or SO_{248}. But SL_{248} does not stabilize any nonzero octic form, and the only octic forms left invariant by SO_{248} are scalar multiples of the fourth power of a quadratic form (the Killing quadratic form for E_8), so we conclude that $S = E_8$. As the full stabilizer of f normalizes S and E_8 has no outer automorphisms, the full stabilizer is contained in the group generated by E_8 and the scalar matrices; the claim follows. □

The compact real form of E_8 discussed in the theorem is the one playing a role in the recent laboratory experiment described in [**Coldea⁺**]; compare [**BoG**].

We will generalize Theorem 3.1 to other fields in Theorem 5.4 and to other groups in Theorem 6.6. Nonetheless, we have included this doubly special result

here for two reasons. First, it is an example where the polynomial function is known explicitly. Second, despite the fact that it is a very special case of our results below, it already sheds new light on the problem posed by Killing more than 125 years ago.

4.   Containment of Lie algebras

We will prove the following result, which will be used to prove that certain group schemes are smooth; see Theorems 5.4 and 8.1.

PROPOSITION 4.1. *Let* $G \leqslant \mathrm{SL}(V)$ *be a simple algebraic group over an algebraically closed field k of prime characteristic such that V is irreducible and tensor indecomposable. Suppose that* $\mathrm{Lie}(G) < \mathfrak{L} \leqslant \mathfrak{sl}(V)$ *are containments of restricted Lie algebras such that $\mathfrak{L}$ is invariant under G, and $\mathfrak{L}$ is minimal with this property. If* $\operatorname{char} k \neq 2, 3$, $\mathrm{Lie}(G)$ *is simple, and the centralizer of $\mathrm{Lie}(G)$ in $\mathfrak{L}$ is zero, then there exist a simple simply connected algebraic group H and a homomorphism* $\phi : H \to \mathrm{SL}(V)$ *such that* $G < \phi(H)$ *and* $\mathfrak{L} = d\phi(\mathrm{Lie}(H))$.

Proof. Write $\mathfrak{g}$ for $\mathrm{Lie}(G)$. Let I be the subalgebra of $\mathfrak{L}$ generated by those $c \in \mathfrak{L}$ such that $[c[c\mathfrak{L}]] = 0$, and suppose that $I \neq 0$. It is G-invariant because G acts by algebra automorphisms on $\mathfrak{L}$; hence $[\mathfrak{g}, I] \leqslant I$, and by minimality $\mathfrak{L} = I + \mathfrak{g}$. Since I is nilpotent, by minimality it is abelian. Moreover, it acts completely reducibly on V because the socle is G-invariant; thus I is a toral subalgebra. As $\mathfrak{g}$ does not centralize I, G acts nontrivially on I, an impossibility, so $I = 0$.

Suppose now that $[\mathfrak{L}\mathfrak{L}]$ is smaller than $\mathfrak{L}$. As $\mathfrak{g}$ is perfect, minimality of $\mathfrak{L}$ implies that $[\mathfrak{L}\mathfrak{L}] = \mathfrak{g}$. But $\mathfrak{L}$ normalizes $[\mathfrak{L}\mathfrak{L}]$, and every derivation of $\mathfrak{g}$ is inner [**Rud**], so $\mathfrak{L}/\mathfrak{g}$ is naturally identified with the centralizer of $\mathfrak{g}$ in $\mathfrak{L}$, that is, 0. This contradicts the hypothesis that $\mathfrak{L} \neq \mathfrak{g}$, so $[\mathfrak{L}\mathfrak{L}] = \mathfrak{L}$.

Now [**Pr 87**, Theorem 3] (for $\operatorname{char} k > 5$) and [**Pr 86**] (for $\operatorname{char} k = 5$) give that $\mathfrak{L}$ is a sum of simple ideals 'of classical type'. Since V is tensor indecomposable for G and V is restricted, it is also tensor indecomposable for $\mathfrak{g}$. It follows that $\mathfrak{L}$ is itself simple of classical type; cf. [**BlZ**, Lemma 3.1]. Steinberg [**St 63**] gives a simple simply connected group H and a homomorphism $\phi : H \to \mathrm{SL}(V)$ such that $\mathfrak{L}$ is the subalgebra of $\mathfrak{sl}(V)$ generated by the images of the root subalgebras of $\mathrm{Lie}(H)$ under $d\phi$.   □

5.   Adjoint representation of E_8

The proof of Theorem 3.1 essentially relied on the nonexistence of overgroups of $E_8(\mathbb{C})$ in $\mathrm{SL}_{248}(\mathbb{C})$. This can be generalized as follows, which exploits the observation that overgroups of simple groups in irreducible representations are

comparatively rare. In the statement, k has characteristic $p \geqslant 0$; when $p = 0$, we set $k[V]^{(p)} = k$.

LEMMA 5.1. *Let $X < \mathrm{SL}(V)$ be a simple algebraic group over an algebraically closed field k such that V is irreducible, restricted, and tensor indecomposable for X. Put q for a nonzero X-invariant quadratic form on V if one exists; otherwise, set $q := 0$. If (X, V) does not appear in Table 1 of [Sei], then, for every $f \in k[V]^X \setminus k[q]$, the stabilizer of f in $\mathrm{GL}(V)$ has identity component X. If additionally $\mathrm{char}\, k \neq 2, 3$ and does not divide $\deg f$, f is not in $k[V]^{(p)}[q]$, and furthermore $\mathrm{char}\, k$ does not divide $n + 1$ if X has type A_n, then the scheme-theoretic stabilizer of f in $\mathrm{GL}(V)$ is smooth with identity component X.*

If $\mathrm{char}\, k = 0$, then V irreducible implies V restricted and tensor indecomposable. If $\mathrm{char}\, k \neq 2, 3$, then V irreducible and restricted implies V tensor indecomposable [Sei, 1.6].

Proof. Put S for the identity component of the scheme-theoretic stabilizer of f in $\mathrm{GL}(V)$, and S_{red} for its reduced subgroup. Because S_{red} contains X and V is an irreducible representation of X, it follows that S_{red} is reductive, and hence semisimple because f is not constant. If $X \neq S_{\mathrm{red}}$, then, as (X, V) is not contained in Table 1 of [Sei], p. 278 of [Sei] gives that S_{red} is the stabilizer of a symplectic (when $\mathrm{char}\, k = 2$) or quadratic form on V, and hence $k[V]^{S_{\mathrm{red}}} = k[q]$, contradicting the existence of f. Therefore, $X = S_{\mathrm{red}}$, and the first claim follows.

For the second claim, we have natural containments $\mathrm{Lie}(X) \leqslant \mathrm{Lie}(S) \leqslant \mathfrak{sl}(V)$ by [SGA3, Section VII$_A$.6]. The hypothesis on the characteristic guarantees that $\mathrm{Lie}(X)$ is a simple Lie algebra [Hog, 2.7a]. Since V is a restricted irrep of X, it is also an irrep of $\mathrm{Lie}(X)$; hence the centralizer of $\mathrm{Lie}(X)$ in $\mathrm{Lie}(S)$ consists of scalar matrices, and so is 0 by Example 2.3. Proposition 4.1 provides a simple, simply connected algebraic group (scheme) H and a homomorphism $\phi : H \to \mathrm{SL}(V)$ with $\mathrm{Lie}(S)$ containing $d\phi(\mathrm{Lie}(H))$.

The image $\phi(H(k))$ of the abstract group of k-points of H is a subgroup of $\mathrm{SL}(V)(k)$ containing $X(k)$, so by Seitz it is $\mathrm{SL}(V)(k)$, $\mathrm{SO}(V)(k)$, or $\mathrm{Sp}(V)(k)$. The map ϕ is a central isogeny by construction [BorT 72, 2.15], and combining this with [BorT 73, (A)] gives that H is isomorphic to $\mathrm{SL}(V)$, $\mathrm{Spin}(V)$, or $\mathrm{Sp}(V)$, respectively. Examining the list of dimensions of the irreps of H from [Lüb01], we see that ϕ is equivalent to V or its dual; hence $\ker \phi$ is zero and the subalgebra $d\phi(\mathrm{Lie}(H))$ of $\mathrm{Lie}(S)$ is $\mathfrak{sl}(V)$, $\mathfrak{so}(V)$, or $\mathfrak{sp}(V)$. But f is Lie invariant under $\mathrm{Lie}(S)$, contradicting Lemma 2.4, so S is smooth. $\square$

REMARK 5.2. Suppose that f is nonconstant homogeneous and that k is algebraically closed. Many of our results show that the naive stabilizer of f in $\mathrm{GL}(V)$ has identity component a simple group G, in which case the naive

stabilizer of kf in $\mathrm{PGL}(V)$ will have identity component the image of G. In Lemma 5.1 and Theorems 5.4 and 8.1, we also prove that the scheme-theoretic stabilizer of f is smooth, in which case its image (the scheme-theoretic stabilizer of kf in $\mathrm{PGL}(V)$) is also smooth.

'Most' pairs (X, V) with V irreducible and tensor indecomposable satisfy the hypotheses of Lemma 5.1 above. Indeed, unless (X, V) appears in (the rather short) Tables C or D, then there exists an $f \in k[V]^X \setminus k[q]$; see Section 12. Furthermore, inspecting the table in Seitz shows that, when char $k = 0$, a randomly selected irreducible representation V of any particular simple X will satisfy the hypotheses of Lemma 5.1 with probability 1, when one defines this probability as a limit over finite sets of weights of increasing size.

We also use Lemma 5.1 to give a version of Theorem 3.1 for any field. Because of the importance of this one example, we give a quick proof of Seitz's result for this case.

LEMMA 5.3. *Let k be an algebraically closed field of characteristic $p \geqslant 0$. Let $G = E_8(k) < \mathrm{SL}_{248}(k)$. If $p \neq 2$, there is a unique proper closed subgroup H with $G < H < \mathrm{SL}_{248}(k)$, and it is isomorphic to $\mathrm{SO}_{248}(k)$. If $p = 2$, there are precisely three proper closed subgroups H_i, $1 \leqslant i \leqslant 3$ of $\mathrm{SL}_{248}(k)$ properly containing G. We have $G < H_1 < H_2 < H_3 < \mathrm{SL}_{248}(k)$ with $H_1 \cong \mathrm{SO}_{248}(k)$, $H_2 \cong \mathrm{O}_{248}(k)$, and $H_3 \cong \mathrm{Sp}_{248}(k)$.*

Proof. It suffices to consider closed connected subgroups. Since G acts tensor indecomposably on the adjoint module, any connected overgroup H of G is simple. Since the rank of H is greater than 8, H must be of type A, B, C, or D. Moreover, as any representation of E_8 has dimension at least 248, the same must be true of H. Thus $H \cong \mathrm{SL}_{248}$, SO_{248}, or Sp_{248}. If $p \neq 2$, G does not preserve an alternating form. In any case, since G acts irreducibly, G preserves a unique (up to scalar) quadratic form (or symplectic form if $p = 2$). The result follows. $\square$

THEOREM 5.4. *Let G be a simple algebraic group of type E_8 over a field k, and put q for a nonzero G-invariant quadratic form on $V := \mathrm{Lie}(G)$. Then there exists a homogeneous polynomial f of degree 8 on V that is G-invariant and does not belong to kq^4. For each such f,*

(1) *the stabilizer of f in $\mathrm{GL}(V)$ is generated by $G(k)$ and the eighth roots of unity in k; and*

(2) *if char $k \neq 2, 3$, the scheme-theoretic stabilizer of f in $\mathrm{GL}(V)$ and the scheme-theoretic stabilizer of kf in $\mathrm{PGL}(V)$ is (the image of) G.*

Proof. Suppose first that k is algebraically closed. Put $\mathscr{E}_8$ for a split group scheme of type E_8 over $\mathbb{Z}$ (and identify $\mathscr{E}_8(k)$ with G) and $\mathfrak{q}$ for an indivisible $\mathscr{E}_8$-invariant quadratic form on $\mathrm{Lie}(\mathscr{E}_8)$. As the space of octic $\mathscr{E}_8(\mathbb{C})$-invariant polynomials on $\mathrm{Lie}(\mathscr{E}_8) \otimes \mathbb{C}$ is two-dimensional, the rank of the corresponding module over $\mathbb{Z}$ is 2. This dimension can only increase when we reduce modulo the characteristic of k, so there is an octic $\mathscr{E}_8$-invariant polynomial f on $\mathrm{Lie}(\mathscr{E}_8) \otimes k$ that is not a multiple of $(\mathfrak{q} \otimes k)^4$ (equivalently, is not a multiple of q^4). The stabilizer of f in $\mathrm{GL}(V)$ is generated by $\mathscr{E}_8(k)$ and the eighth roots of unity by Lemma 5.1 or Lemma 5.3. Claim (2) is a direct application of Lemma 5.1 and Remark 5.2.

