S. Kondo
Nagoya Math. J.
Vol. 99 (1985), 11–30

TYPE II DEGENERATIONS OF $K3$ SURFACES

SHIGEYUKI KONDO

Introduction

A *degeneration* of $K3$ surfaces (over the complex number field) is a proper holomorphic map $\pi: X \to \Delta$ from a three dimensional complex manifold to a disc, such that, for $t \neq 0$, the fibres $X_t = \pi^{-1}(t)$ are smooth $K3$ surfaces (i.e. surfaces X_t with trivial canonical class $K_{X_t} = 0$ and $\dim H^1(X_t, \mathcal{O}_{X_t}) = 0$).

Recently, Kulikov [7], Persson and Pinkham [12] have classified the semi-stable degenerations of $K3$ surfaces into three types and Friedman [2], [3] has studied the local moduli problem for D-semi-stable $K3$ surfaces. On the other hand, Piatetskii-Shapiro and Shafarevich [13], Burns and Rapoport [1] proved the Torelli theorem for Kaehler $K3$ surfaces. One of the next steps for the study of the moduli problem for $K3$ surfaces is to extend the theory of the period of smooth $K3$ surfaces to the degenerate case.

From the point of view of the moduli problem, the following surfaces are fundamental (see (1.6)): A *stable* $K3$ *surface of type* II is a surface $X = X_1 \cup X_2$ with normal crossings such that; (i) X_i is a smooth rational surface ($i = 1, 2$) and $E = X_1 \cap X_2$ is a smooth elliptic curve, (ii) the dualizing sheaf ω_X on X is trivial, (iii) the line bundle $N_{E/X_1} \otimes N_{E/X_2}$ over E is trivial, where N_{E/X_i} is the normal bundle of E in X_i ($i = 1, 2$).

In this paper we define the periods of stable $K3$ surfaces of type II and prove the Torelli theorem for them. Let $X = X_1 \cup X_2$ be a stable $K3$ surface of type II. Then the component X_i is not always minimal and there happens a birational modification between the stable $K3$ surfaces of type II, which is called a modification of type I in [7]. Let $L(X)$ denote the lattice $\{(x_1, x_2) \in H^2(X_1; \boldsymbol{Z}) \oplus H^2(X_2; \boldsymbol{Z}); (x_1, [E_1])_{X_1} = (x_2, [E_2])_{X_2}\}/\boldsymbol{Z}([E_1] - [E_2])$, where $[E_i]$ ($\in H^2(X_i; \boldsymbol{Z})$) is the cohomology class of the double curve $E = X_1 \cap X_2$. Then $L(X)$ is an even unimodular lattice of signature $(1, 17)$.

Received October 27, 1983.

https://doi.org/10.1017/S0027763000021462 Published online by Cambridge University Press

We define the period of the stable $K3$ surface X by a homomorphism ω_X: $L(X) \to J(E)$: = Jacobian variety of E (see (2.8)). The idea of our definition is due to Y. Namikawa [11].

Roughly speaking, our main result is as follows: let X and X' be two stable $K3$ surfaces of type II with the "same" period. Then there is a bimeromorphic map $X \dashrightarrow X'$ which is a composite of modifications of type I (see (2.14)).

The plan of this paper is as follows: in Section 1 we collect the known facts about the semi-stable degenerations of $K3$ surfaces, in Section 2 we state our main results (Theorems (2.14), (2.15)), and Section 3 is devoted to their proofs.

I would like to express my thanks to Professor Yukihiko Namikawa whose insight and encouragement are invaluable.

§1. Semi-stable degenerations of $K3$ surfaces

(1.1) A *semi-stable degeneration* of surfaces (resp. $K3$ surfaces) is a proper holomorphic map π: $X \to \Delta$ from a three dimensional complex manifold to a disc such that: (i) the fibres $X_t = \pi^{-1}(t)$ are smooth surfaces (resp. smooth $K3$ surfaces) for $t \neq 0$; (ii) the central fibre $X_0 = \pi^{-1}(0)$ is a divisor with normal crossings; (iii) all components of X_0 have multiplicity one in the fibre.

If a degeneration of surfaces is projective, it becomes bimeromorphic to a semi-stable one after a base change ([5]).

(1.2) Let π: $X \to \Delta$ be a semi-stable degeneration of surfaces. The *dual graph* of $X_0 = \pi^{-1}(0)$ is the following simplicial complex: (i) The set of vertices is the set of irreducible components of X_0; (ii) The set of edges is the set of components of double curves of X_0; (iii) The set of faces is the set of triple points of X_0.

(1.3) A degeneration of surfaces π: $X \to \Delta$ is *weakly Kaehler* if there exists a bimeromorphic map ϕ: $X \dashrightarrow X'$ such that ϕ is biholomorphic on $X - \pi^{-1}(0)$, the diagram

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ \Delta & = & \Delta \end{array}$$

is commutative and such that X' is a Kaehler manifold.

In the study of the degenerations of $K3$ surfaces, the following results are essential.

(1.4) THEOREM (Kulikov [7], Persson and Pinkham [12]). *Let $\pi: X \to \varDelta$ be a degeneration of $K3$ surfaces. If all components of the central fibre $X_0 = \pi^{-1}(0)$ are algebraic, then X is bimeromorphic to a semi-stable degeneration $\pi': X' \to \varDelta$ with $K_{X'} \equiv \mathcal{O}_{X'}$, where $K_{X'}$ is the canonical line bundle of X'.*

(1.5) THEOREM (Kulikov [7]). *Let $\pi: X \to \varDelta$ be a weakly Kaehler, semi-stable degeneration of $K3$ surfaces with $K_X \equiv \mathcal{O}_X$. Then $X_0 = \pi^{-1}(0)$ is one of the following three types*:

(Type I) *X_0 is a smooth $K3$ surface*;

(Type II) *$X_0 = V_1 + V_2 + \cdots + V_{n-1} + V_n$, where V_1 and V_n are rational surfaces, $V_2, \cdots, V_{n-1}$ are elliptic ruled surfaces and $V_i \cap V_{i+1}$, $i = 1, \cdots, n-1$, are smooth elliptic curves. The dual graph of X_0 is as follows*:

(Type III) *$X_0 = V_1 + \cdots + V_n$, where all V_i's are rational surfaces and the double curves $V_i \cap V_j$ on V_j are smooth rational curves forming a cycle. The dual graph of X_0 is a triangulation of 2-sphere S^2.*

(1.6) *Remark.* In this paper we study the type II degenerations in the above Theorem (1.5). Among them, the type II degenerations without the elliptic ruled components are fundamental in the following sense: let $\pi: X \to \varDelta$ be as in Theorem (1.5). Suppose the central fibre $X_0 = V_1 + V_2 + \cdots + V_{n-1} + V_n$ is of type II. By performing some birational modifications, we can assume that the elliptic ruled components $V_2, \cdots, V_{n-1}$ are minimal. Then we can contract $V_2, \cdots, V_{n-1}$ along the rulings for which the double curves are sections (cf. [2], [4]). This produces a new threefold X' mapping to $\varDelta$, and X' has a curve of A_{n-2} surface singularities. Moreover the new central fibre X'_0 is a surface of type II without the elliptic ruled components. This is similar to the case of degenerations of elliptic curves of type I_b ([6], p. 604).

(1.7) LEMMA. *Let $\pi: X \to \varDelta$ be as in Theorem* (1.5). *Suppose X_0 is of type* II *and without the elliptic ruled component*: *i.e. $X_0 = X_1 \cup X_2$, where*

X_i is a rational surface ($i = 1, 2$) and $E = X_1 \cap X_2$ is a smooth elliptic curve. Then

(i) $E \in |-K_{X_i}|$ $(i = 1, 2)$;

(ii) $N_{E/X_1} \otimes N_{E/X_2} \cong \mathcal{O}_E$;

(iii) $(E^2)_{X_1} + (E^2)_{X_2} = 0$.

Proof. By the adjunction formula and $\mathcal{O}_X(X_0) = \mathcal{O}_X$, we have $K_{X_1} = [K_X + X_1]|_{X_1} = [X_1]|_{X_1} = -[X_2]|_{X_1}$. Hence $E \in |-K_{X_1}|$. Since $N_{E/X_1} = \mathcal{O}_X(X_2)|_E$ and $N_{E/X_2} = \mathcal{O}_X(X_1)|_E$,

$$N_{E/X_1} \otimes N_{E/X_2} = \mathcal{O}_X(X_1)|_E \otimes \mathcal{O}_X(X_2)|_E = \mathcal{O}_X(X_0)|_E = \mathcal{O}_E .$$

Now the statement (iii) is obvious.