Now let k be arbitrary. The natural homomorphism $k[V]^G \otimes k_{\mathrm{alg}} \to k_{\mathrm{alg}}[V \otimes k_{\mathrm{alg}}]^{G \times k_{\mathrm{alg}}}$ is an isomorphism [**Ses**, Lemma 2], so there exists an $f \in k[V]^G \setminus k[q]$. As $G(k) = G(k_{\mathrm{alg}}) \cap \mathrm{GL}(V)$, claim (1) follows. Claim (2) is obvious because it can be verified after base change to k_{alg}. $\qquad\square$

We conjecture that the scheme-theoretic stabilizer of kf in $\mathrm{PGL}(V)$ is smooth for all k, and that the scheme-theoretic stabilizer S of f is also smooth when $\mathrm{char}\,k = 3$. However, if $\mathrm{char}\,k = 2$, then S is not smooth, because $\mathrm{Lie}(S)$ contains both $\mathrm{Lie}(G)$ and the scalar matrices (because they are the Lie algebra of the group scheme of eighth roots of unity), so $\dim \mathrm{Lie}(S) > \dim \mathrm{Lie}(G) = \dim G = \dim S$.

REMARK 5.5. The method of proof used in this section can be applied more generally to argue for example that G is the identity component of the stabilizer of a subset of $V^{\otimes r} \otimes (V^*)^{\otimes s}$ for some r, s. As a concrete illustration, consider V the minuscule 56-dimensional representation of a group G of type E_7 over an algebraically closed field k of characteristic 2. Then G stabilizes a nonzero quadratic form q, and $k[V]^G = k[q]$ (see Proposition 12.1), so G is not the identity component of the stabilizer of a homogeneous form. But there is a nonsymmetric 4-linear form Ψ on V whose stabilizer has identity component G; see [**Lurie**, Section 6] and [**Luz**]. This claim could be proved following the methods of this section by checking that $\mathrm{SO}(q)$ does not stabilize Ψ.

The case where k has characteristic different from 2 is easier. Then $k[V]^G = k[f]$ for a quartic form f and Lemma 5.1 says that the stabilizer of f has identity component G. So in any characteristic G is the identity component of the stabilizer of a degree-4 element in the tensor algebra on V.

6. Adjoint groups are stabilizers of canonical homogeneous forms

In this section, we show that each split adjoint group, roughly speaking, can be realized as the identity component of the stabilizer of a *canonical* homogeneous form on its Lie algebra.

Table A. Torsion primes and primes that are not very good.

Type of $\mathscr{R}$	A_ℓ	B_ℓ ($\ell \geqslant 3$) or D_ℓ ($\ell \geqslant 4$)	C_ℓ	E_6, E_7, F_4	E_8	G_2
Torsion primes	$\emptyset$	2	$\emptyset$	2, 3	2, 3, 5	2
Not very good primes	Divisors of $\ell + 1$	2	2	2, 3	2, 3, 5	2, 3

To see this, fix a simple root system $\mathscr{R}$, and put A for the ring obtained by adjoining to $\mathbb{Z}$ the inverses of the torsion primes listed in Table A, and also adjoining $1/2$ if $\mathscr{R}$ has type C_ℓ for some $\ell \geqslant 1$. (This list is chosen in order to apply the results of [De 73].) This data uniquely determines a split adjoint algebraic group $\mathscr{G}$ over A of type $\mathscr{R}$ [SGA3, Section XXV.1]. Let $\mathscr{T}$ be a split maximal A-torus in the simply connected cover $\tilde{\mathscr{G}}$ of $\mathscr{G}$, and put W for the Weyl group $N_{\tilde{\mathscr{G}}}(\mathscr{T})/\mathscr{T}$.

The free module $\mathrm{Lie}(\mathscr{T})$ is naturally identified (via a pinning) with $Q^\vee \otimes A$, where $Q^\vee$ denotes the root lattice of the dual root system $\mathscr{R}^\vee$. It is classical that $\mathbb{R}[Q^\vee \otimes \mathbb{R}]^W$ is a polynomial ring with homogeneous generators $p_1, p_2, \ldots, p_\ell$, where

$$2 = \deg p_1 < \deg p_2 \leqslant \cdots \leqslant \deg p_{\ell-1} < \deg p_\ell = (\text{Coxeter \# of } \mathscr{R}), \quad (6.1)$$

and that these degrees are all distinct unless $\mathscr{R}$ has type $D_{2\ell'}$, in which case both $p_{\ell'}$ and $p_{\ell'+1}$ have degree ℓ'. These generators are not uniquely determined.

Example 6.2 (Flat bases). Whatever the type of $\mathscr{R}$, one can impose an additional condition on the generators of the real ring of invariants. Write $\langle\,,\,\rangle$ for a Weyl-invariant symmetric bilinear form on $P \otimes \mathbb{R}$, where $P = (Q^\vee)^*$ is the weight lattice with basis the fundamental dominant weights $\omega_1, \ldots, \omega_\ell$, and define a bilinear map b on polynomials in $P \otimes \mathbb{R}$, $b : \mathbb{R}[Q^\vee \otimes \mathbb{R}] \times \mathbb{R}[Q^\vee \otimes \mathbb{R}] \to \mathbb{R}[Q^\vee \otimes \mathbb{R}]$ by

$$b(p, p') = \sum_i \sum_j \frac{\partial p}{\partial \omega_i} \frac{\partial p'}{\partial \omega_j} \langle \omega_i, \omega_j \rangle.$$

In [SaiYS], the generators $p_1, \ldots, p_\ell$ are said to be a *flat basis* if $(\partial/\partial p_\ell)b(p, p')$ belongs to $\mathbb{R}$ for all p, p'. (This definition was motivated by the study of logarithmic poles; see [Sai].) Flat bases were constructed in [Ta] for types E_7 and E_8, and in [SaiYS] for the remaining types. The latter paper also proved that there is a *unique* flat basis up to scaling the elements by nonzero real numbers, or interchanging the two invariants of degree ℓ' in the case when $\mathscr{R}$ has type $D_{2\ell'}$.

LEMMA 6.3. *$A[\mathrm{Lie}(\mathscr{T})]^W$ is a polynomial ring with homogeneous and indivisible generators $p_1, \ldots, p_\ell$ with degrees as in (6.1). For every homomorphism of A into a field k, the natural map $A[\mathrm{Lie}(\mathscr{T})]^W \otimes k \to k[\mathrm{Lie}(\mathscr{T}) \otimes k]^W$ is an isomorphism, and $k[\mathrm{Lie}(\mathscr{T}) \otimes k]^W$ is a polynomial ring with generators the images of $p_1, \ldots, p_\ell$.*

Proof. The main result of [**De 73**] says that the arrow in the statement of the lemma is an isomorphism and that the rings are graded polynomial rings. Taking $k = \mathbb{R}$ and tracing through the proof of [**De 73**, Lemma 6], shows that $A[\mathrm{Lie}(\mathscr{T})]^W$ has homogeneous indivisible generators of the same degrees as those of $\mathbb{R}[\mathrm{Lie}(\mathscr{T}) \otimes \mathbb{R}]^W$. □

In view of Lemma 6.3, we may:

$$\text{Choose indivisible homogeneous } p_1, \ldots, p_\ell \in A[\mathrm{Lie}(\mathscr{T})]^W$$
$$\text{whose images in } \mathbb{R}[\mathrm{Lie}(\mathscr{T}) \otimes \mathbb{R}]^W \text{ are a generating set.} \qquad (6.4)$$

Example 6.5 (Type A_ℓ). For $\mathscr{R}$ of type A_ℓ, we may identify $\mathrm{Lie}(\mathscr{T})$ with the space of $(\ell + 1)$-vectors whose coordinates sum to zero, which identifies $A[\mathrm{Lie}(\mathscr{T})]^W$ with $A[x_1, \ldots, x_{\ell+1}]/(\sum x_i)$. The fundamental theorem of symmetric polynomials gives that one may take p_i to be the elementary symmetric polynomial in $x_1, \ldots, x_{\ell+1}$ of degree $i + 1$.

The generators p_1, p_2 of degrees 2, 3, respectively, are the same as those in the flat basis in Example 6.2, but p_i for $i \geqslant 3$ are different in the two cases, as follows from [**SaiYS**, 2.5.4].

Fix a homomorphism of A to a field k. The natural map $k[\mathrm{Lie}(\tilde{\mathscr{G}}) \otimes k]^{\mathscr{G}} \to k[\mathrm{Lie}(\mathscr{T}) \otimes k]^W$ is an isomorphism by [**SpSt**, Section II.3] or [**KacW**, Theorem 4(i)], and we write f_i for the pullback of the element p_i chosen in (6.4). Note that f_1 is a $\mathscr{G}$-invariant quadratic form, a scalar multiple of the Killing form.

If $\mathscr{R}$ has type A_1, then $\mathscr{G}$ is SO_3, the identity component of the stabilizer of f_1. For $\mathscr{R}$ of higher rank, we have the following.

THEOREM 6.6. *Suppose that $\mathscr{R}$ is not of type A_1 or C_2, char k is very good for $\mathscr{R}$, and $f \in \{f_2, \ldots, f_\ell\}$ satisfies the following.*

(i) *If $\mathscr{R}$ has type C_ℓ for $\ell \geqslant 3$, then $f \neq f_\ell$.*

(ii) *If $\mathscr{R}$ has type D_ℓ for $\ell \geqslant 4$, then $\deg f \neq \ell$.*

(iii) *If $\mathscr{R}$ has type A_3, then $f = f_3$ (hence $\deg f = 4$).*

If k is infinite, then the naive stabilizer of f has identity component $\mathscr{G}(k)$. If char $k \neq 2, 3$, and does not divide $\deg f$, then the scheme-theoretic stabilizer of f has identity component $\mathscr{G} \times k$.

Proof. We reduce the proof to Lemma 5.1. Because the characteristic is very good, $\text{Lie}(\tilde{\mathscr{G}})$ is a restricted irreducible representation of $\mathscr{G}$ [**Hi**], and is tensor indecomposable [**Sei**, 1.6(i)]. This representation appears only in lines I_1, I_2, I_4, and I_6 of Seitz's table.

Line I_1 says that PSp_{2n} is contained in $S^2 \text{SL}_{2n}$; this larger group has a unique invariant of degree $2n$, hence exception $f \neq f_n$ in (i). Line I_2 says that SO_{2n+1} is contained in $\wedge^2 \text{SL}_{2n+1}$, which has no nonconstant invariants so this gives no exceptions. Line I_4 says that PSO_{2n} is contained in $\wedge^2 \text{SL}_{2n}$ which has a degree n invariant, hence the exception $\deg f \neq n$ in (ii). Finally, line I_6 says that PGL_4 is contained in $\wedge^2 \text{SL}_6$, which has a degree 3 invariant, hence the exception $f \neq f_2$ in (iii).

Suppose now that additionally char $k \neq 2, 3$. The restriction of f to $\text{Lie}(\tilde{\mathscr{T}})$ $\otimes k$ cannot be in the k-span of $h^p q^r$ for some nonconstant $h \in k[\text{Lie}(\tilde{\mathscr{T}}) \otimes k]$ and $r \geqslant 1$, for if it were then h^p would also belong to $k[\text{Lie}(\tilde{\mathscr{T}}) \otimes k]^W$, which would contradict the fact that the restriction of f is a generator.

Applying Lemma 5.1 completes the proof of the proposition in the case when k is algebraically closed. The claim for arbitrary k follows. $\qquad\square$

Although type C_2 is excluded from the theorem, in that case $\mathscr{G} = \text{PSp}_4 = \text{SO}_5$ is the identity component of the stabilizer of the degree-8 homogeneous polynomial $f_1 f_2$, the degree-4 (but inhomogeneous) $f_1 + f_2$, or anything in $k[f_1, f_2] \setminus (k[f_1] \cup k[f_2])$.

Example 6.7 (E_8 octic). In the case of E_8, we choose $p_1, \ldots, p_8$ as in Example 6.2, so that their images over $\mathbb{R}$ are rational multiples of the flat basis $\hat{q}_1, \ldots, \hat{q}_8$ from [**Ta**, page 15]. Taking $p_1 := \hat{q}_1$ and $p_2 := 15\hat{q}_2/8$ gives indivisible polynomials with integer coefficients; to see this it is helpful to refer to [**Meh**, page 1089]. In particular,

$$p_2 = 8x_1^8 - 28x_1^4 x_2^4 - 14x_1^4 x_2^2 x_3^2 + \cdots, \quad \text{and} \quad p_1^4 = x_1^8 + 19x_1^4 x_2^4 + 72x_1^4 x_2^2 x_3^2 + \cdots,$$

where the $x_1, \ldots, x_8$ are a basis for the weight lattice defined in [**Meh**]. From this, it is clear that the image of p_2 in $k[\text{Lie}(\tilde{\mathscr{T}}) \otimes k]^W$ is not in the k-span of p_1^4, not even when char $k = 2, 3$, which we have excluded. (Although $p_2 - 8p_1^4$ is divisible by 5.) The pullback f of p_2 to $\text{Lie}(\tilde{\mathscr{G}}) \otimes k$ then provides an octic form whose scheme-theoretic stabilizer has identity component $E_8 \times k$, and this octic form is *canonical* in the sense that it is determined up to multiplication by a unit

in $\mathbb{Z}[1/30]$ by the property of restricting to be an element in the flat basis for the Weyl invariants.

7.   3875-dimensional representation of E_8

LEMMA 7.1. *Let G be a simple algebraic group of type E_8 over a field k, and let V be the second smallest faithful irreducible representation of G. There exist nonzero G-equivariant bilinear maps $b : V \times V \to k$ and $\star : V \times V \to V$ satisfying*

$$v_1 \star v_2 = v_2 \star v_1 \quad and \quad b(v_1 \star v_2, v_3) = b(v_{\pi(1)} \star v_{\pi(2)}, v_{\pi(3)})$$

for all $v_1, v_2, v_3 \in V$ and every permutation π of $\{1, 2, 3\}$, and these properties determine b and $\star$ uniquely up to multiplication by an element of $k^\times$.

The representation V in the lemma has dimension 3875 if $\operatorname{char} k \neq 2$ and 3626 if $\operatorname{char} k = 2$; cf. [**Lüb01**, A.53]. In either case, the highest weight is the one denoted ω_1 in [**Bou**].

Proof. Put $\mathscr{E}_8$ for a split semisimple group scheme of type E_8 over $\mathbb{Z}$ and $\mathscr{V}$ for a Weyl module of $\mathscr{E}_8$ over $\mathbb{Z}$ with highest weight λ_1. Then $\mathscr{E}_8 \times \mathbb{C}$ is the complex group E_8 and $\mathscr{V} \otimes \mathbb{C}$ is its second smallest faithful irreducible representation. Note that $\mathscr{V} \otimes \mathbb{C}$ is orthogonal and has a unique E_8-invariant line in $(V \otimes \mathbb{C})^{\otimes 3}$; this line consists of symmetric tensors. It follows that the same is true for the representation $\mathscr{V} \otimes \mathbb{Q}$ of $\mathscr{E}_8 \times \mathbb{Q}$, and we find a symmetric bilinear form b on $\mathscr{V}$ and a bilinear map $\star : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$ which are both indivisible and $\mathscr{E}_8$-equivariant. We can interpret $\star$ as a (not necessarily associative) product operation on $\mathscr{V}$, and we define corresponding operations on $\mathscr{V} \otimes k$ by base change. Because the invariant tensor in $(\mathscr{V} \otimes \mathbb{C})^{\otimes 3}$ is symmetric, the displayed equations hold when $k = \mathbb{C}$, and it follows by base change that they hold also for arbitrary k.

Suppose that G is k-split. If $\operatorname{char} k \neq 2$, then $\mathscr{V} \otimes k$ is irreducible, that is, is V (because G is split), and the proof is complete. If $\operatorname{char} k = 2$, then $\mathscr{V} \otimes k$ is reducible, and the second displayed identity implies that the maximal proper submodule of $\mathscr{V} \otimes k$ is an ideal for the multiplication $\star$. It follows that $\star$ and b factor through to give a multiplication and a nondegenerate symmetric bilinear form on the irreducible quotient V, both of which are nonzero and $(\mathscr{E}_8 \times k)$-invariant.

In the general case, G is isomorphic to a Galois twist of $\mathscr{E}_8 \times k$ by a 1-cocycle $z \in Z^1(k, \mathscr{E}_8 \times k)$. Using z and Galois descent from a separable closure of k gives G-equivariant maps b and $\star$ defined on V over k. Uniqueness and the identities follow because they hold after base change to a separable closure. $\square$

We offer the following observations about the multiplication $\star$ on V. If char $k \neq 2, 3$, then the automorphism group scheme of $(V, \star)$ is G by Lemma 5.1.

If char $k = 3$, then Lemma 5.1 gives that the automorphism group of this multiplication has k-points $G(k)$.

If char $k = 2$, then, by [**Sei**], the only other closed connected overgroups of $\mathscr{E}_8(k)$ in $\mathrm{SL}(V)$ are $H = \mathrm{SO}(V)$ or $\mathrm{Sp}(V)$, but these cannot stabilize $\star$: the highest weight λ of the defining representation of such an H is not in the root lattice but 2λ is, so there is no nonzero H-invariant multiplication. Therefore $G(k)$ is the naive automorphism group of the multiplication $\star$ on V. Alexander Premet asks: Does this multiplication satisfy the Jacobi identity?

8. Simple groups as stabilizers of cubic forms

Groups of type B and D over an algebraically closed field k are isogenous to SO_n for some n; that is, they are isogenous to the identity component of the stabilizer in GL_n of a quadratic form. Analogous statements hold for type E_6 with a cubic form and type E_7 for a quartic form (as long as char $k \neq 2$). What about types C, G_2, F_4, E_8, and also A? We observe now that all of these, and E_7 also, can almost always be obtained as stabilizers of *cubic* forms. This result is new even in the case when $k = \mathbb{C}$.

THEOREM 8.1. *Let G be a simple and simply connected algebraic group over an algebraically closed field k with (a) char $k = 0$ or (b) char $k > 2$ rank $G+1$. There exist an irreducible kG-module V and a homogeneous polynomial $f \in k[V]$ of degree 3 such that the image of G in $\mathrm{GL}(V)$ is the identity component of the scheme-theoretic stabilizer of f.*

At the cost of replacing cubic forms in some cases with quadratic forms, we can ease the hypothesis on the characteristic.

THEOREM 8.2. *Let G be a simple and simply connected algebraic group over an algebraically closed field k of characteristic $p \neq 2$. Assume that, if G is of type A_{n-1} or C_n, then p does not divide n. There exist an irreducible and tensor indecomposable kG-module V and a homogeneous polynomial $f \in k[V]$ of degree 2 or 3 such that the image of G in $\mathrm{GL}(V)$ is the identity component of the naive stabilizer of f.*

We postpone the proofs of these theorems until after the following examples, which will be used also in the proof of Theorem 13.1.

Example 8.3 (Trace zero matrices)**.** Consider the conjugation action of SL_n (equivalently, of PGL_n) on the space M of n-by-n matrices over an algebraically closed field k, for some $n \geqslant 2$. Because the matrices with distinct eigenvalues are dense and the normalizer of the diagonal matrices equals the monomial matrices in SL_n, the ring $k[M]^{\mathrm{SL}_n}$ equals the symmetric polynomials in n variables. That is, it is a polynomial ring with generators of degrees $1, 2, \ldots, n$, the coefficients of the characteristic polynomial.

Put M_0 for the Lie algebra of SL_n, that is, the trace zero subspace of M. Tracking the proof of [**N**, 4.1] shows that $k[M_0]^{\mathrm{SL}_n}$ is polynomial, with generators the restrictions of the generators of $k[M]^{\mathrm{SL}_n}$ of degrees $2, 3, \ldots, n$.

Finally, suppose that char $k \mid n$, and put $\bar{M}_0$ for M_0 modulo the scalar matrices. Then $k[\bar{M}_0]^{\mathrm{SL}_n}$ is identified with the subring of elements $f \in k[M_0]^{\mathrm{SL}_n}$ such that $f(t I_n + m) = f(m)$ for all $m \in M_0$ and $t \in k$. It is easy to see that this ring has transcendence degree $n - 2$.

For later use, we note that when k has characteristic 2 and $n = 4$, one checks that the coefficients c_2, c_3 of degrees $2, 3$ of the characteristic polynomial belong to $k[\bar{M}_0]^{\mathrm{SL}_4}$, but that $\det(t I_4 + m) = t^4 + $ (terms of lower degree in t). So $k[\bar{M}_0]^{\mathrm{SL}_4} = k[c_2, c_3]$.

REMARK 8.4. The previous example noted that $k[\mathfrak{sl}_n]^{\mathrm{PGL}_n}$ is a polynomial ring. In contrast, $k[\mathfrak{pgl}_n]^{\mathrm{PGL}_n}$ is isomorphic to the Weyl-group-invariant subspace of the symmetric algebra on the A_{n-1} root lattice tensored with k [**SpSt**, page 199], that is, with the S_n-invariant subalgebra of the symmetric algebra on the obvious $(n - 1)$-dimensional subspace of the n-dimensional permutation representation of S_n. When char k divides n and $n \geqslant 5$, this ring is not polynomial, by [**KeM**, 5.2] or [**N**, 4.3]; cf. [**SpSt**, Problem II.3.18].

Example 8.5 (Self-adjoint endomorphisms)**.** Let k be an algebraically closed field. Fix a $2n$-dimensional k-vector space W for some $n \geqslant 3$ and a nondegenerate alternating bilinear form b on W; write $\mathrm{Sp}(W)$ for the isometry group of b. Define Y to be the space of *self-adjoint* endomorphisms of W, that is, the collection of endomorphisms T so that $b(Tw, w) = 0$ for all $w \in W$. Note that $\mathrm{Sp}(W)$ acts on Y by conjugation; cf. [**GoGu**, Section 2].

It is shown in [**GoGu**, Theorem 2.7] that any self-adjoint operator leaves invariant a pair of totally singular complementary spaces with respect to b. With respect to an appropriate basis, it follows that a self-adjoint operator corresponds to $\mathrm{diag}(A, A^{\top})$, and the stabilizer of a pair of such spaces in $\mathrm{Sp}(W)$ is GL_n, which acts via conjugation. Thus, the $\mathrm{Sp}(W)$-orbits in Y can be identified with similarity classes of n-by-n matrices, and $k[Y]^{\mathrm{Sp}(W)}$ is generated by the coefficients of the characteristic polynomial of a generic self-adjoint operator; that is, it is a polynomial ring generated in degrees $1, \ldots, n$. A generic element will be one in which the minimal polynomial of A has degree n and has

distinct roots. (Such an element has stabilizer isomorphic to SL_2^n.) It follows that $k[Y]^{Sp(W)}$ is a polynomial ring in generators of degrees $1, \ldots, n$.

Write Y_0 for the subspace of Y on which the linear invariant vanishes. The same proof shows that $k[Y_0]^{Sp(W)}$ is isomorphic to the ring of PGL_n-invariant functions on M_0 as in the preceding example.

Now let X denote the space of alternating $2n$-by-$2n$ matrices, that is, those matrices L so that $L_{ij} = -L_{ji}$ and $L_{ii} = 0$ for all i, j. Then G acts on X via $g \cdot L = gLg^\top$, and this representation is isomorphic to Y, via sending L to JL, where J is the alternating matrix defining b; cf. [**GoGu**, Section 2]. We write X_0 for the submodule of X corresponding to Y_0.

We now describe $k[V]^{Sp(W)}$ for V the irreducible representation with highest weight λ_2. If n is not divisible by char k, then X_0 is V, and the claim follows from above. If n is divisible by char k, then V is isomorphic to X_0/k, and the pullback $k[V]^{Sp(W)} \hookrightarrow k[X_0]^{Sp(W)}$ identifies $k[V]^{Sp(W)}$ with the ring of PGL_n-invariants on the space $\bar{M}_0$ from the previous example.