(1.8) DEFINITION. *A stable $K3$ surface of type* II (resp. *a quasi-stable $K3$ surface of type* II) is a surface $X = X_1 \cup X_2$ with normal crossings such that X_i is a smooth rational surface, $E = X_1 \cap X_2$ is a smooth elliptic curve and satisfies the following conditions: (i) $E \in |-K_{X_i}|$ $(i = 1, 2)$; (ii) $N_{E/X_1} \otimes N_{E/X_2} \cong \mathcal{O}_E$ (resp. (i) $E \in |-K_{x_i}|$ $(i = 1, 2)$; (ii′) degree $(N_{E/X_1} \otimes N_{E/X_2}) = 0$).

(1.9) *Remark.* More generally, Friedman ([2], [3]) has defined the D-semi-stable $K3$ surfaces (cf. [3], (5.5)). We remark here that a quasi-stable $K3$ surface of type II is D-semi-stable if and only if it satisfies the condition (ii) in (1.8) (i.e. stable).

Every stable $K3$ surface of type II is obviously quasi-stable. In Sections 2 and 3, we shall treat a quasi-stable $K3$ surface of type II rather than a stable one. The following result states that every stable $K3$ surface of type II is nothing but a degenerate fibre of a semi-stable degeneration of $K3$ surfaces.

(1.10) THEOREM (Friedman [2], [3]). *Let X be a stable $K3$ surface of type* II. *Then the Kuranishi space of X looks like $V_1 \cup V_2$, here*

(1) *V_1, V_2 are smooth and meet transversally;*

(2) $\dim V_1 = \dim H^1(X, \theta_X) = 20$, $\dim V_2 = 20$ *and* $\dim (V_1 \cap V_2) = 19$, *where θ_X is a sheaf of derivations of $\mathcal{O}_X$;*

(3) *V_1 is a space corresponding to the topologically trivial deformations;*

(4) *Let X_t be a surface corresponding to a point $t \in V_1 \cup V_2$. Then*

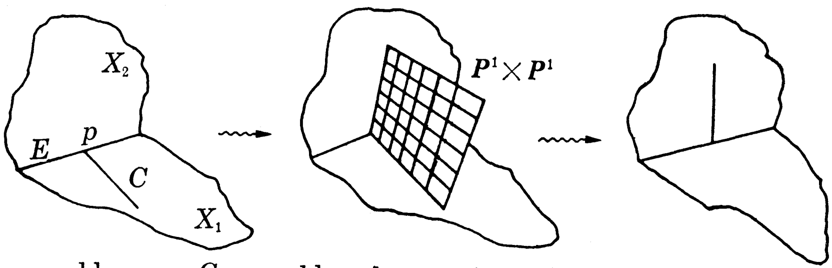
(i) *X_t is a smooth $K3$ surface if $t \in V_2 - V_1$.*

(ii) *X_t is a quasi-stable $K3$ surface of type* II *if $t \in V_1$,*

(iii) *X_t is a stable $K3$ surface of type* II *if and only if $t \in V_1 \cap V_2$.*

(1.11) *Remark.* In [2], [3], Friedman has showed the similar results for every D-semi-stable $K3$ surface.

(1.12) *A modification of type* I is a birational modification of a stable $K3$ surface as follows: Let $X = X_1 \cup X_2$ be a stable $K3$ surface of type II, $E = X_1 \cap X_2$ the double curve and C an exceptional curve of the first kind on X_1. Note that C intersects at exactly one point with E (see (1.13)). By (1.10) we regard X as a cental fibre of a semi-stable degeneration of $K3$ surfaces. Then C can be moved to the adjacent component X_2;



blow up C blow bown $P^1 \times P^1$ in the other direction

For quasi-stable $K3$ surfaces, the modification of type I is defined as follows: on X_1, contracting C to a point, and on X_2, blowing up at $p = E \cap C$.

We close this section with two lemmas for quasi-stable $K3$ surfaces.

(1.13) LEMMA. *Let S be a component of a quasi-stable $K3$ surface of type* II *and C an irreducible curve on S with $E \neq C$ and $(C^2)_S < 0$. Then C is a smooth rational curve such that either*

(1) $(C^2)_S = -1$, $(C, E)_S = 1$, *or*

(2) $(C^2)_S = -2$, $(C, E)_S = 0$,

where $E \in |-K_S|$ is the double curve.

Proof. By $E \in |-K_S|$, the arithmetic genus of C can be computed as follows: $2p_a(C) - 2 = (C^2)_S - (C, E)_S$. The lemma (1.13) can be easily deduced from this formula.

(1.14) LEMMA. *Let $X = X_1 \cup X_2$ be a quasi-stable $K3$ surface of type* II. *Then the possible types for the relatively minimal model of X_i are as follows*: (a) P^2, *or* (b) F_n, $n = 0, 2$.

Proof. Let E be the double curve of X. Let $\overline{X}_i$ be a relatively minimal model of X_i $(i = 1, 2)$. By the classification of surfaces, $\overline{X}_i$ is either P^2 or F_n, $n \geqq 0$, $n \neq 1$. We note that $\overline{X}_i$ has an anti-canonical divisor which is a smooth elliptic curve. If $\bar{X}_i \cong F_n$, then $-K_{F_n} = 2s_n + (n+2)R$, where R is a fibre and s_n is the section with $(s_n^2) = -n$. By the above remark, we have

$$0 \leqq (-K_{F_n}, s_n) = -2n + n + 2 = 2 - n\,, \quad \text{and} \quad n \leqq 2\,.$$

Hence we have proved (1.14).

(1.15) *Remark.* In the following sections, we assume that the self-intersection number $(E^2)_{X_i}$ is equal to zero, $i = 1, 2$ (see (2.4)). In this case, by (1.14), we can choose $\boldsymbol{P}^2$ as a relatively minimal model of X_i ($i = 1, 2$).

§ 2. Periods of stable $K3$ surfaces and Torelli theorem

In this section, we define the period of (quasi-) stable $K3$ surfaces and we state the Torelli theorem. Our statement may be regarded as a degenerate case of the Torelli theorem for Kaehler $K3$ surfaces ([1]). In the following, we shall deal with quasi-stable $K3$ surfaces of type II. For stable $K3$ surfaces of type II, theorems (2.14), (2.15) are also true with some modifications of the period domain (see Remark (2.9), (ii)). For simplicity, we say a quasi-stable $K3$ surface for a quasi-stable $K3$ surface of type II.

(2.1) Let $X = X_1 \cup X_2$ be a quasi-stable $K3$ surface with the double curve E. The Mayer-Vietoris cohomology exact sequence is as follows:

$$0 \longrightarrow H^1(E;\boldsymbol{Z}) \longrightarrow H^2(X;\boldsymbol{Z}) \longrightarrow H^2(X_1;\boldsymbol{Z}) \oplus H^2(X_2;\boldsymbol{Z}) \longrightarrow H^2(E;\boldsymbol{Z}) \longrightarrow 0\,.$$

Put ${}^0W_2(X) := H^2(X;\boldsymbol{Z})$, ${}^0W_1(X) := H^1(E;\boldsymbol{Z})$, and we let ${}^0L(X)$ denote the quotient module ${}^0W_2(X)/{}^0W_1(X)$. Then

$${}^0L(X) \cong \operatorname{Ker}\{H^2(X_1;\boldsymbol{Z}) \oplus H^2(X_2;\boldsymbol{Z}) \longrightarrow H^2(E;\boldsymbol{Z})\}\,.$$

Under this isomorphism, we always regard an element of ${}^0L(X)$ as a class in $H^2(X_1;\boldsymbol{Z}) \oplus H^2(X_2;\boldsymbol{Z})$. Let D_i be a divisor on X_i and denote by $[D_i]$ the cohomology class of D_i. If an element $(\alpha_1, \alpha_2) \in {}^0L(X)$ such that α_i is represented by $[D_i]$, we often denote (α_1, α_2) by $[D_1] + [D_2]$. Let E_i be the double curve on X_i ($i = 1, 2$), then $[E_1] - [E_2]$ is contained in ${}^0L(X)$ for $(E_1^2)_{X_1} + (E_2^2)_{X_2} = 0$.

A *lattice* H is a free abelian group of finite rank endowed with a integral quadratic form. The group $H^2(X_1; \boldsymbol{Z}) \oplus H^2(X_2; \boldsymbol{Z})$ admits a canonical structure of a lattice induced from the cup product. Note that ${}^0L(X)$ inherits a lattice structure from that of $H^2(X_1; \boldsymbol{Z}) \oplus H^2(X_2; \boldsymbol{Z})$. We denote its bilinear form by $\langle\, , \rangle$.