Proof of Theorem 8.2. By Lemma 5.1, it suffices to produce, for G of each of the types A_2, A_n ($n \geqslant 4$), C_n ($n \geqslant 3$), E_7, E_8, F_4, and G_2, a *restricted* dominant weight λ so that the corresponding irrep $L(\lambda)$ has a G-invariant cubic form and λ does not appear in Seitz's Table 1.

For the groups G and weights λ listed in Table B, the Weyl module of highest weight λ is irreducible (with the restrictions on the characteristic in the table) by [**Lüb01**], and the corresponding irrep for a split complex Lie group has a G-invariant cubic form. Therefore, the irrep $L(\lambda)$ of G over k also has a G-invariant cubic form, and we are done with this case.

Example 8.3 gives a restricted irrep for type A_n ($n \geqslant 4$, p not dividing $n+1$) with an invariant cubic form, and this irrep does not appear in Seitz's Table 1; this proves Theorem 8.2 for those groups. Similarly, Example 8.5 handles type C_n for $n = 3$ and $n \geqslant 5$ (for p not dividing n). □

The theorem holds also in characteristic 2 for many types by the same proof. However, the argument fails in particular for types A_1, C_4, and G_2. For example, the only nontrivial restricted irrep of A_1 does not support any invariant nonconstant forms, so in this case one must consider irreps that are tensor decomposable.

Proof of Theorem 8.1. In Theorem 8.2, the only places where the polynomial is quadratic is for types B_n ($n \geqslant 1$) and D_n ($n \geqslant 3$). Such a group is isogenous to SO_r for $r = 3$ or $r \geqslant 5$. By the hypothesis on the characteristic, the irrep V with highest weight $2\lambda_2$ is the vector space of trace zero r-by-r symmetric matrices where SO_r acts by conjugation. The degree-3 coefficient of the characteristic polynomial is invariant under SO_r and V does not appear in Seitz's Table 1, so

Table B. Examples of irreducible representations $L(\lambda)$ over a field k such that $L(\lambda)$ has a nonzero G-invariant cubic form and there is no overgroup H that stabilizes a cubic form and lies properly between G and $SL(L(\lambda))$.

Group G	Highest weight λ	dim $L(\lambda)$	char k
A_2	$2\lambda_1$	6	$\neq 2$
C_4	$2\lambda_2$	308	$\neq 2, 3, 5$
	$2\lambda_4$	594	$\neq 2, 5, 7$
	$\lambda_2 + \lambda_4$	792	$\neq 2, 3, 7$
E_7	$2\lambda_1$	7 371	$\neq 2, 5, 19$
	$\lambda_2 + \lambda_7$	40 755	$\neq 2, 3, 7$
E_8	λ_1	3 875	$\neq 2$
F_4	λ_4	26	$\neq 3$
	$\lambda_1 + \lambda_4$	1 053	$\neq 2$
G_2	$2\lambda_1$	27	$\neq 2, 7$
	$2\lambda_2$	77	$\neq 2, 3$

Lemma 5.1 gives that G is the identity component of the (naive) stabilizer, and further that the scheme-theoretic stabilizer is smooth. □

9. There are only finitely many overgroups

For the proof of the main result Theorem 15.1, we need the statement that a semisimple irreducible subgroup of $SL(V)$ is contained in only finitely many closed subgroups of $SL(V)$. We prove instead Proposition 9.2, which is much more general. For x, y in a group G, we write ${}^{x}y := xyx^{-1}$, and for $A \subseteq G$ we put ${}^{A}y := \{{}^{a}y \mid a \in A\}$ and ${}^{x}A := \{{}^{x}a \mid a \in A\}$.

LEMMA 9.1. *Let $G < H < X$ be groups. If $[N_X(G) : G]$ is finite and the number of H-conjugacy classes of subgroups ${}^{x}G$, $x \in X$ which are contained in H is finite, then G has finitely many fixed points on X/H.*

Proof. Suppose that $Gx^{-1}H = x^{-1}H$ (equivalently that ${}^{x}G < H$). Let $G_i = {}^{x_i}G$, $1 \leqslant i \leqslant m$ be representatives for the H-conjugacy classes of $\{{}^{x}G \mid {}^{x}G < H\}$. So ${}^{x}G = {}^{h_i x_i}G$ for some i and some $h_i \in H$. Thus $y := x_i^{-1}h_i^{-1}x$ is in $N_X(G)$.

Further,

$$yGx^{-1}H = Gyx^{-1}H = Gx_i^{-1}H,$$

and so $N_X(G)x^{-1}Y = N_X(G)x_i^{-1}H$. Of course $N_X(G)x_i^{-1}H$ is a finite union of $G\backslash X/H$ double cosets, whence the result. □

PROPOSITION 9.2. *Let X be a semisimple algebraic group over an algebraically closed field k. If G is a closed, connected subgroup of X not contained in a proper parabolic subgroup of X, then G is semisimple, $C_X(G)$ is finite, $[N_X(G) : G]$ is finite, and there are only finitely many closed subgroups H of X with $G \leqslant H \leqslant X$.*

Proof. As G is not contained in a proper parabolic subgroup of X, it cannot normalize a nontrivial unipotent subgroup, nor can it centralize a nontrivial torus in X [**BorT 65**, Theorem 4.15a]. This implies that G is semisimple and that the identity component of $C_X(G)$ is trivial; hence $C_X(G)$ is finite. Since the outer automorphism group of a semisimple group is finite, it follows that $[N_X(G) : G]$ is finite.

We next show that, for any connected, closed subgroup H such that $G \leqslant H \leqslant X$, the following holds (see also [**Mar 13**, Lemma 11]).

The set $\{^xG \mid {}^xG \leqslant H\}$ is a finite union of H-conjugacy classes. (9.3)

First assume that k is not the algebraic closure of a finite field. Then we can choose $g_1, g_2 \in G$ so that G is the Zariski closure of $\langle g_1, g_2 \rangle$. Let $f : H^2/H \to X^2/X$ be the morphism of varieties induced by the inclusion of H^2 in X^2, where H, X act on H^2, X^2 by simultaneous conjugation. By [**Mar 03**, Theorem 1.1], f is a finite morphism. Let $p_H : H^2 \to H^2/H$ and $p_X : X^2 \to X^2/X$ be the canonical projections. If two elements in H^2 generate H-cr subgroups and both belong to the same fiber of p_H, then they are H-conjugate by [**BaMR**, Theorem 3.1], or see [**Ri 88**, Theorem 6.4].

If $x \in X$ and $^xG \leqslant H$, then, since G is not contained in a parabolic subgroup of X, ^xG cannot be contained in a parabolic subgroup of H. In particular, ^xG is an H-cr subgroup of H and $p_X(^xg_1, {}^xg_2) = p_X(g_1, g_2)$. Thus, $p_H(^xg_1, {}^xg_2)$ lies in the fiber of f above $p_X(g_1, g_2)$. Since f is a finite map, this fiber is finite. So, by the previous paragraph, $(^xg_1, {}^xg_2)$ lies in one of finitely many H-conjugacy classes.

If k is the algebraic closure of a finite field, let k' be a bigger algebraically closed field containing k. If $G_1(k), G_2(k)$ are closed subgroups of $H(k)$, let $G_i(k')$ be the Zariski closures of $G_i(k)$ in $H(k')$. Then $\{g \in H(k') \mid {}^gG_1(k') = G_2(k')\}$ is a variety defined over k. So, if there are k'-points, there are k-points. So, if the $G_i(k')$ are conjugate in $H(k')$, then the $G_i(k)$ are conjugate in $H(k)$, completing the verification of (9.3).

By Lemma 9.1, it follows that the closed overgroups of G in X contain only finitely many subgroups in a given X-conjugacy class of subgroups.

We now prove that there are only finitely many closed subgroups H lying between G and X. By the first paragraph of the proof, any such H is semisimple. By induction on the codimension of G in X, we are reduced to proving that G is contained in only finitely many semisimple maximal subgroups of X. Further, (9.3) and the fact that $[N_X(G) : G]$ is finite reduce us to showing there are only finitely many conjugacy classes of maximal closed subgroups containing G. If X is simple, then in fact X only has finitely many conjugacy classes of maximal closed subgroups, by representation theory in the case of classical groups and by [**LieS**] for exceptional groups.

Suppose that $X = X_1 \times X_2 \times \cdots \times X_m$ with $m > 1$ and that the X_i are simple. There are only finitely many conjugacy classes of maximal subgroups of the form $Y_1 \times \cdots \times Y_m$ for $Y_j = X_j$ for all $j \neq i$, and Y_i is maximal in X_i, and so we can ignore these.

The other possible maximal closed subgroups of X are 'diagonal', that is, up to reordering of the factors in X are of the form $Y \times X_3 \times \cdots \times X_m$, where there is a bijective morphism $\phi : X_1 \to X_2$, and Y is the image of X_1 under $\mathrm{Id}_{X_1} \times \phi$. There are countably many conjugacy classes of such subgroups (essentially corresponding to Frobenius morphisms and outer automorphisms). It is straightforward to see that the intersection of any infinite collection of nonconjugate diagonal subgroups of $X_1 \times X_2$ is a finite group. (Indeed, as long as we take more than $\mathrm{Out}(X_1)$ such subgroups, the intersection is finite.) Thus, G cannot be contained in infinitely many nonconjugate maximal closed subgroups of X, for then up to reordering G would be contained in $A \times X_3 \times \cdots \times X_m$ with A finite and so in $X_3 \times \cdots \times X_m$, a contradiction. $\qquad\square$

10. Generic stabilizers

The purpose of this section is to prove the following theorem.

THEOREM 10.1. *Let G be a closed, simple, and irreducible subgroup of* $\mathrm{SL}(V)$ *over an algebraically closed field k. If $\dim V > \dim G$ (in particular if V is tensor decomposable), then, for a generic $v \in V$, the identity component of G_v is unipotent.*

In characteristic 0, irreducible implies tensor indecomposable, so the hypothesis is that $\dim V > \dim G$, and it is already known that G_v is finite (that is, the identity component is trivial) and, with a small number of exceptions, G_v is itself trivial; see [**AnVE**] or [**PoV**, Theorem 7.11]. This seems likely to be true in positive characteristic as well, and will be the subject of future work.

For our applications, Theorem 10.1 more than suffices. Note that, if V is tensor decomposable, then, since the minimal dimension d of a representation of G satisfies $d^2 > \dim G$, $\dim V > \dim G$. Note also that, if $\dim G > \dim V$, then $\dim G_v \geqslant \dim G - \dim V > 0$, and almost always G_v will contain a torus for generic v. (In characteristic 0, only the natural representation of SL_2 has generic stabilizer with an identity component that is nontrivial unipotent.)

We give the proof at the end of the section. A key part of the proof is the main result of [**Ken**] that in most cases, if $\dim V > \dim G + 2$, then the identity component of G_v contains only unipotent elements.

For G an algebraic group acting on a variety V, let V^x be the fixed space of x on V and $V(x) := \{v \in V \mid {}^g x \cdot v = v \text{ for some } g \in G\}$.

LEMMA 10.2. *Let G be a reductive algebraic group acting on an irreducible variety V over an algebraically closed field k.*

(1) *If $\dim G_v = 0$ for some $v \in V$, then, for all sufficiently large primes r, $\dim V(x) < \dim V$ for every $x \in G$ of order r.*

(2) *If $\dim V(x) < \dim V$ for all nonidentity $x \in G$, then, for v in a nonempty open subvariety of V, G_v is trivial.*

(3) *If r is prime, $r \neq \operatorname{char} k$, and $\dim V(x) < \dim V$ for all x of order r, then, for v in a nonempty open subvariety, the identity component of G_v is unipotent.*

Proof. (1): By Lemma 2.2 there is a nonempty open subvariety U of V and there is a positive integer n such that $|G_v(k)| = n$ for all $v \in U$. For every $v \in U$ and prime r not dividing n, the conjugacy class of x does not meet G_v, so $\dim V(x) < \dim V$, proving (1).

Now let $X \subset G$ be the union of finitely many conjugacy classes and let it satisfy $\dim V(x) < \dim V$ for all $x \in X$. Then, as V is irreducible, the finite union $\bigcup_{x \in X} V(x)$ is contained in a proper closed subvariety Z of V, and, for every v in the nonempty open set $V \setminus Z$, the stabilizer G_v does not meet X.

Suppose that $\dim V(x) < \dim V$ for all nonidentity $x \in G$, and take X to be the union of the nonidentity unipotent elements in G and the elements of order r, for some prime r not equal to $\operatorname{char} k$. As G is reductive, X consists of a finite number of conjugacy classes, and the previous paragraph gives that G_v is finite for generic v. Let n be such that $|G_v| = n$ for v in a nonempty open subvariety of V. Repeating the argument of the previous paragraph with X the set of elements of G whose order divides n completes the proof of (2).

Taking X to be the set of elements of G of order r gives (3).   □

Next we note how to pass from characteristic 0 to positive characteristic for semisimple elements. One can obviously generalize the result, but we just state it in the form we need. Fix a simple Chevalley group G over $\mathbb{Z}$ and a representation of G on $V := \mathbb{Z}^n$ for some n. Fix algebraically closed fields K and k so that char $K = 0$ and char $k = p > 0$.

LEMMA 10.3. *Maintain the notation of the previous paragraph. If $G(K)_v$ is finite for generic $v \in V \otimes K$, then the identity component of $G(k)_v$ is unipotent for generic $v \in V \otimes k$.*

Proof. Since $G(K)_v$ is finite for generic v, there exists a prime $r \neq p$ so that $G(K)_v$ contains no elements of order r; as in Lemma 10.2(1), dim $V(x) <$ dim V for every $x \in G(K)$ of order r.

Let C be a conjugacy class of elements of order r in $G(k)$. This class is actually defined over the ring of algebraic integers R, and consequently $C(K)$ and $C(k)$ are both irreducible and have the same dimension. Choose $x \in C(K) \cap G(R)$. Consider the morphism from $G \times V^x \to V$ defined by $\alpha : (g, v) \mapsto gv$; the image of this morphism is $V(x)$. Note that this map is actually defined over R. As the image of $\alpha(K)$ is contained in a proper closed subvariety of $V \otimes K$, then the same is true of $\alpha(k)$, as any hypersurface of V can be defined by an equation $f = 0$ for some polynomial f over R, and then we can reduce modulo p. Thus, dim $V(\bar{x}) <$ dim V, where $\bar{x}$ is the reduction of x and is an element of $C(k)$, and dim $V(y) <$ dim V for any element y of order r in $G(k)$. Now apply Lemma 10.2(3). $\qquad\square$

We need to deal with a few special cases.

LEMMA 10.4. *Suppose that the hypotheses of Theorem 10.1 hold. If $V \cong W \otimes W'$ for a representation W of G, where W' is a nontrivial Frobenius twist of W or W^*, then G_v is finite for generic $v \in V$.*

Proof. It suffices to consider the case that $G = \mathrm{SL}(W)$, in which case G has finitely many orbits on $\mathbb{P}(V)$ by [**GuLMS**, Lemma 2.6], and the result follows since dim $\mathbb{P}(V) = $ dim G. $\qquad\square$

LEMMA 10.5. *Let G be an algebraic group acting on an irreducible variety V over an algebraically closed field k. For $x \in G$, if dim ${}^G x + $ dim $V^x <$ dim V, then $V(x)$ is contained in a proper closed subvariety of V.*

Proof. Define $\alpha : G \times V^x \to V$ by $\alpha(g, w) = gw$, so the image of α is precisely $V(x)$. The fiber over gw contains (gc^{-1}, cw) for $c \in C_G(x)$, and so has dimension at least that of $C_G(x)$, whence dim $V(x) \leqslant$ dim ${}^G x + $ dim V^x. $\qquad\square$

LEMMA 10.6. *Suppose that the hypotheses of Theorem 10.1 hold, and that G has type A_1. If $\dim V > 5$, then G_v is trivial for generic v. If $\dim V = 4$ or 5, then G_v is finite for generic v.*

Proof. Suppose that $\dim V > 5$. For nonidentity $x \in G$,

$$\dim V^x \leqslant (1 + \dim V)/2 < \dim V - 2.$$

Since $\dim {}^G x = 2$ for any noncentral $x \in G$, Lemmas 10.5 and 10.2(2) give that G_v is trivial for generic v.

If $\dim V = 4$ and V is not a twist of a restricted module, then apply Lemma 10.4. Finally, suppose that $\dim V = 4$ or 5 and that V is restricted (in particular, the characteristic is at least 5). Any nontrivial unipotent element x has a one-dimensional fixed space, and so $\dim V(x) \leqslant 3$. Suppose that x is semisimple but noncentral. If $\dim V = 4$, then $\dim V^x \leqslant 1$ unless x has order 3, whence G_v has exponent 3, and so is finite. If $\dim V = 5$, then $\dim V^x = 1$ unless x has order 4 (and so is acting as an involution on V). Again, we see that G_v is finite. $\qquad\square$

Example 10.7 ($S^2\mathrm{SO}_n$). Let $G = \mathrm{SO}_n(k)$ for some $n \geqslant 4$, $V = L(2\omega_1)$, and $p \neq 2$. We claim that the generic stabilizer is elementary abelian of order 2^n.

Let W be the natural module for G, and consider $V' := \mathrm{Sym}^2(W)$. If p does not divide n, then $V' \cong k \oplus L(2\omega)$. Thus, we see the stabilizer of a generic point is the intersection of G with some conjugate of G in $\mathrm{SL}(W)$. Since $\mathrm{SO}(W)$ is the centralizer of an involution in $\mathrm{SL}(W)$ and generically the product of two such involutions is a regular semisimple element, it follows that the intersection will generically be the group of involutions in a maximal torus.

If p does divide n, then V' is a uniserial module for G with three composition factors with a trivial socle and head. Let V'' be the radical of this module, and so $V \cong V''/k$. Clearly, a generic point $v \in V''$ corresponds to a nondegenerate quadratic form. Thus, the stabilizer of v in $\mathrm{SL}(W)$ is precisely a conjugate of $\mathrm{SO}(W)$. Since $p \neq 2$, $\mathrm{SO}(W)$ is a maximal closed subgroup of $\mathrm{SL}(W)$. Thus, if $g \in \mathrm{SL}(W)$ and $gv - v \in k$, then g normalizes the stabilizer of v. Since $\mathrm{SO}(W)$ is self-normalizing in $\mathrm{SL}(W)$, this implies that g already fixed v. The argument of the previous paragraph still applies to give the claim.

Proof of Theorem 10.1. Suppose first that $\dim V > \dim G + 2$. By [**Ken**, page 15], Lemma 10.4, and Example 10.7, it suffices to consider the following cases.

(a) $G = \mathrm{SL}_8(k) = \mathrm{SL}(W)$ and $V = \wedge^4 W$.

(b) $G = C_4(k)$ and $V = L(\lambda_4)$, $p \neq 2$.

(c) $G = \mathrm{HSpin}_{16}$ and V is a half-spin representation of dimension 128.

(d) $G = \mathrm{SL}_9(k) = \mathrm{SL}(W)$ and $V = \wedge^3 W$.

(e) $G = C_4(k)$ and $V = L(\lambda_3)$, $p = 3$.

The first four cases follow from [**V**] for char $k \neq 2, 3, 5$, as we now explain using [**Levy**]. Set $e = 1$ for cases (a)–(c), and $e = 2$ for case (d). Then there is a subspace $\mathfrak{c}$ of V such that $\dim \mathfrak{c} \leqslant (1/e) \operatorname{rank} G$, and there is a finite and surjective morphism of varieties $\mathfrak{c} \to V/G$; see [**Levy**, Lemma 2.7 and page 432], respectively. Therefore, $\dim V - \dim G + \dim G_v \leqslant (1/e) \operatorname{rank} G$. For each of the possibilities for G and V, one checks that $e(\dim V - \dim G) = \operatorname{rank} G$, so G_v is finite for generic v if char $k \neq 2, 3, 5$.

In any characteristic, for (a), (c), and (d), the representation is minuscule, and so equals the Weyl module of the same highest weight over k, and Lemma 10.3 gives that generic stabilizers have unipotent identity component. For (b) and $p \geqslant 5$, $L(\lambda_4)$ is the Weyl module, and the same argument works; for $p = 3$, we find that the generic stabilizer for the Weyl module V' has unipotent identity component, and it is easy to see that the same holds for $L(\omega_4) = V'/k$.

For (e), $V = L(\omega_3)$ has dimension 40. Let W be the natural eight-dimensional module for G. Then $\wedge^3 W$ has composition factors W, W, V. Now compute for each involution $x \in \mathrm{Sp}(W)$ that $\dim V^x + \dim {}^G x < \dim V$. Thus, for a generic v, G_v contains no involutions, and so no torus.

We are reduced to considering the cases $\dim V = \dim G + 1$ or $\dim G + 2$. First suppose that G is tensor decomposable. If d is the minimal dimension of an irreducible representation of G, then $\dim V \geqslant d^2 > \dim G + 2$ unless $G = A_n$, and up to a twist $V \cong W \otimes W'$, where W is the natural module for G and W' is a Frobenius twist of W or W^*. By Lemma 10.4, the result holds in this case.

So we may assume that V is tensor indecomposable and in particular is restricted if $p \neq 0$. Inspection of the tables in [**Lüb01**] leave the following possibilities.

(f) $G = A_1$ and $V = L(3\omega_1)$ or $L(4\omega_1)$ with $p \geqslant 5$.

(g) $G = A_2$, $V = L(3\omega_1)$ and $p \geqslant 5$.

(h) $G = A_3$, $V = L(\omega_1 + \omega_2)$ and $p = 3$.

(i) $G = B_2$, $V = L(\omega_1 + \omega_2)$ and $p = 5$.

In case (f), G_v is finite for generic v by Lemma 10.6. In case (g), the result follows by Lemma 10.3. In case (h), [**CohW**] shows that G_v is finite. It remains to consider case (i). Note that the nonidentity central element z of G acts nontrivially. Let $x \in G$ be a noncentral involution (there is precisely one such conjugacy class). Since x and xz are conjugate, it follows that $\dim V^x = 6 =$

$(1/2) \dim V$. Since $\dim {}^G x = 4$, we see that, for generic $v \in V$, G_v contains no involutions, whence the result. $\qquad\square$

11. Same rings of invariants: examples

In this section, k is a field of characteristic $p \geqslant 0$. We give examples of simple algebraic groups $G < H < \mathrm{GL}(V)$ so that $k[V]^G = k[V]^H$.

Example 11.1 ($\mathrm{Spin}_{11} \subset \mathrm{HSpin}_{12}$). Let V be a half-spin module for $H = \mathrm{HSpin}_{12}$, and consider the subgroup $G = \mathrm{Spin}_{11}$. Suppose first that $\mathrm{char}\, k \neq 2$. Igusa calculates in [**Ig**] that $k[V]^G = k[V]^H = k[f]$ for a homogeneous quartic form f on V. (Alternatively, this can be understood from the point of view of internal Chevalley modules as in [**AzBS**] and [**Rub**, Theorem 4.3(3)]. For the determination of the ring of invariants, it suffices to note that there is an H-invariant quartic form on V, which can be constructed from the root system as in [**He**].)

Now let $\mathrm{char}\, k = 2$. The representation V of H is obtained by base change from a representation defined over $\mathbb{Z}$, and the quartic form as in the previous paragraph reduces to the square of a quadratic form q, that is, $k[V]^H$ contains $k[q]$, and in fact we have equality because $\dim k[V]^H = 1$ by [**BeGL**, 6.2]. Now, $\dim k[V]^G = 1$ by [**GuLMS**, 2.11], so $k[V]^G = k[V]^H = k[q]$.

Example 11.2 ($\mathrm{PGL}_3 \subset G_2$ in char. 3). Suppose that $p = 3$, and put V for the seven-dimensional irreducible representation of H split of type G_2. The short root subgroups of H generate a subgroup G isomorphic to PGL_3 (see, for example, [**ConGP**, Section 7.1]) such that the restriction of V to G is the irreducible part of the adjoint representation [**Sei**]. (Alternatively, the inclusion $\mathrm{PGL}_3 \leqslant G_2$ can be viewed from the perspective of octonion algebras as in [**MaV**].)

There is a nonzero G_2-invariant quadratic form q on V, and G_2 acts transitively on the nonvanishing set of q in $\mathbb{P}(V)$; it follows that $k[V]^H = k[q]$. As A_2 has finitely many orbits on $\mathbb{P}(V)$ [**GuLMS**, 2.5], it follows that $k[V]^G$ must equal $k[q]$ too.