(2.2) *Remark.* ${}^0W_2(X)$, ${}^0W_1(X)$ are the weight filtrations of the mixed Hodge structure on X ([7], p. 960).

In our study, the problem is how to interpret the modifications of type I in the language of cohomology groups. The following lemma will be needed.

(2.3) LEMMA ([2]). *Let $X = X_1 \cup X_2$ be a quasi-stable $K3$ surface with the double curve E and let C be an exceptional curve of the first kind on X_1.* Let *$X' = X_1' \cup X_2'$ be the quasi-stable $K3$ surface with the double curve E' obtained by the modification of type* I *along C. We denote this modification by ϕ_C. Then*

$$N_{E'/X_1'} \otimes N_{E'/X_2'} \cong N_{E/X_1} \otimes N_{E/X_2}$$

and ϕ_C induces a lattice isometry

$$\phi_C^*\colon {}^0L(X') \longrightarrow {}^0L(X)\ .$$

Proof. The first statement follows easily from definition. Let C' be the exceptional curve on X_2' created by ϕ_C. Denote by

$$\pi_1\colon X_1 \longrightarrow X_1' \qquad (\text{resp. } \pi_2\colon X_2' \longrightarrow X_2)$$

the blowing up at $p' = E' \cap C'$ (resp. $p = E \cap C$). For $([D_1'], [D_2']) \in {}^0L(X')$ such that $(D_2', C')_{X_2'} = r$, we define ϕ_C^* by

$$\phi_C^*(([D_1'], [D_2'])) = (\pi_1^*([D_1']) + [rC],\ [(\pi_2)_* D_2']).$$

Then we can easily check that $\phi_C^*([D_1'] + [D_2'])$ is contained in ${}^0L(X)$ and ϕ_C^* is isometric. We leave the proof to the reader.

(2.4) ASSUMPTION. From now on, we assume that the self-intersection number $(E_i^2)_{X_i}$ is equal to zero ($i = 1, 2$). Since $(E_1^2)_{X_1} + (E_2^2)_{X_2} = 0$, every quasi-stable $K3$ surface satisfies this assumption, after performing some modifications of type I.

(2.5) DEFINITION. We keep the notation of (2.1). Let $\pi_i\colon X_i \to \overline{X}_i$ be a relatively minimal model ($i = 1, 2$). Here we choose $\overline{X}_i \cong \boldsymbol{P}^2$ (see (1.15)). By the assumption (2.4), $\pi_i\colon X_i \to \boldsymbol{P}^2$ is the blowing up of $\boldsymbol{P}^2$ at nine points

https://doi.org/10.1017/S0027763000021462 Published online by Cambridge University Press

on a smooth elliptic curve. We denote the distinct exceptional curves of π_i (not necessarily irreducible) which meet E_i by $L_i^1, \cdots, L_i^9$. We suppose that they are indexed in such a way that $L_i^k \subset L_i^{k'}$ implies that $k \geqq k'$. Let H_i be the total transform of the line in $\bar{X}_i = \boldsymbol{P}^2$ which passes through $\pi_i(L_i^1)$ and $\pi_i(L_i^2)$ (at least when $\pi_i(L_i^1) \neq \pi_i(L_i^2)$; otherwise take the tangent line of $\pi_i(E_i)$ at $\pi_i(L_i^1) = \pi_i(L_i^2)$). Note that the set $\{[H_i], [L_i^1], \cdots, [L_i^9]\}$ is a basis of $H^2(X_i; \boldsymbol{Z})$ $(i = 1, 2)$. Any indexed set of exceptional curves $\{L_i^k\}$ thus obtained will be called an *exceptional-configuration* of X. As L_i^k is the unique effective divisor within its cohomology class $[L_i^k]$, we use the same terminology for the corresponding collection $\{[L_i^k]\}$.

(2.6) A basis of ${}^0L(X)$ is given by $\{[E_1], [E_2], [L_1^9] + [L_2^9], [L_i^k] - [L_i^{k+1}], [H_i] - [L_i^1] - [L_i^2] - [L_i^3]; i = 1, 2, k = 1, \cdots, 7\}$. We note that for all quasi-stable $K3$ surfaces, their corresponding lattices ${}^0L(X)$ are isometric each other. Let L (resp. F) be an abstract lattice which is isometric to ${}^0L(X)$ (resp. ${}^0W_1(X)$) for some reference quasi-stable $K3$ surface X with the double curve E and let θ be a vector in L corresponding to $[E_1] - [E_2] \in {}^0L(X)$.

(2.7) DEFINITION. A *marking* of a quasi-stable $K3$ surface X with the double curve E is a lattice isometry

$$\alpha_X\colon {}^0L(X) \oplus {}^0W_1(X) \longrightarrow L \oplus F$$

such that $\alpha_X({}^0W_1(X)) = F$ and $\alpha_X([E_1] - [E_2]) = \pm\theta$. We call the pair (X, α_X) a *marked* quasi-stable $K3$ surface of type II.

Now we define the periods of quasi-stable $K3$ surfaces. The idea of our definition is due to Y. Namikawa ([11]).

(2.8) Let X be a quasi-stable $K3$ surface with the double curve E. Let $\underline{\omega}_X$ be the dualizing sheaf of X (i.e. let $f\colon \bar{X} = X_1 \amalg X_2 \to X$ be the normalization of X, with E_i being the smooth elliptic curve on X_i such that $f(E_i) = E (i = 1, 2)$. Then $\underline{\omega}_X$ is the sheaf of 2-forms ω on $\bar{X}$ holomorphic except for simple poles at E_i $(i = 1, 2)$ and with $\mathrm{Res}_{E_1}\omega + \mathrm{Res}_{E_2}\omega = 0$). By definition (1.8), there is a nowhere vanishing section ω_X of $H^0(X, \underline{\omega}_X)$. Consider the exact homology sequence of the pair $(X_i, X_i - E)$: $\cdots \to H_3(X; \boldsymbol{Z}) \to H_3(X_i, X_i - E; \boldsymbol{Z}) \xrightarrow{\partial} H_2(X_i - E; \boldsymbol{Z}) \to H_2(X_i; \boldsymbol{Z}) \to H_2(X_i, X_i - E; \boldsymbol{Z}) \to \cdots$. We identify $H_k(X_i, X_i - E; \boldsymbol{Z})$ with $H^{4-k}(E; \boldsymbol{Z})$ by the Lefschetz duality. The connecting morphism $\partial\colon H_1(E; \boldsymbol{Z}) \to H_2(X_i - E; \boldsymbol{Z})$ is then dual to the residue homomorphism; in particular, for a cycle $\gamma \in H_1(E; \boldsymbol{Z})$ we have

https://doi.org/10.1017/S0027763000021462 Published online by Cambridge University Press

$$\int_{\partial\gamma} \omega_{X_i} = \int_{\gamma} \mathrm{Res}_E \omega_{X_i} ,$$

where $\omega_{X_i} \in H^0(X_i, \Omega^2_{X_i}(E))$ is a nowhere vanishing section induced from ω_X $(i = 1, 2)$. Let $\{\alpha, \beta\}$ be a basis of $H_1(E; \boldsymbol{Z})$. If necessary, changing α and β, we can normalize ω_X by the condition

$$\int_\alpha \mathrm{Res}_E\, \omega_{X_1} = \tau , \quad \mathrm{Im}\, \tau > 0 , \quad \text{and} \quad \int_\beta \mathrm{Res}_E\, \omega_{X_1} = 1 .$$

Now we regard ${}^0L(X)$ as a subgroup of $\mathrm{Pic}\,(X_1) \oplus \mathrm{Pic}\,(X_2)$ under the canonical isomorphism $H^0(X_i; \boldsymbol{Z}) \cong \mathrm{Pic}\,(X_i)$, $i = 1, 2$. Let ι be a group homomorphism from $\mathrm{Pic}\,(X_1) \oplus \mathrm{Pic}\,(X_2)$ to $\mathrm{Pic}\,(E)$ defined as follows: for $(\alpha_1, \alpha_2) \in \mathrm{Pic}\,(X_1) \oplus \mathrm{Pic}\,(X_1)$,

$$\iota((\alpha_1, \alpha_2)) = j_1^* \alpha_1 \otimes j_2^* \alpha_2^{-1} ,$$

where j_i is an inclusion $E \subset X_i$, $i = 1, 2$. Then, by definition, $\iota({}^0L(X)) \subset \mathrm{Pic}^0\,(E)$ (= the group of divisors of degree zero on E). So we get a group homomorphism ι: ${}^0L(X) \to \mathrm{Pic}^0\,(E)$. On the other hand, we define an Abel-Jacobi isomorphism

$$\xi \colon \mathrm{Pic}^0\,(E) \longrightarrow J(E) := \boldsymbol{C}/\{\boldsymbol{Z} + \boldsymbol{Z}\tau\}$$

by $\xi(\gamma) = \int_\gamma \mathrm{Res}_E\, \omega_{X_1}$ for $\gamma \in \mathrm{Pic}^0\,(E)$.