Example 11.3 (SO_{2n} in Sp_{2n} in char. 2). Let W be a $2n$-dimensional vector space over k algebraically closed of characteristic 2, for some $n \geqslant 3$. Write q for a nondegenerate quadratic form on W and b for the alternating form $b(v, w) := q(v + w) + q(v) + q(w)$ for $v, w \in W$. Put $\mathrm{SO}(W)$ and $\mathrm{Sp}(W)$ for the special orthogonal group of q and the symplectic group of b, respectively. Set V to be the irrep of $\mathrm{Sp}(W)$ with highest weight ω_2 as in Example 8.5. We now sketch a proof that $k[V]^{\mathrm{SO}(W)} = k[V]^{\mathrm{Sp}(W)}$.

Recall the definition of the space Y of self-adjoint operators from Example 8.5. We claim that for semisimple self-adjoint operators T, $\mathrm{Sp}(W) \cdot T = \mathrm{SO}(W) \cdot T$. To see this, note that T is a polynomial in a semisimple self-adjoint operator S whose minimal polynomial has degree n. Thus $W = W_1 \perp W_2 \perp \cdots \perp W_n$ where the W_i are two-dimensional nondegenerate (and are the distinct eigenspaces of S). Since $\mathrm{SO}(W)$ is transitive on such decompositions, the $\mathrm{SO}(W)$ and $\mathrm{Sp}(W)$ orbits of T agree. As the semisimple elements are dense in Y, it follows that $k[Y]^{\mathrm{Sp}(W)} = k[Y]^{\mathrm{SO}(W)}$, and the same proof shows that the rings of invariants also coincide for Y_0, and if n is even for $Y_0/Y_0^{\mathrm{Sp}(W)}$.

Note that this representation of $\mathrm{SO}(W)$ is the irreducible part of the adjoint module, and that roughly speaking the generators of the ring of invariants have half the degree one finds in characteristic zero.

Example 11.4 (F_4 in char. 2). Let $H = F_4$, and take V to be the 26-dimensional irreducible representation for k a field of characteristic 2. Then H contains subgroups G of type C_4 and adjoint D_4, and the restriction of V to G is the representation studied in the previous example; see [**Sei**, Section 4]. Viewing V as the space of trace-zero elements in an Albert algebra A and H as the automorphism group of A, we see that H preserves the coefficients of the generic minimal polynomial on A whose restrictions to V are algebraically independent functions of degree 3 (the norm) and 2 (sometimes called the 'quadratic trace'). The last paragraph of Example 8.3 gives that $k[V]^G$ is a polynomial ring with generators of degrees 3 and 2; as it is contained in $k[V]^H$, we conclude that $k[V]^H = k[V]^G$.

Example 11.5 (Tensor decomposable in positive characteristic). Let $G = \mathrm{SL}(W)$, where W is a k-vector space and char $k \neq 0$. Let σ be a Frobenius twist on H. Let V be either $W \otimes W^\sigma$ or $W^* \otimes W^\sigma$. Then G is contained in $H = \mathrm{SL}(W) \otimes \mathrm{SL}(W)$, and G and H both act irreducibly on V. Identifying V with $\mathrm{End}(W)$, we see that H leaves invariant $\det$. Since G has a dense orbit on $\mathbb{P}(V)$ [**GuLMS**, Lemma 2.6], it follows that $k[V]^G = k[V]^H = k[\det]$. Note also that the generic point of V has a finite stabilizer in G, as follows by dimension and the fact that G has a dense orbit on $\mathbb{P}(V)$.

12. Representations with few invariants

We now prove the following result, which is most interesting in prime characteristic. We use what is known in characteristic zero (as in [**SaK**], [**Kac**], or [**PoV**]) in our proof.

PROPOSITION 12.1. *Let* $G \leqslant \mathrm{SL}(V)$ *be a simple algebraic group over an algebraically closed field* k, *such that* V *is an irreducible* G-*module. Then, up to a Frobenius twist or a twist by a graph automorphism, we have the following.*

Table C. Simple $G \leqslant \mathrm{SL}(V)$ with $k[V]^G = k$.

G	V	$\dim V$	Symplectic?	char k
A_n	λ_1	$n+1$	No	All
A_n (even $n \geqslant 4$)	λ_2	$\binom{n+1}{2}$	No	All
B_n	λ_1	$2n$	Yes	2
C_n	λ_1	$2n$	Yes	All
D_5	Half-spin	16	No	All
G_2	λ_1	6	Yes	2

Table D. Simple $G \leqslant \mathrm{SL}(V)$ with $k[V]^G = k[q]$ for a nonzero quadratic form q.

G	V	$\dim V$	char k	G	V	$\dim V$	char k
B_n	λ_1	$2n+1$	$\neq 2$	A_1	$\lambda_1 + p^i\lambda_1$ $(i \geqslant 1)$	4	$= p \neq 0$
D_n	λ_1	$2n$	All	A_2	$\lambda_1 + \lambda_2$	7	3
A_1	$2\lambda_1$	3	$\neq 2$	A_3	λ_2	6	All
A_5	λ_3	20	2	B_4	λ_4	16	All
B_3	λ_3	8	All	B_5	λ_5	32	2
C_3	λ_3	8	2	C_3	λ_2	13	3
D_6	Half-spin	32	2	G_2	λ_1	7	$\neq 2$
E_7	λ_7	56	2	F_4	λ_4	25	3

(1) $k[V]^G = k$ *if and only if* (G, V) *appears in Table* C.

(2) $k[V]^G = k[q]$ *for a nonzero quadratic form* q *if and only if* (G, V) *appears in Table* D.

Proof. The dimension of $k[V]^G$ is at most 1 if and only if G has an open orbit in $\mathbb{P}(V)$. Therefore, to identify all pairs (G, V) with $k[V]^G = k$ or $k[q]$, it suffices to examine the list of (G, V) with finitely many orbits in $\mathbb{P}(V)$ from Tables I and II of [**GuLMS**]. Some of the entries in Table I are excluded because they have $\dim k[V]^G = 1$ (for example, because they are defined over $\mathbb{Z}$ and $\dim \mathbb{C}[V]^G = 1$) and are not self-dual. The spin representation of B_4 is defined over $\mathbb{Z}$, and $\mathbb{C}[V]^{B_4}$ is generated by a quadratic form, so the same holds over k. The representations (A_5, λ_3) and (E_7, λ_7) from Table I behave like (D_6, λ_6) as described in Example 11.1; when char $k = 2$, they belong to Table D. The half-spin representation of D_5 has an open orbit in characteristic $\neq 2$ by [**Ig**] and in characteristic 2 by [**Lie**, 2.9], so it belongs to Table C. The invariants of $(A_1, \lambda_1 + p^i\lambda_1)$, $(A_2, \lambda_1 + \lambda_2)$, (C_3, λ_2), (F_4, λ_4), and (B_5, λ_5) were determined in Section 11. The ring of invariants of $(A_3, \lambda_1 + \lambda_2)$ is generated by an octic form

Table E. Representations excluded from Theorem 15.1.

G	H	dim V	char k	Degrees	See
Spin_{11}	HSpin_{12}	32	All	$\begin{cases} 4 & \text{if char } k \neq 2 \\ 2 & \text{if char } k = 2 \end{cases}$	11.1
PGL_3	G_2	7	3	2	11.2
SO_{2n} ($n \geqslant 3$)	Sp_{2n}	$\begin{cases} 2n^2 - n - 2 & \text{if } n \text{ even} \\ 2n^2 - n - 1 & \text{if } n \text{ odd} \end{cases}$	2	$\begin{cases} ? \\ 2, 3, \ldots, n \end{cases}$	8.5, 11.3
SO_8 or Sp_8	F_4	26	2	2, 3	11.4
SL_n	$\mathrm{SL}_n \otimes \mathrm{SL}_n$	n^2	$\neq 0$	n	11.5

[**Chen**]. The representation (C_3, λ_3) with char $k = 2$ is in Table II, so its ring of invariants is one dimensional; on the other hand, this representation is identified with the spin representation of B_3, so it leaves a quadratic form invariant.   □

13.   Same transcendence degree

The proof of our main result, Theorem 15.1, relies on showing that there are very few inclusions of groups $G < H \leqslant \mathrm{SL}(V)$ where the rings of invariant functions $k[V]^G$ and $k[V]^H$ have the same (Krull) dimension (equivalently $k(V)^G$ and $k(V)^H$ have the same transcendence degree). We actually prove a stronger result, namely that, when the dimensions are the same, the rings are actually the same.

THEOREM 13.1. *Suppose that $G < H \leqslant \mathrm{SL}(V)$ with G a simple algebraic group over an algebraically closed field k acting irreducibly on V, and H closed in $\mathrm{SL}(V)$. If $\dim k[V]^G = \dim k[V]^H$, then $k[V]^G = k[V]^H$ and one of the following holds, up to a Frobenius twist and/or a twist by a graph automorphism.*

(a) *$H = \mathrm{SL}(V)$, (G, V) is in Table C and $k[V]^G = k$.*

(b) *$H = \mathrm{Sp}(V)$, char $k = 2$, $G = G_2$, dim $V = 6$, and $k[V]^G = k$.*

(c) *$H = \mathrm{SO}(V)$, (G, V) is in Table D and $k[V]^G = k[q]$.*

(d) *(G, H, V) is in Table E.*

The case when $k = \mathbb{C}$ of the theorem, under the additional hypothesis that $k[V]^G = k[V]^H$, was previously investigated in [**So**] and [**Sc 08**]. Of course, in the positive characteristic case, we can always replace V by a Frobenius twist (this does change the module but not the subgroup of $\mathrm{SL}(V)$). We will ignore

this distinction in what follows. In particular, if V is tensor indecomposable, we will assume that V is restricted.

We remark that in cases (a)–(d), we do have $k[V]^G = k[V]^H$.

In our theorem, we find *a fortiori* that, in almost all of the examples with $\dim k[V]^G = \dim k[V]^H$, the rings of invariants are polynomial rings. Such representations have been called variously regular, coregular, or cofree; they have been classified in characteristic zero in the papers [**KacPV**], [**Sc 78**], etc.; cf. the books [**PoV**] or [**Po 92**]. The proofs below do not rely on the full strength of those results, but rather only on the determination of those representations with $\dim k[V]^G \leqslant 1$, which can be found in [**SaK**] or [**Kac**] in the case when $k = \mathbb{C}$ or in [**GuLMS**] for arbitrary k; see Section 12. Note that, if k does not have characteristic 2, then all examples have $k[V]^G$ of dimension at most 1.

We start with some lemmas.

LEMMA 13.2. *Let k be an algebraically closed field with H a connected reductive group over k. Suppose that G is a proper reductive subgroup of H and that U is a connected unipotent subgroup of H. Then GU is not dense in H.*

Proof. Suppose that GU is dense in H; we may assume that U is a maximal connected unipotent subgroup, that is, the unipotent radical of a Borel subgroup. By [**Ri 77**, Theorem A], H/G is an affine variety; hence every orbit of U on H/G is closed (as follows from the Lie–Kolchin theorem). So if U has a dense orbit on H/G, it has only one orbit; hence $H = GU$.

Then $(U \cap G)^\circ$ is a *maximal* connected unipotent subgroup of G. Indeed, let V be a maximal connected unipotent subgroup of G; so $V \leqslant U^g$ for some $g \in H$. However, $H = GU$ implies that $V = G \cap U^g$ is conjugate in G to $G \cap U$.

Thus $\dim GU = \dim G + \dim U - \dim V = \mathrm{rank}(G) + \dim V + \dim U < \mathrm{rank}(H) + 2 \dim U = \dim H$. $\qquad\square$

LEMMA 13.3. *Suppose that $G < H < \mathrm{SL}(V)$, where G and H are connected reductive. If $k[V]^G$ and $k[V]^H$ have the same transcendence degree, then, for generic $v \in V$, GH_v is dense in H, and the identity component of H_v is not unipotent.*

Proof. The transcendence degree of $k[V]^H$ is the codimension of the highest-dimensional orbit of H on V. Thus, for v generic, Gv is dense in Hv, whence GH_v is dense in H. Now, by Lemma 13.2, this cannot happen if the identity component of H_v is unipotent. $\qquad\square$

LEMMA 13.4. *Suppose that $G < H < \mathrm{SL}(V)$ and that $k[V]^G$ and $k[V]^H$ have the same transcendence degree. Assume further that G is a simple algebraic group and that V is a tensor decomposable irreducible G-module. Then*

$G = \mathrm{SL}(W)$, *up to a twist* $V = W \otimes W'$, *where* W' *is a (nontrivial) Frobenius twist of* W *or* W^*, *and* $H = \mathrm{SL}(W) \otimes \mathrm{SL}(W)$.

Proof. Note that H is semisimple and $\dim V$ is not prime. Suppose that H is simple and that V is tensor decomposable for H. Then it follows by Theorem 10.1 that the generic stabilizer in H of an element of V has unipotent identity component, contradicting Lemma 13.3.

If H is simple and V is tensor indecomposable for H, then, by [**Sei**, Theorem 1], H is $\mathrm{SL}(V)$, $\mathrm{SO}(V)$, or $\mathrm{Sp}(V)$. In particular, $k[V]^H = k$ or $k[q]$ with q quadratic, so $\dim k[V]^G \leqslant 1$ as well. On the other hand, as we have already noted, $\dim V > \dim G + 1$ unless $G = \mathrm{SL}(W)$ and $V = W \otimes W'$, where W' is a twist of W or W^*. In that case $k[V]^G = k[f]$ with f of degree $\dim W$. Thus, $\dim V = 4$, and $H = \mathrm{SO}(V)$ is tensor decomposable.

So we may assume that H is not simple.

First consider the case that G is maximal in H. It follows that H is a central product of two copies of G, and G embeds diagonally in H. For convenience we consider $\tilde{H} = G \times G$. Let π_i denote the projection onto the ith factor.

Thus, $V = W_1 \otimes W_2$, where the first copy of G acts trivially on W_2 and the second copy acts trivially on W_1. Assume that $\dim W_1 \leqslant \dim W_2$. A straightforward computation shows that, for a generic $v \in V$, $\pi_2(\tilde{H}_v)$ is an injection into $\mathrm{SL}(W_2)$. Moreover, if $\dim W_1 < \dim W_2$, the $\pi_2(\tilde{H}_v)$ is proper in G. In particular, $\dim H_v < \dim G$, whence $\dim(G H_v) < 2 \dim G = \dim H$. Thus $\dim Gv < \dim Hv$, and so $\dim k[V]^G > \dim k[V]^H$.

If $\dim W_1 = \dim W_2$, we can view $G \leqslant \mathrm{SL}(W)$ and $G \leqslant \mathrm{SL}(W) \leqslant H \leqslant Y := \mathrm{SL}(W) \times \mathrm{SL}(W) < \mathrm{SL}(V)$. Note that, if v is a generic point of V, then Y_v is a diagonal subgroup of Y. If G is proper in $\mathrm{SL}(W)$, then $\dim H_v$ is generically less than $\dim G$ since the intersection of H with a generic Y_v will not be a full diagonal subgroup of H. In that case, $\dim G + \dim H_v < \dim H$, whence $\dim k[V]^G > \dim k[V]^H$.

If $G = \mathrm{SL}(W)$, then (since V is irreducible) we see that V is a twist of $W \otimes W'$, where W' is a Frobenius twist of W or W^* as allowed in the conclusion.

If G is not maximal in H, then we can choose Y with $G < Y < H$ with G maximal in Y, whence by induction we are in the one case allowed. Then Y is maximal in $\mathrm{SL}(V)$ [**Sei**, Theorem 3], a contradiction. $\square$

We need one more preliminary result.

LEMMA 13.5. *Let* $G = \mathrm{SL}(W)$ *with* $\dim W = 2m > 2$. *Then* $\mathrm{SO}(W) \, g \, \mathrm{Sp}(W)$ *is not dense in* G *for any* $g \in G$.

Proof. If G is in characteristic 2, then $\mathrm{SO}(W) \leqslant \mathrm{Sp}(W)$. Hence by [**GoGu**], $\dim(\mathrm{Sp}(W) \cap {}^g \mathrm{Sp}(W)) \geqslant 3m$, whence the result. In any other characteristic,

we can take $\mathrm{SO}(W) = C_G(\tau)$, where τ is the inverse transpose map and $^g \mathrm{Sp}(W) = C_G(\tau J)$, where J is a skew-symmetric matrix. Thus, $\mathrm{SO}(W) \cap {}^g \mathrm{Sp}(W) = C_G(\tau, \tau J)$. Generically, J will be a semisimple regular element, and so this intersection is the centralizer of τ in $\mathrm{SL}(W) \cap k[J]^\times$, which has dimension $m - 1$, whence the result. $\square$

Proof of Theorem 13.1. If V is tensor decomposable for G, the result follows by Lemma 13.4; this case is in Table E. So we assume that V is tensor indecomposable for G (and so also for H). This forces H to be a simple algebraic group as well.

If $H = \mathrm{Sp}(V)$, then (G, V) is symplectic and appears in Table C, giving the claim in (b). As the representations in (a) and (c) have already been listed in Tables C and D, we assume that $H \neq \mathrm{SL}(V), \mathrm{Sp}(V), \mathrm{SO}(V)$, and Theorem 1 in [**Sei**] asserts that (G, H) appears in Table 1 in [**Sei**].

If $\dim V > \dim H$, then H_v° is unipotent for generic v, a contradiction by Lemma 13.3. So we may assume that $\dim V \leqslant \dim H$, and therefore V is one of the representations enumerated in [**Lüb01**].

Suppose that V is the irreducible part of the adjoint module for H. If $\dim V \geqslant \dim H - 2$, then G will not act irreducibly. Otherwise, as in [**Hi**], H has type B, C, or F_4 and $p = 2$, or H has type G_2 and $p = 3$. Consulting Seitz's table, the only possibility for G is $G = D_n$ in the case when $H = C_n$ and $p = 2$; but in this case the representation is a twist of the natural module, and we have $k = k[V]^H \neq k[V]^G$.

We can exclude many cases by exploiting equation (2.1) and its analog for H. Combining these, we obtain

$$\dim G + \dim k[V]^H \geqslant \dim V. \tag{13.6}$$

If H has type E_n for some n, then the only representations in [**Lüb01**] not already considered are the cases $H = E_6$ or E_7 and V the minuscule representation of H of dimension 27 or 56, respectively. In either case, $\dim k[V]^H = 1$. According to Seitz's table, the only possibility for G is C_4 and $p \neq 2$ (and $H = E_6$), in which case the restriction of V to C_4 is $L(\omega_2)$, which has $\dim k[V]^{C_4} \geqslant 2$ by [**GuLMS**].

If $H \cong F_4$, then the only remaining choice for V from [**Lüb01**] is that $\dim V = 26$ (if $p \neq 3$) or 25 (if $p = 3$). Note that the identity component of H_v for generic v is D_4, so $\dim k[V]^H$ is 2 (if $p \neq 3$) or 1 (if $p = 3$). By Seitz's table, the only possibilities for G are G_2 with $p = 7$ (which is too small by (13.6)), or D_4 or C_4 with $p = 2$ (which are in Table E).

If $H \cong G_2$, then $\dim V = 7$ (6 if $p = 2$) is the only remaining possibility, and $\dim k[V]^H \leqslant 1$; hence $\dim G \geqslant 5$. By [**Sei**], the only possibility for G is A_2, with $p = 3$ and V the irreducible part of the adjoint representation of G, as in Table E.

Thus, H is a classical group of rank at least 2, and V is not the natural module.

Comparing the tables in [**Lüb01**] and [**GuLMS**] shows that most of the possibilities for V have $\dim k[V]^H \leqslant 1$. (This includes such cases as $H = \mathrm{SL}_4(k)$ and $V = L(\lambda_1 + \lambda_2)$ for $p = 3$.) Of the possibilities for G listed in Seitz's table, most fail condition (13.6). The only interesting cases are where H has type D and V is a half-spin representation, or $H = \mathrm{SL}(W)$ and $V = \wedge^2 W$ or $L(2\lambda_1)$.

So suppose that $H = \mathrm{HSpin}_n$ and $G = \mathrm{Spin}_{n-1}$ for $n = 10$ or 14. If $\mathrm{char}\,k \neq 2$, Igusa showed in [**Ig**] that $\dim k[V]^G = \dim k[V]^H + 1$. Without restriction on the characteristic, for $n = 14$, $\dim k[V]^H = 1$ by [**BeGL**, Proposition 6.2], whereas $\dim k[V]^G \geqslant 2$ by [**GuLMS**]. For $n = 10$ and $\mathrm{char}\,k = 2$, $k[V]^H = k$ by [**Lie**, 2.9], whereas $k[V]^G$ contains a quadratic form (by reduction from $\mathbb{Z}$). So these cases do not occur. The case when $n = 12$ is in Table E.

Suppose that $H = \mathrm{SL}(W)$ with $\dim W > 3$ and $V = L(\lambda_2)$. If $\dim W$ is odd, then $k[V]^H = k$, and there are no possibilities for G by Table C. Suppose that $\dim W$ is even. Then $\dim k[V]^H = 1$, and the only semisimple maximal subgroups G with large enough dimension to possibly satisfy (13.6) are $\mathrm{Sp}(W)$ or $\mathrm{SO}(W)$. Of course $\mathrm{Sp}(W)$ is not irreducible on V. It follows by Lemma 13.5 that $G\,\mathrm{SO}(W)$ is not dense in $\mathrm{SL}(W)$.

Suppose that $H = \mathrm{SL}(W)$ and $V = L(2\lambda_1)$. As V is restricted, we have $p \neq 2$. The generic stabilizer is $\mathrm{SO}(W)$. There is no semisimple maximal subgroup G of H such that $G\,\mathrm{SO}(W)$ is dense in H, whence the result holds in this case (the only semisimple subgroup of sufficiently large dimension is $\mathrm{Sp}(W)$, but by Lemma 13.5 $\mathrm{Sp}(W)g\,\mathrm{SO}(W)$ is never dense in $\mathrm{SL}(W)$).

We now consider the remaining representations from [**Lüb01**] with $\dim k[V]^H > 1$. Suppose that $H = \mathrm{Sp}_{2m}(W)$ and $V = L(\lambda_2)$: this is the irreducible part of $\wedge^2 W$. By Example 11.3, $\dim k[V]^H = m - 1$ or $m - 2$, depending upon whether p divides m or not. Thus, $\dim G \geqslant \dim V - (m-1)$ or $(m-2)$. The only semisimple maximal subgroup of $\mathrm{Sp}(W)$ with such a dimension is $\mathrm{SO}(W)$ with $p = 2$. This case is in Table E.

The tables in [**Lüb01**] leave only $V = L(\lambda_m)$ for $H = \mathrm{Sp}_{2m}(k)$ for $m = 3$ (all p), or 4, 5, or 6 ($p = 2$ only). But, for these cases, [**Sei**] shows there are no irreducible simple algebraic subgroups. $\qquad\square$

Here is another result, where we allow G to be semisimple.

PROPOSITION 13.7. *Suppose that $G < H \leqslant \mathrm{SL}(V)$ with H a simple algebraic group and G a semisimple irreducible subgroup of $\mathrm{SL}(V)$. If H acts tensor indecomposably on V and $\dim k[V]^G = \dim k[V]^H$, then either G is simple, or $k[V]^G = k$, or $\dim V = 8$ with $G = \mathrm{Sp}_2 \times \mathrm{Sp}_4$ and $H = \mathrm{SO}_8(V)$.*

Proof. Assume that G is not simple. By [**Sei**, Theorem 1], it follows that $H = \mathrm{SO}(V)$, $\mathrm{Sp}(V)$, or $\mathrm{SL}(V)$. In the last two cases, $k[V]^H = k$, whence the result holds.

Thus, we may assume that $H = \mathrm{SO}(V)$ and $\dim V \geqslant 8$. Then $G \leqslant Y :=$ $\mathrm{Sp}(W_1) \otimes \mathrm{Sp}(W_2)$ or $\mathrm{SO}(W_1) \otimes \mathrm{SO}(W_2)$, and moreover G has a dense orbit on $\mathbb{P}(V)$. Assume that $m_1 = \dim W_1 \leqslant m_2 = \dim W_2$. It follows that the stabilizer Y_v of a generic $v \in V$ has dimension equal to $\dim \mathrm{SO}_{m_2-m_1}$ in the second case and equal to $(3/2)(\dim W_1) + \dim \mathrm{Sp}_{m_2-m_1}$ in the first case.