We define a group homomorphism ω_X: ${}^0L(X) \to J(E)$ by the composite of ι and ξ; $\omega_X = \xi \circ \iota$. Since $\mathrm{Res}_E\, \omega_{X_1} + \mathrm{Res}_E\, \omega_{X_2} = 0$, the above definition is independent of selecting the component X_i of X. Put $L^* := \mathrm{Hom}\,(L, \boldsymbol{Z})$. Let H^+ be the upper-half plane and $\boldsymbol{Z}^{2\times 19}$ a lattice in $L^*_{\boldsymbol{C}} := L^* \otimes \boldsymbol{C}$ which acts on $H^+ \times L^*_{\boldsymbol{C}}$ as follows: for $(\tau, (z_j)_{1 \leq j \leq 19}) \in H^+ \times L^*_{\boldsymbol{C}}$ and $(m^1_j, m^2_j)_{1 \leq j \leq 19} \in \boldsymbol{Z}^{2\times 19}$,

$$(m^1_j, m^2_j) \colon (\tau, (z_j)_j) \longrightarrow (\tau, (z_j + m^1_j \cdot \tau + m^2_j)_j) .$$

Let $\Omega := \{H^+ \times L^*_{\boldsymbol{C}}\}/\boldsymbol{Z}^{2\times 19}$ denote the quotient space. Let (X, α_X) be a marked quasi-stable $K3$ surface with the double curve E. Then the period of smooth elliptic curve E determines a point in H^+ as usual; we denote it by $\alpha_X(\tau_X) \in H^+$. As mentioned above ω_X is now considered as a homomorphism from ${}^0L(X)$ to $\boldsymbol{C}$ modulo $\boldsymbol{Z} + \boldsymbol{Z}\alpha_X(\tau_X)$. Hence we think of ω_X as a homomorphism from L to $\boldsymbol{C}$ modulo $\boldsymbol{Z} + \boldsymbol{Z}\alpha_X(\tau_X)$; we denote it by $\alpha_X(\omega_X)$. In this way each marked quasi-stable $K3$ surface (X, α_X) determines a point $[(\alpha_X(\tau_X), \alpha_X(\omega_X))] \in \Omega$. We call $[(\alpha_X(\tau_X), \alpha_X(\omega_X))]$ the *period* of (X, α_X)

and Ω the *period domain* for quasi-stable $K3$ surfaces.

(2.9) *Remark.* (i) The homomorphism ω_X coincides with the extension class of mixed Hodge structure on X in the sense of Carlson's (cf. [2]).

(ii) The condition $N_{E/X_1} \otimes N_{E/X_2} \cong \mathcal{O}_E$ implies that $\omega_X([E_1]-[E_2]) \equiv 0$ in $J(E)$. Hence if we take the quotient lattice ${}^0L(X)/\boldsymbol{Z}([E_1]-[E_2])$ for L, we can construct the period domain for stable $K3$ surfaces.

(iii) We can easily check that the periods of quasi-stable $K3$ surfaces are invariant under the modifications of type I in the following sense: let X be a quasi-stable $K3$ surface with the double curve E and $\phi\colon X \dashrightarrow X'$ a modification of type I. We also think of E as the double curve of X'.

Then for any $([C_1], [C_2]) \in {}^0L(X')$,

$$\omega_X(\phi^*(([C_1], [C_2]))) = \omega_{X'}(([C_1], [C_2])) \quad \text{in } J(E)$$

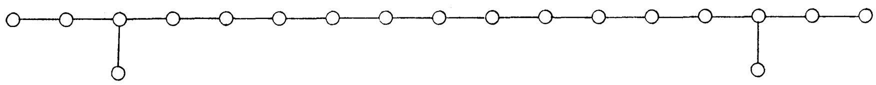
(see (2.3)).

Before stating the Torelli theorem, we need some definitions. In the following, we refer to [1], [8], [9] and [14] for the reflection groups and its geometric applications.

(2.10) We keep the notation of (2.1). Let X be a quasi-stable $K3$ surface with the double curve E. We fix an exceptional configuration $\{L_i^k\}$ of X. Let $L(X)$ denote the quotient module ${}^0L(X)/\boldsymbol{Z}([E_1]-[E_2])$. Then $L(X)$ has a lattice structure induced from that of ${}^0L(X)$. Moreover, by the expression of (2.6), $L(X)$ is isometric to $H \oplus (-E_8) \oplus (-E_8)$, where H is the lattice of rank 2 with the corresponding matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E_8 is the lattice of rank 8 with the Cartan matrix of the root system E_8. For brevity, we also denote the bilinear form of $L(X)$ by $\langle\, ,\rangle$ and denote an element $[D_1] + [D_2] \bmod ([E_1]-[E_2])$ of $L(X)$ by $[D_1]+[D_2]$.

We let Δ_X denote the set $\{[L_1^9]+[L_2^9], [L_i^k]-[L_i^{k+1}], [H_i]-[L_i^1]-[L_i^2]-[L_i^3];\ \mathrm{i}=1, 2,\ k=1, 2, \cdots, 8\}$. As mentioned above, we regard Δ_X as a subset $L(X)_{\boldsymbol{R}}$. Any class $\delta \in \Delta_X$ determines an automorphism s_δ of $L(X)_{\boldsymbol{R}}$ defined by $s_\delta(x) = x + \langle x, \delta\rangle\delta$ for $x \in L(X)_{\boldsymbol{R}}$. Note that s_δ is a reflection for the hyperplane orthogonal to δ. Since the signature of $L(X)$ is $(1, 17)$, the set $\{x \in L(X)_{\boldsymbol{R}};\ \langle x, x\rangle > 0\}$ has two connected components; write $P_X^+ \cup P_X^- = \{x \in L(X)_{\boldsymbol{R}};\ \langle x, x\rangle > 0\}$. Here P_X^+ is the component which contains an element (κ_1, κ_2), where κ_i is the cohomology class of the 2-form corresponding to a Kaehler metric on X_i $(i=1, 2)$ and satisfies the condition $\langle\kappa_1, [E_1]\rangle = \langle\kappa_2, [E_2]\rangle$. The following result is known.

(2.11) PROPOSITION (cf. [14]). *Let* W_X *be the reflection group generated by* Δ_X *and* C_X *denote the set* $\{x \in P_X^+;\ \langle x, \delta\rangle > 0$ *for all* $\delta \in \Delta_X\}$. *Then* W_X *acts on* P_X^+ *and* the *closure of* C_X *in* P_X^+ *is a fundamental domain for this action. Moreover, the Coxeter diagram of* W_X *is as follows*:



(2.12) PROPOSITION. *Let* R_X *denote the set* $W_X \cdot \Delta_X$. *Then* R_X *agrees with the set of all elements* $\alpha \in L(X)$ *with* $\langle \alpha, \alpha\rangle = -2$.

Proof. Let Γ be the subgroup of the group of isometries of $L(X)$ generated by the reflections $\{s_\delta;\ \delta \in L(X),\ \langle \delta, \delta\rangle = -2\}$. Then $\{s_\delta;\ \delta \in \Delta_X\}$ is a generator of Γ (see [14], § 3). Hence we have $W_X = \Gamma$. Let α be an element of $L(X)$ with $\langle \alpha, \alpha\rangle = -2$ and let denote H_α the hyperplane $\{x \in L(X)_R;\ \langle x, \alpha\rangle = 0\}$. By (2.11), we can choose $w \in W_X$ such that $w(H_\delta) = H_\alpha$ for some $\delta \in \Delta_X$. Since $H_{w(\delta)} = w(H_\delta)$, we have $H_\alpha = H_{w(\delta)}$. So $\alpha = r \cdot w(\delta)$ for some $r \in \boldsymbol{R}$. It then follows that $\alpha = \pm w(\delta)$.