Note that in the second case $m_1 \geqslant 3$. Thus, we see that $\dim k[V]^Y = \dim V - \dim Y + \dim Y_v > 1 = \dim k[V]^H$ unless we are in the first case with $m_1 = 2$ and $m_2 = 2$ or 4. If $m_2 = 2$, then $H = \mathrm{SO}(V)$ is not simple. If $m_2 = 4$, then in fact $\mathrm{Sp}_2 \otimes \mathrm{Sp}_4$ does have a dense orbit on $\mathbb{P}(V)$ because V is an internal Chevalley module [**GuLMS**, Table I]. The only possible proper irreducible subgroup of Y is $\mathrm{SL}_2 \times \mathrm{SL}_2$ and is too small to have a dense orbit on $\mathbb{P}(V)$. $\qquad\square$

14. Same rings of invariants, but without containment of groups

We close our discussion of groups with the same invariants with an easy consequence of Theorem 13.1, where we drop the hypothesis that G is contained in H, but we strengthen the hypothesis on the invariants to be that the rings $k[V]^G$ and $k[V]^H$ are the same.

COROLLARY 14.1. *Assume that G is a simple algebraic group acting irreducibly on V and that $\dim k[V]^G > 1$. If H is a connected algebraic subgroup of $\mathrm{SL}(V)$ acting irreducibly on V with $k[V]^H = k[V]^G$, then $H = G$ or one of the following holds, up to a Frobenius twist and/or a graph automorphism.*

(1) $\mathrm{char}\, k = 2$, $H = \mathrm{Sp}_{2n}$, $G = \mathrm{SO}_{2n}$ *with $n \geqslant 3$, and $V = L(\lambda_2)$.*

(2) $\mathrm{char}\, k = 2$, $\dim V = 26$, G *has type D_4 or C_4, and H has type F_4.*

Proof. Let Y be the group generated by G and H (which we assume is distinct from G). Then $G < Y$ have the same invariants, and so Theorem 13.1 implies that $\mathrm{char}\, k = 2$ and $Y = \mathrm{Sp}$ and $V = L(\lambda_2)$, or $Y = F_4$ and $\dim V = 26$. Also by Theorem 13.1, there is no other simple H, and by Proposition 13.7 (since G is simple) there is no semisimple example either. $\qquad\square$

15. Main theorem

We now fix a pair (G, V) and ask whether there is some $f \in k[V]^G$ such that G is the identity component of the stabilizer of f. Trivially, we must exclude those representations where $k[V]^G = k$ or $k[q]$ for a quadratic form q (when $G \neq \mathrm{SO}(V)$); these make a short list that we provide in Tables C and D. These lists are well known in characteristic zero: a convenient reference is the table at the end of [**PoV**].

In the following theorem, we write k_{alg} for an algebraic closure of a field k.

THEOREM 15.1. *Let $G < \mathrm{GL}(V)$ be an absolutely simple algebraic group over a field k such that V is absolutely irreducible and $(G(k_{\mathrm{alg}}), V \otimes k_{\mathrm{alg}})$ is not in Table C or Table D up to Frobenius twists and graph automorphisms. Then exactly one of the two following statements holds.*

(1) *There exists a homogeneous $f \in k[V]^G$ such that the naive stabilizer of f in $\mathrm{GL}(V \otimes k_{\mathrm{alg}})$ has identity component $G(k_{\mathrm{alg}})$.*

(2) *$(G(k_{\mathrm{alg}}), V \otimes k_{\mathrm{alg}})$ is a Frobenius twist of a representation from Table E.*

The theorem gives tight control over the stabilizer of f, because the normalizer N of G in $\mathrm{GL}(V)$ is known precisely. It has identity component N° generated by G and the scalar matrices, and N itself can hardly be much larger. Indeed, there is an inclusion of N/N° into the automorphism group of the Dynkin diagram of G, which is 1 (type A_1, B, C, E_7, E_8, F_4, or G_2), $\mathbb{Z}/2$ (type A_n for $n \geqslant 2$, D_n for $n \geqslant 5$, or E_6), or has order 6 (type D_4); see for example [**Sp**, Section 16.3]. In any concrete case, one is reduced to checking which representatives of N/N° in N stabilize f.

The shortness of the proof hides the fact that it relies on all of the results from Sections 9–13.

Proof of Theorem 15.1. We assume that $(G(k_{\mathrm{alg}}), V \otimes k_{\mathrm{alg}})$ does not appear in Tables C, D, or E, and we will produce an f as in (1).

Suppose first that k is perfect. By Proposition 9.2 there are only finitely many closed connected proper overgroups of $G \times k_{\mathrm{alg}}$ in $\mathrm{SL}(V \otimes k_{\mathrm{alg}})$; write $\mathscr{O}$ for the set of such. The k-automorphisms of k_{alg} permute the elements of $\mathscr{O}$, so there is a finite Galois extension L of k such that all L-automorphisms of k_{alg} fix every element of $\mathscr{O}$. Write $H_1, \ldots, H_r$ for representatives of the orbits in $\mathscr{O}$ under the group Γ of k-automorphisms of L.

As (G, V), by hypothesis, does not appear in Table E, Theorem 13.1 gives that $\dim L[V]^{H_i} < \dim L[V]^G$ for each i, and we can pick a homogeneous polynomial $f_i \in k[V]^G \setminus k[V]^{H_i}$. Define

$$f := \prod_{i=1}^{r} \prod_{\sigma \in \Gamma} \sigma(f_i).$$

As H_i is semisimple, it has no nontrivial characters, so the irreducible factors (in $L[V]$) of a H_i-invariant function are also H_i-invariant, and hence f is not invariant under any of the H_i. As f is fixed by every element of Γ, it belongs to $k[V]$, and the theorem is proved in this case.

Now suppose that k is imperfect. The case where k is prefect provides an $f \in k^{p^{-\infty}}[V]^G$ as in (1). Writing f as a polynomial in a dual basis for V, we find finitely many coefficients in $k^{p^{-\infty}}$, so there is a positive integer s such that f^{p^s} belongs to $k[V]$ and has stabilizer with identity component $G(k_{\mathrm{alg}})$. $\qquad\square$

Theorem 15.1 can be compared with the following famous theorem of Chevalley [**Sp**, 5.5.3]: if G is a subgroup of a linear algebraic group G', then there is a finite-dimensional representation V of G' so that G is precisely the stabilizer in G' of a point in $\mathbb{P}(V)$. Chevalley's result holds in much greater generality (G need not be simple, or even reductive), but Theorem 15.1 says that for a typical irreducible and tensor indecomposable representation V there is a G-fixed point $[f]$ in $\mathbb{P}(\mathrm{Sym}^d(V))$ for some d such that the identity component of the stabilizer of $[f]$ is G.

16. Realizations of simple groups as automorphism groups

There are several general-purpose mechanisms for realizing a semisimple group (up to isogeny, taking identity components, and avoiding some bad characteristics or special cases) as automorphism groups of some algebraic structure, namely as the automorphism group of

- the spherical building associated with G [**Ti 74**];

- a twisted flag variety of G [**De 77**]; or

- the Lie algebra of G [**St 61**].

These three interpretations are easiest to understand when the group G is split, but it is easy to see via twisting that these interpretations extend to describing also nonsplit semisimple algebraic groups over any field. These bullets only give *adjoint* groups. One can also interpret G as the automorphism group of

- some finite-dimensional k-algebra [**GorP**].

This construction is more precise than the previous one in that it gives G on the nose (and not just up to isogeny) but also less precise in that one does not have control of the algebra. For E_8, this interpretation just gives that E_8 is the automorphism group of its Lie algebra, as in the previous bullet.

Our Theorem 15.1 adds an additional item to this list:

- a homogeneous function f on a representation V of G.

Here one can pick a faithful and absolutely irreducible representation V (if one exists), and get G on the nose (and not just up to isogeny).

Although f is not uniquely determined, there is still enough of a connection between G and f so as to play properties of one off against the other. Here are three examples of such.

First, we can relate isotropy of G (that is, whether G contains a nonzero split k-torus) with isotropy of f (whether there is a nonzero $v \in V$ such that $f(v) = 0$).

LEMMA 16.1. *Suppose that G is a linear algebraic group acting with finite kernel on a vector space V over an infinite field k and that $f \in k[V]^G$ is homogeneous and nonconstant. If G is isotropic, then f is isotropic.*

Proof. Put T for the nontrivial k-split torus in G. Its image in $\mathrm{GL}(V)$ is also a nontrivial split k-torus in $\mathrm{GL}(V)$, so there is a nonzero $v \in V$ that is not fixed by the action of T. But f is homogeneous and T-invariant, so $f(v) = 0$. □

This relationship between the isotropy of G and isotropy of f is similar to what one finds for the Tits algebras of G. (Recall that the Tits algebras defined in [Ti 71] or [KMRT] are classes in the Brauer group of finite separable extensions of k corresponding to dominant weights of G. When G is the spin or special orthogonal group of a quadratic form, the only possibly nontrivial Tits algebra is the Brauer class of the Clifford algebra or the even part of the Clifford algebra.) One knows that, if G is isotropic, then the Tits algebras cannot have too large an index, where the precise bounds depend on the maximal split torus in G and the dominant weight corresponding to the Tits algebra. There is no corresponding converse implication, in that the Tits algebras may all be zero and yet the group can be anisotropic, which occurs for example when G is any of the compact real groups $\mathrm{Spin}(8n)$ for $n \geqslant 1$, G_2, F_4, or E_8. Nonetheless, the one-way implication between isotropy of G and small indexes for the Tits algebras, notably exploited by Merkurjev in [Mer] to disprove Kaplansky's conjecture, is now a standard tool in the study of semisimple algebraic groups, quadratic forms, division algebras, etc., over arbitrary fields. See the survey [Hof 00] or papers such as [Hof 98, Hof 99, Iz, GaS], and [Mey].

Second, Theorem 15.1 gives a way to study k-forms of G. Recall that an algebraic group G' over k (respectively, $f' \in k[V]$) is called a k-*form of G* (respectively, *of f*) if there is an extension field L/k so that $G' \times L$ is isomorphic to $G \times L$ (respectively, there is a $g \in \mathrm{GL}(V \otimes L)$ so that $f' = f \circ g$ in $L[V]$). If f is k-*similar* to a homogeneous form f' (that is, if there is a $g \in \mathrm{GL}(V)$ and $\mu \in k^\times$ so that $f' = \mu f \circ g$ in $k[V]$) then obviously f and f' have isomorphic stabilizers in $\mathrm{GL}(V \otimes L)$ for every extension L of k. That is, taking identity components of stabilizers gives an arrow

$$
\boxed{\begin{array}{c} k\text{-forms of } f \\ \text{up to } k\text{-similarity} \end{array}} \rightarrow \boxed{\begin{array}{c} k\text{-forms of } G \\ \text{up to } k\text{-isomorphism} \end{array}} \tag{16.2}
$$

that is functorial in k. This arrow is well known to be injective (but typically not surjective) in the case where G is the special orthogonal group of a quadratic form f. It is bijective when G has type E_8 and f is the octic form studied in Sections 3 and 5 or the cubic form studied in Section 7. In general, the arrow (16.2) may be injective, surjective, both, or neither, and a careful choice of V and f can guarantee that it is bijective; this question is studied in [**BeR**].

Third, for S the scheme-theoretic stabilizer of f, faithfully flat descent trivially gives a bijection

$$H^1(k, S) \leftrightarrow \boxed{k\text{-forms of } f \text{ up to } k\text{-isomorphism}} \qquad (16.3)$$

that is functorial in k, where H^1 denotes flat cohomology. By examining the number of independent parameters appearing in explicit k-forms of f, one can in principle give an upper bound on the essential dimension of the group S as defined in [**Re 10**] or [**Re 12**]. This is a usual method for giving an upper bound on the essential dimension of an orthogonal group. Our results here give an effective means for describing S as an algebraic group, thereby allowing one to use (16.3) to prove statements about essential dimensions of familiar groups. For example, can studying k-forms of an octic as in Section 3 or 5 give a better upper bound on the essential dimension of E_8 over $\mathbb{C}$? The strongest result currently known is that the essential dimension is at most 231 [**Lem**, Corollary 1.4], which is quite far from the lower bound of 9 [**Re 10**].

We expect that the homogeneous forms provided by our Theorem 15.1 will provide many new avenues for studying simple algebraic groups over arbitrary fields, since the three relationships we have chosen to highlight here are analogous to previously known tools that have already been widely exploited.

Acknowledgements

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