(2.13) We call C_X in (2.11) the *fundamental chamber* of X endowed with the exceptional configuration $\{L_i^k\}$. The convex polyhedron C_X defines the partition $R_X = R_X^+ \amalg R_X^-$, where $R_X^+ = \{\delta \in R_X;\ \langle \delta, x\rangle > 0$ for all $x \in C_X\}$. This partition has the property that

(*) If $\alpha_1, \cdots, \alpha_n \in R_X^+$, and $\alpha = \sum_{i=1}^n r_i \alpha_i \in R_X$ ($r_i > 0$ integers), then $\alpha \in R_X^+$ (e.g. [1], p. 241).

An element $\alpha \in R_X$ is called *nodal* if either α is represented by a smooth rational curve with self-intersection number -2 or there is a sequence $\{X \xrightarrow[\phi_1]{} X_1 \longrightarrow \cdots \xrightarrow[\phi_r]{} X_r\}$ of modifications of type I such that α is represented by $\phi_1^* \circ \cdots \circ \phi_r^*([C])$ (see (2.3)), where C is a smooth rational curve on X_r with self-intersection -2. We denote the set of all nodal classes by Δ_X^n. Let W_X^n be the reflection group generated by Δ_X^n and put $R_X^n := W_X^n \cdot \Delta_X^n$. Note that if $\alpha \in R_X^n \cap \Delta_X$, then α is of one of the following types; (a) $\alpha = [L_1] + [L_2]$, where L_i is an exceptional curve of the first kind on X_i $(i = 1, 2)$, (b) $\alpha = [C_1] + \cdots + [C_k]$ $(k \geqq 1)$, where C_i is a smooth rational curve with self-intersection -2 and $(C_i, C_{i+1}) = 1$, $(C_i, C_j) = 0$ for $i \neq j \pm 1$. Moreover, in Section 3, we shall characterize R_X^n as follows (see 3.4)): $R_X^n = \{\alpha \in R_X;\ \omega_X(\alpha) = 0$ in $J(E)\}$. Let C_X^n be the set $\{x \in P_X^+;\ \langle x, \delta\rangle > 0$ for all $\delta \in \Delta_X^n\}$. Then Proposition (2.11) holds for the action of W_X^n on P_X^+

and a fundamental domain C^n_X. We remark that C^n_X is independent of the choice of an exceptional configuration of X. We call C^n_X the *nodal chamber* of X. Now we formulate our main results.

(2.14) THEOREM. *Let* $X = X_1 \cup X_2$ *and* $X' = X'_1 \cup X'_2$ *be two quasi-stable* $K3$ *surfaces of type* II *with the double curves* E, E', *respectively. Let* ϕ^*: ${}^0L(X') \oplus {}^0W_1(X') \to {}^0L(X) \oplus {}^0W_1(X)$ *be an isometry such that* (i) $\phi^*({}^0W_1(X')) = {}^0W_1(X)$, (ii) $\phi^*([E'_1] - [E'_2]) = \pm([E_1] - [E_2])$ *(By* (ii), ϕ^* *induces an isometry from* $L(X')$ *to* $L(X)$. *For simplicity, we also denote it by* ϕ^*), (iii) $\phi^*(P^+_{X'}) = P^+_X$ *and* $\phi^*(C^n_{X'}) = C^n_X$, (iv) ϕ^* *sends* $H^{1,0}(E', \boldsymbol{C})$ *to* $H^{1,0}(E, \boldsymbol{C})$ *and* $\omega_X(\phi^*((\alpha_1, \alpha_2))) = J(\phi^*)(\omega_{X'}((\alpha_1, \alpha_2)))$ *(in* $J(E)$) *for* $(\alpha_1, \alpha_2) \in {}^0L(X')$, *where* $J(\phi^*)$ *is the isomorphism of Jacobian varieties induced from* ϕ^*: $H^{1,0}(E') \to H^{1,0}(E)$. *Then there is a sequence*

$$\{X \xrightarrow[\phi_1]{} X_1 \xrightarrow[\phi_2]{} \cdots \longrightarrow X_{r-1} \xrightarrow[\phi_r]{} X_r\}$$

of modifications of type I *and an isomorphism* ψ: $X_r \to X'$ *such that the associated isometry*

$$\phi_1^* \circ \cdots \circ \phi_r^* \circ \psi^* \colon {}^0L(X') \oplus {}^0W_1(X') \longrightarrow {}^0L(X) \oplus {}^0W_1(X)$$

agrees with ϕ^*.

(2.15) THEOREM. *For every point* $[(\tau, \omega)] \in \Omega$, *there is a marked quasi-stable* $K3$ *surface of type* II *with the period* $[(\tau, \omega)]$.

Proofs of (2.14), (2.15) will be given in Section 3.

(2.16) *Remark.* Let D be the period domain for smooth $K3$ surfaces. Let us recall that there is an étale covering $\tilde{D} \to D$ such that $\tilde{D}$ is a relevant moduli space for marked Kaehler $K3$ surfaces ([1], p. 239, or [10], Theorem (10.5)).

In our case, the corresponding situation is as follows: We let N denote the lattice $H \oplus (-E_8) \oplus (-E_8)$. As remarked in (2.9), (ii), we can construct the period domain $\Omega_0 := \{H^+ \times N_C\}/\boldsymbol{Z}^{2\times 18}$ for stable $K3$ surfaces by the same way for Ω. Here we select the lattice $\boldsymbol{Z}^{2\times 18} \subset N_C$ which contains N. Let W be the reflection group generated by $\Delta := \{\delta \in N;\ \langle \delta, \delta \rangle = -2\}$ and consider the space Ω'_0 consisting of pairs $([(\tau, \omega)], \kappa) \in \Omega_0 \times N_R$ satisfying $\langle \kappa, \kappa \rangle > 0$. Naturally W acts on Ω'_0: for $\delta \in \Delta$,

$$s_\delta \colon ([(\tau, \omega)], \kappa) \longrightarrow ([(\tau, s_\delta(\omega))], s_\delta(\kappa)) .$$

Let $\Omega''_0 \subset \Omega'_0$ denote the complement of the set of fixed points of reflections.

We define an equivalence relation $\sim$ on Ω_0'' by letting $([(\tau, \omega)], \kappa) \sim ([(\tau', \omega')], \kappa')$ if and only if $[(\tau, \omega)] = [(\tau', \omega')]$ and κ and κ' belong to the same connected component of $\Omega_0'' \cap ([(\tau, \omega)] \times N_{\boldsymbol{R}})$. Let $\tilde{\Omega}_0 := \tilde{\Omega}_0''/\sim$ denote the quotient space. It is provided with a canonical projection

$$\pi\colon \tilde{\Omega}_0 \longrightarrow \Omega_0 .$$

Then $\tilde{\Omega}_0$ receives the structure of analytic space, étale over Ω_0 ([10], Lemma (10.4)).

Let M be the set of isomorphism classes of marked stable $K3$ surfaces (with isomorphisms defined in the obvious manner). Then we associate a map $p\colon M \to \tilde{\Omega}_0$ which assigns to the isomorphism class of marked stable $K3$ surface (X, α_X) the equivalence class of $((\alpha_X(\tau_X), \alpha_X(\omega_X))$, the nodal chamber of X). Here we use the following fact which will be proved in Section 3, Proposition (3.4); If $\delta \in R_X$, then $\delta \in R_X^n$ if and only if $\omega_X(\delta) \equiv 0$ in $J(E)$.

In this situation we reformulate theorems (2.14), (2.15) as follows:

(i) The map $p\colon M \to \tilde{\Omega}_0$ is surjective.

(ii) Let (X, α_X) and $(X', \alpha_{X'})$ be two marked stable $K3$ surfaces whose images by the map p are contained in the same fibre of π, then there is a bimeromorphic map $X \dashrightarrow X'$ which is a composite of modifications of type I.

§ 3. Proofs of (2.14), (2.15)

Let $X = X_1 \cup X_2$ be a quasi-stable $K3$ surface and E the double curve. We fix an exceptional configuration $\{L_i^k\}$ of X. First we shall prove the following two lemmas. We keep the notation in Section 2.

(3.1) LEMMA. *If $\alpha \in \Delta_X - R_X^n$, then either $\{s_\alpha([L_i^k])\}$ is an exceptional configuration of X or there is a composition $\{\phi := \phi_1 \circ \phi_2\colon X' \underset{\phi_2}{\dashrightarrow} X'' \underset{\phi_1}{\dashrightarrow} X\}$ of modifications of type* I *such that $\{\phi^* \circ s_\alpha([L_i^k])\}$ is an exceptional configuration of X'. (We remark here that every reflection s_α, $\alpha \in \Delta_X$, is defined on $H^2(X_1; \boldsymbol{R}) \oplus H^2(X_2; \boldsymbol{R})$, and the expression $s_\alpha([L_i^k])$ is in this meaning.)*

(3.2) LEMMA. *If $w \in W_X$ is such that $w(C_X) \subset C_X^n$, then either $\{w([L_i^k])\}$ is also an exceptional configuration of X or there is a composition $\{X_r \underset{\phi_r}{\dashrightarrow} X_{r-1} \underset{\phi_{r-1}}{\dashrightarrow} \cdots \dashrightarrow X_1 \underset{\phi_1}{\dashrightarrow} X\}$ of modifications of type* I *such that $\{\phi_r^* \circ \cdots \circ \phi_1^* \circ w([L_i^k])\}$ is an exceptional configuration of X_r.*

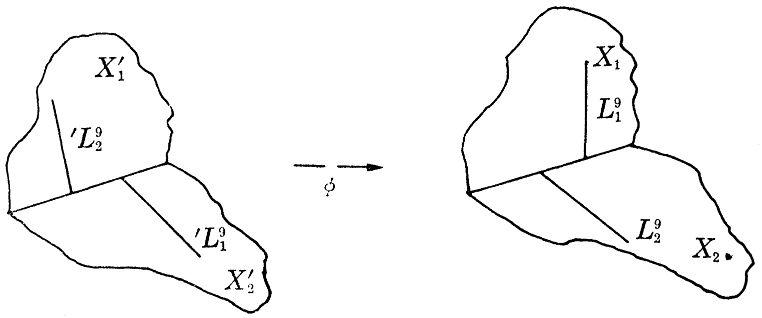
(3.3) *Remark.* Looijenga [9] has deeply studied rational surfaces with

https://doi.org/10.1017/S0027763000021462 Published online by Cambridge University Press

anti-canonical cycle. Our process in the above lemmas is similar to his method, but in our case modifications of type I occur, which make the argument more complicated (see [9], § 4).

Proof of (3.1) (see [9], § 3). Let α be an element of Δ_X. If $\alpha = [L_i^{k-1}] - [L_i^k]$, then $\alpha \notin R_X^n$ if and only if L_i^k is not contained in L_i^{k-1}, hence L_i^k and L_i^{k-1} are disjoint (see (2.13)). Since s_α interchanges L_i^{k-1} and L_i^k and leaves all other $L_i^{k'}$ fixed, everything is obvious in this case. Next if $\alpha = [H_i] - [L_i^1] - [L_i^2] - [L_i^3]$, then the condition that $\alpha \notin R_X^n$ implies that $\pi_i(L_i^1)$, $\pi_i(L_i^2)$, $\pi_i(L_i^3)$ are not collinear. (Here π_i: $X_i \to \overline{X}_i$ is a relatively minimal model of X_i (see (2.5)).) Suppose that $\pi_i(H_i)$ is not a tangent line of E. Then $\pi_i(L_i^1)$, $\pi_i(L_i^2)$ and $\pi_i(L_i^3)$ are distinct. Moreover, by the assumption of the indices of the exceptional configuration, each L_i^k is a maximal exceptional curve in the sense that $L_i^k = \pi_i^{-1} \circ \pi_i(L_i^k)$. Now $s_\alpha([L_i^1]) = [H_i] - [L_i^2] - [L_i^3]$ is represented by the total transform of the line $\overline{X}_i = \boldsymbol{P}^2$ which passes through $\pi_i(L_i^2)$ and $\pi_i(L_i^3)$ minus $L_i^2 + L_i^3$. If we denote this representative $'L_i^1$ (and $'L_i^2$, resp. $'L_i^3$, the corresponding representatives of $s_\alpha([L_i^2])$, resp. $s_\alpha([L_i^3])$), then it is clear that $'L_i^1$, $'L_i^2$, $'L_i^3$ are disjoint and that any L_i^μ $(\mu > 3)$ which meets $'L_i^k$ $(1 \leqq k \leqq 3)$ is actually contained in $'L_i^k$. So $\{s_\alpha([L_i^k])\}$ is an exceptional configuration of X. The proof for the case that $\pi_i(H_i)$ is a tangent line is similar.

Last of all, if $\alpha = [L_1^9] + [L_2^9]$, the condition that $\alpha \notin R_X^n$ just means that the points $L_1^9 \cap E$ and $L_2^9 \cap E$ are distinct. Note that $s_\alpha([L_1^9] + [L_2^9]) = -[L_1^9] - [L_2^9]$, $s_\alpha([L_1^8] - [L_1^9]) = [L_1^8] + [L_2^9]$, $s_\alpha([L_2^8] - [L_2^9]) = [L_2^8] + [L_1^9]$ and $s_\alpha([L_i^k]) = [L_i^k]$ for $i = 1, 2$, $1 \leqq k \leqq 8$. Let ϕ: $X' \dashrightarrow X$ be the birational map obtained by the modifications of type I along the exceptional curves of the first kind L_1^9, L_2^9 (by the assumption of indices of $\{L_i^k\}$, L_1^9 and L_2^9 are first kind). Let $'L_1^9$ (resp. $'L_2^9$) be the exceptional curve on X' obtained by blowing up the point $L_1^9 \cap E$ (resp. $L_2^9 \cap E$).



Put $[{}'L_i^k] = \phi^*([L_i^k])$, $i = 1, 2$, $1 \leqq k \leqq 8$, Then we have that $\phi^*(-[L_1^9] - [L_2^9]) = [{}'L_1^9] + [{}'L_2^9]$, $\phi^*([L_1^8] + [L_2^9]) = [{}'L_1^8] - [{}'L_2^9]$ and $\phi^*([L_2^8] + [L_1^9]) = [{}'L_2^8] - [{}'L_1^9]$. Now it is easily check that $\{[{}'L_i^k]\}$ is an exceptional configuration. We leave the proof to the reader.

Proof of Lemma (3.2) (see [9], (3.5), (4.2)). First we claim that C_X^n contains C_X. For this purpose, it is sufficient to prove that Δ_X^n is contained in R_X^+ (see (2.13)). Let $\delta = [D]$ ($\in \Delta_X^n$) be a nodal class. If D has a component which is not contained in L_i^k, then obviously $\langle \delta, [H_1] + [H_2] \rangle > 0$. Since $[H_1] + [H_2] \in \bar{C}_X$, δ is contained in R_X^+. Now we assume that δ is represented by a divisor D whose each irreducible component is contained in some L_i^k. Since D is connected (by definition), D is one of the following two types: (i) D is a smooth rational curve with self-intersection -2, (ii) $D = C_1 + C_2$, where C_i is an exceptional curve of the first kind on X_i $(i = 1, 2)$ with $C_1 \cap E = C_2 \cap E$. If D is a smooth rational curve with self-intersection -2, then D is represented by $L_i^k - L_i^{k'}$ $(1 \leqq k < k' \leqq 9)$. Hence $[D]$ is a positive linear combination of elements of Δ_X:

$$[D] = ([L_i^k] - [L_i^{k+1}]) + \cdots + ([L_i^{k'-1}] - [L_i^{k'}])\ .$$

By (2.13), (*), $[D]$ is contained in R_X^+.

Next, if $D = C_1 + C_2$, where C_i is an exceptional curve of the first kind on X_i $(i = 1, 2)$ with $C_1 \cap E = C_2 \cap E$, then $[C_1] + [C_2]$ is a positive linear combination of elements of Δ_X:

$$\begin{aligned}[C_1] + [C_2] = ([L_1^j] - [L_1^{j+1}]) + \cdots + ([L_1^8] - [L_1^9]) + ([L_1^9] + [L_2^9])\\ + ([L_2^8] - [L_2^9]) + \cdots + ([L_2^k] - [L_2^{k+1}]).\end{aligned}$$

So in this case $[C_1] + [C_2] \in R_X^+$, too. Hence C_X is contained in C_X^n. Note that there are no hyperplane separating C_X from $w(C_X)$ which is orthogonal to some $\alpha \in \Delta_X^n$.

Now we prove (3.2). We pick a point $x_0 \in w(C_X)$ and denote the set $\{\alpha \in R_X^+; \langle \alpha, x_0 \rangle < 0\}$ by Φ_{x_0}. Then Φ_{x_0} corresponds to the set of hyperplanes orthogonal to $\alpha \in R_X$ which separate C_X and $w(C_X)$. By [15] Lemma 9 (in § 3), Φ_{x_0} is a finite set. Hence we can index the elements of Φ_{x_0} as follows: $\Phi_{x_0} = \{\alpha_1, \cdots, \alpha_k\}$ (with $k = \operatorname{card} \Phi_{x_0}$) such that the set

$$\{C_i = s_{\alpha_i} \circ \cdots \circ s_{\alpha_1}(C_X);\ i = 1, \cdots, k\}$$

is a chain of the fundamental chambers from C_X to $w(C_X)$; more precisely the intersection of C_{i-1} and $C_i = s_{\alpha_i}(C_{i-1})$ is a non-empty open set in the hyperplane $H_{\alpha_i} = \{x \in L(X)_R;\ \langle x, \alpha_i \rangle = 0\}$. Note that α_1 is contained in

$\varDelta_X$. Since C_X is contained in C_X^n, the condition that $\alpha_i \in R_X^+$ implies that $\alpha_i \notin R_X^n$ $(i = 1, \cdots, k)$. With induction on i, Lemma (3.2) now follows easily from (3.1).

Proof of Theorem (2.14). Let $\{L_i^k\}$ (resp. $\{'L_i^k\}$) be an exceptional configuration of X (resp. X'). Let C_X (resp. $C_{X'}$) be the fundamental chamber of X (resp. X') endowed with the exceptional configuration $\{L_i^k\}$ (resp. $\{'L_i^k\}$). By the assumptions in (2.14), we have that $\phi^*(P_{X'}^+) = P_X^+$, $\phi^*(C_{X'}^n) = C_X^n$. On the other hand, by (2.12), we have $\phi^*(R_{X'}) = R_X$. Hence both C_X and $\phi^*(C_{X'})$ are fundamental domains for the action of W_X on P_X^+ (see (2.11)). In particular, $w(C_X) = \phi^*(C_{X'})$ for some $w \in W_X$. It then follows that $w(\varDelta_X) = \phi^*(\varDelta_{X'})$. Hence, if necessary, changing the indices of X_1 and X_2 (or equivalently, replacing ϕ^* by $\iota \circ \phi^*$, where ι is the symmetry of the Coxeter diagram of W_X (see (2.11)), we can assume that

$$\begin{gathered} w([L_i^k] - [L_i^{k+1}]) = \phi^*(['L_i^k] - ['L_i^{k+1}])\,, \\ w([L_1^9] + [L_2^9]) = \phi^*(['L_1^9] + ['L_2^9]) \quad \text{and} \\ w([H_i] - [L_i^1] - [L_i^2] - [L_i^3]) = \phi^*(['H_i] - ['L_i^1] - ['L_i^2] - ['L_i^3])\,, \\ i = 1, 2, \quad k = 1, 2, \cdots, 8\,. \end{gathered}$$

Since $w(C_X)$ is contained in $\phi^*(C_{X'}^n) = C_X^n$, by applying Lemma (3.2), we get a sequence $\{X_r \xrightarrow[\phi_r]{} X_{r-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow[\phi_1]{} X_0 = X\}$ of modifications of type I such that $\{\phi_r^* \circ \cdots \circ \phi_1^* \circ w([L_i^k])\}$ is an exceptional configuration of $X_r = X_{1,r} \cup X_{2,r}$. We denote $\phi_r^* \circ \cdots \circ \phi_1^* \circ \phi^*$ by ψ^* and $\{\phi_r^* \circ \cdots \circ \phi_1^* \circ w([L_i^k])\}$ by $\{[L_{i,r}^k]\}$. Let $\pi_{i,r}\colon X_{i,r} \to \bar{X}_{i,r}$ be a relatively minimal model and let E_r be the double curve of X_r. Then we have that

$$\begin{gathered} \psi^*(['L_i^k] - ['L_i^{k+1}]) = [L_{i,r}^k] - [L_{i,r}^{k+1}]\,, \\ \psi^*(['L_1^9] + ['L_2^9]) = [L_{1,r}^9] + [L_{2,r}^9] \quad \text{and} \\ \psi^*(['H_i] - ['L_i^1] - ['L_i^2] - ['L_i^3]) = [H_{i,r}] - [L_{i,r}^1] - [L_{i,r}^2] - [L_{i,r}^3] \\ (i = 1, 2,\ k = 1, 2, \cdots, 8)\,, \end{gathered}$$

where $H_{i,r}$ is the total transform of the line in $\bar{X}_{i,r} = \boldsymbol{P}^2$ which passes through $\pi_{i,r}(L_{i,r}^1)$ and $\pi_{i,r}(L_{i,r}^2)$ (at least when $\pi_{i,r}(L_{i,r}^1) \neq \pi_{i,r}(L_{i,r}^2)$; otherwise take the tangent line of $\pi_{i,r}(E_r)$ at $\pi_{i,r}(L_{i,r}^1) = \pi_{i,r}(L_{i,r}^2)$). Let u_i^k (resp. $'u_i^k$) be the point $E_r \cap L_{i,r}^k$ (resp. $E' \cap {'L_i^k}$). Then by the equations $\psi^*(['L_1^k] - ['L_1^{k+1}]) = [L_{1,r}^k] - [L_{1,r}^{k+1}]$ and the assumption (iv) in (2.14), there is an isomorphism

$$\psi_0\colon E_r \longrightarrow E'$$

such that $\psi_0(u_1^k) = {}'u_1^k$ $(k = 1, \cdots, 9)$. Moreover, from $\psi^*([{}'L_1^9] + [{}'L_2^9]) = [L_{1,r}^9] + [L_{2,r}^9]$, we obtained

$$\psi_0(u_1^9) - \psi_0(u_2^9) = {}'u_1^9 - {}'u_2^9$$

(Here we consider E_r (resp. E') a group with the identity element u_1^1 (resp. ${}'u_1^1$)). So we get $\psi_0(u_2^9) = {}'u_2^9$. By the equations $\psi^*([{}'L_2^k] - [{}'L_2^{k+1}]) = [L_{2,r}^k] - [L_{2,r}^{k+1}]$, we conclude that $\psi_0(u_i^k) = {}'u_i^k$ $(i = 1, 2,\ k = 1, 2, \cdots, 9)$.

Let u_i (resp. ${}'u_i$) denote the third point at which $H_{i,r}$ intersects E_r (resp. ${}'H_i$ intersects E'). Then the linear system $|u_i + u_i^1 + u_i^2|$ (resp. $|{}'u_i + {}'u_i^1 + {}'u_i^2|$) gives an embedding $E_r \to P^2 = \overline{X}_{i,r}$ (resp. $E' \to P^2 = \overline{X}'_i$), $i = 1, 2$. By the above equation w.r.t. ψ^*, it follows that

$$\psi_0^* |{}'u_i + {}'u_i^1 + {}'u_i^2| = |u_i + u_i^1 + u_i^2| \qquad (i = 1, 2)\ .$$

Hence ψ_0 can extend to an isomorphism $\bar{\psi}_i\colon \overline{X}_{i,r} \to \overline{X}'_i$ $(i = 1, 2)$. Obviously $\bar{\psi}_i$ induces an isomorphism $\psi_i\colon X_{i,r} \to X'_i$ $(i = 1, 2)$. Moreover $\psi := \psi_1 \cup \psi_2\colon X_{1,r} \cup X_{2,r} \to X'_1 \cup X'_2$ is an isomorphism and by construction, $\psi^* = \phi_r^* \circ \cdots \circ \phi_1^* \circ \phi^*$ agrees with an isomorphism induced from ψ.

Proof of Theorem (2.15). Let $[(\tau, \omega)] \in \Omega$ be given. Let E be a smooth elliptic curve with the period $\{1, \tau\}$ and ω_E a holomorphic 1-form on E such that

$$\int_\alpha \omega_E = \tau\ , \qquad \int_\beta \omega_E = 1$$

for a suitable basis $\{\alpha, \beta\}$ of $H_1(E; Z)$. We regard a basis of L as a coordinate system of L_C and write

$$\omega = [(t_1, t_2, \cdots, t_{19})] \in L_C^*/Z^{2\times 19}\ ,$$

where $(t_1, t_2, \cdots, t_{19}) \in L_C^*$. Now we consider the following equations in the points $z_{\mu,j}$, $\mu = 1, 2$, $j = 0, 1, \cdots, 9$, on E: modulo $Z + Z\tau$,

(i) $\sum_{j=1}^9 \int_{z_{1,j}}^{z_{1,0}} \omega_E \equiv t_1\ , \qquad \sum_{j=1}^9 \int_{z_{2,0}}^{z_{2,j}} \omega_E \equiv t_2$

(ii) $\int_{z_{2,9}}^{z_{1,9}} \omega_E \equiv t_3$

(iii) $\sum_{j=1}^3 \int_{z_{1,i}}^{z_{1,0}} \omega_E \equiv t_4\ , \qquad \int_{z_{1,i+1}}^{z_{1,i}} \omega_E \equiv t_{i+4}\ (i = 1, \cdots, 7)$

(iv) $\sum_{j=1}^3 \int_{z_{2,0}}^{z_{2,j}} \omega_E \equiv t_{12}\ , \qquad \int_{z_{2,i}}^{z_{2,i+1}} \omega_E \equiv t_{i+12}\ (i = 1, \cdots, 7)\ .$

These equations are correspond to the expression (2.6) of the basis of ${}^0L$. By Jacobi's inversion theorem, we can solve these equations as follows:

First we take a point $z_{1,0} = p_0 \in E$, arbitrarily. Applying Jacobi theorem on the equation (iii), we can find $p_1, \cdots, p_8 \in E$ such that $\{z_{1,i} = p_i;\ i = 1, \cdots, 8\}$ satisfies the equation (iii). Next from the equation (i), we can find a point $p_9 \in E$ such that $\{z_{1,i} = p_i;\ i = 1, \cdots, 8, 9\}$ satisfies the first of the equation (i). Similarly from the equation (ii), there is a point $q_9 \in E$ such that

$$\int_{q_9}^{p_9} \omega_E \equiv t_3 \bmod \boldsymbol{Z} + \boldsymbol{Z}\tau .$$

Moreover, by the relations

$$\begin{aligned}&(z_{1,1} + \cdots + z_{1,9}) - (z_{2,1} + \cdots + z_{2,9}) \\ &\qquad = 9(z_{1,9} - z_{2,9}) + \textstyle\sum_{i=1}^{8} i(z_{1,i} - z_{1,i+1}) + \sum_{i=1}^{8} i(z_{2,i+1} - z_{2,i})\end{aligned}$$

and

$$\begin{aligned}&9z_{i,0} - z_{i,1} - \cdots - z_{i,9} \\ &\qquad = 3(3z_{i,0} - z_{i,1} - z_{i,2} - z_{i,3}) + 2(z_{i,1} - z_{i,2}) + 4(z_{i,2} - z_{i,3}) \\ &\qquad\quad + \textstyle\sum_{k=1}^{6} (7 - k)(z_{i,k+2} - z_{i,k+3}) ,\end{aligned}$$

we can write that

$$\int_{z_{2,0}}^{z_{1,0}} \omega_E \equiv \text{a linear combination of } \{t_1, t_2, \cdots, t_{19}\} .$$

Again, applying Jacobi theorem on this equation and the equation (iv), we can find $q_0, q_1, \cdots, q_8 \in E$ such that $\{z_{2,j} = q_j;\ 0 \leqq j \leqq 9\}$ is a solution of the equation (iv) and the second of the equation (i). Consequently we obtain the solution $\{z_{1,i} = p_i,\ z_{2,j} = q_j;\ 0 \leqq i, j \leqq 9\}$ of the equations (i)–(iv).

Let $\bar{X}_i$ be a copy of $\boldsymbol{P}^2$ $(i = 1, 2)$. Now we consider the embeddings $|3p_0|\colon E \to \bar{X}_1 = \boldsymbol{P}^2$, $|3q_0|\colon E \to \bar{X}_2 = \boldsymbol{P}^2$. Let X_1 (resp. X_2) be the surface obtained from $\bar{X}_1$ (resp. $\bar{X}_2$) by taking successive blowing ups at $p_1, \cdots, p_9$ (resp. $q_1, \cdots, q_9$). Let E_1 (resp. E_2) be the proper transform of E by the above blowing ups. We denote the induced isomorphism from E_1 to E_2 by ϕ. Then the surface X obtained from X_1 and X_2 by patching through E_1 and E_2 under the isomorphism ϕ is the required one.

Lastly we prove the following proposition which has been used in the reformulation of theorems (2.14), (2.15) (see (2.16)).

For $\alpha \in \Delta_X$, we can define $\omega_X(\alpha)$ by regarding α as an element in ${}^0L(X)$. We can also define $\omega_X(\alpha)$ for $\alpha \in R_X$, since α is represented by an element of $\boldsymbol{Z}\cdot\Delta_X$.

(3.4) PROPOSITION. *If $\alpha \in R_X$, then $\alpha \in R_X^n$ if and only if $\omega_X(\alpha) = 0$ in $J(E)$.*

Proof. Let $\delta(\in \Delta_X^n)$ be a nodal class. Then, by definition and (2.9), (iii), $\omega_X(\delta) = 0$ in $J(E)$. If $\alpha \in R_X^n$, then $\alpha \equiv 0 \bmod \boldsymbol{Z} \cdot \Delta_X^n$ and so $\omega_X(\alpha) = 0$ in $J(E)$. Conversely, if $\alpha \in R_X$ such that $\omega_X(\alpha) = 0$ in $J(E)$, then $\alpha = w(\beta)$ for some $\beta \in \Delta_X$, $w \in W_X$. Write $w = w'' \circ w'$ with $w'(C_X) \subset C_X^n$ and $w'' \in W_X^n$. According to (3.2), there is a sequence

$$\{X_r \xrightarrow[\phi_r]{} X_{r-1} \longrightarrow \cdots \xrightarrow[\phi_1]{} X_0 = X\}$$

of modifications of type I such that $\phi_r^* \circ \cdots \circ \phi_1^* \circ w'(\beta) = [L] + \xi[L']$ for some exceptional curves L, L' on X_r $(r \geqq 0)$, where $\xi = -1$ (resp. $\xi = 1$) if and only if L and L' lie on the same component of X_r (resp. on the distinct component). Let E_r be the double curve of X_r. Since $w'' \circ w'(\beta) \equiv w'(\beta) \bmod \boldsymbol{Z} \cdot \Delta_X^n$, $\omega_X(w'(\beta)) = \omega_X(w'' \circ w'(\beta)) = \omega_X(\alpha) = 0$ in $J(E)$. Hence, by (2.9), (iii), $\omega_{X_r}([L] + \xi[L']) = 0$ in $J(E_r)$. This implies $L \cap E_r = L' \cap E_r$ by Abele's theorem. It follows that either L contains L' or L does not lie on the component of X_r on which L' lies. Hence $[L] + \xi[L'] \in R_{X_r}^n$. By definition, each ϕ_i preserves the nodal classes and so $w'(\beta) \in R_X^n$. Consequently $\alpha = w'' \circ w'(\beta) \in R_X^n$.

REFERENCES

[1] Burns, D. and M. Rapoport, On the Torelli problem for $K3$ surfaces, Ann. Sci. École Norm. Sup., **4** (1975), 235–274.
[2] Friedman, R. D., Hodge theory, degenerations, and the global Torelli problem, Harvard Thesis (1981).
[3] ——, Global smoothings of varieties with normal crossings, Ann. of Math., **118** (1983), 75–114.
[4] Friedman, R. D. and D. Morrison editors, Birational geometry of degenerations, Progress in Math. (1983), Birkhauser.
[5] Kempf, G. et al., Toroidal embeddings, I, Lecture Note in Math., vol. **339**, Springer (1973).
[6] Kodaira, K., On compact analytic surface, II, Ann. of Math., **77** (1963), 563–626.
[7] Kulikov, V., Degenerations of $K3$ surfaces and Enriques surfaces, Math. USSR-Izv, **11** (1977), 957–989.
[8] Looijenga, E., Invariant theory for generalized root systems, Invent. Math., **61** (1980), 1–32.
[9] ——, Rational surfaces with anti-canonical cycle, Ann. of Math., **114** (1981), 267–322.
[10] Looijenga, E. and C. Peters, Torelli theorem for $K3$ surfaces, Compositio Math., **42** (1980), 145–186.
[11] Namikawa, Y., Type I degenerations of $K3$ surfaces. (in Japanese), Proc. Symp. Algebraic Geometry, Tohoku University (1980).

[12] Persson, U. and H. Pinkham, Degenerations of surfaces with trivial canonical bundle, Ann. of Math., **113** (1981), 45–66.
[13] Piatetskii-Shapiro, I. and I. Shafarevich, A Torelli theorem for algebraic surfaces of type $K3$, Math. USSR-Izv., **35** (1971), 530–572.
[14] Vinberg, E., Some arithmetical discrete groups in Lobatchevsky spaces, in Discrete Subgroups of Lie groups, Bombay, Oxford Univ. Press (1973), 323–348.
[15] ——, Discrete linear groups generated by reflections, Math. USSR-Izv., **5** (1971), (1971), 1083–1119.

Department of Mathematics
Faculty of Sciences
Nagoya University
Nagoya, 464 Japan

https://doi.org/10.1017/S0027763000021462 Published online by Cambridge University Press