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On Arithmetic Means of Sequences Generated by a Periodic Function

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Abstract. In this paper we prove the convergence of arithmetic means of sequences generated by a periodic function $\varphi(x)$, moreover if $\varphi(x)$ satisfies a suitable symmetry condition, we prove that their limit is $\varphi(0)$. Applications of previous results are given to study other means of sequences and the behaviour of a class of recursive series.

Introduction

Let Φ_{2T} be the class of real functions defined in R and periodic of period $2T$. For every $\varphi(x) \in \Phi_{2T}$ we consider the sequence $\{u_n\}$ given by

$$u_n = \varphi(n) \quad \forall n \in \mathrm{N};$$

We denote by $\{\mathcal{A}_{n,\varphi}\}$, $\{\mathcal{G}_{n,\varphi}\}$ and $\{\mathcal{M}^p_{n,\varphi}\}$, respectively, the sequence of arithmetic means, the sequence of geometric means and the sequence of power means of order p ($p \in \mathrm{R}^+$) of $\{u_n\}$.

Using a known fact we prove that the sequences $\{\mathcal{A}_{n,\varphi}\}$, $\{\mathcal{G}_{n,\varphi}\}$ and $\{\mathcal{M}^p_{n,\varphi}\}$ are convergent; moreover, if $\varphi(x)$ satisfies a suitable symmetry condition, the limit of these sequences is $\varphi(0)$. From these theorems we deduce, then, the convergence or the divergence of the class of recursive series $\sum_\lambda^\varphi$ that we have considered in [3] and in [4]; finally we give some examples to complete the theory.

We introduce, now, other notations. If x is a real number we denote, as usual, by $[x]$ the greatest integer less than or equal to x; we denote by $\{\tau_n\}$ the sequence defined by setting

$$\tau_n = n - \left[\frac{n}{2T}\right] 2T \quad \forall n \in \mathrm{N}.$$

If $T \in \mathrm{Q}^+$, we set $2T = \frac{u}{v}$ and suppose that u is prime to v.

1 Arithmetic Means and Applications

Theorem 1 *Let $\varphi(x) \in \Phi_{2T}$, and, if $T \in \mathrm{R}^+ - \mathrm{Q}$, let $\varphi(x)$ be bounded and Riemann-integrable in $[0, 2T]$. Then*[1]

$$\lim_{n\to\infty} \mathcal{A}_{n,\varphi} = \begin{cases} \frac{1}{2T}\int_0^{2T} \varphi(x)\,dx & \text{if } T \in \mathrm{R}^+ - \mathrm{Q} \\ \dfrac{\varphi(1) + \varphi(2) + \cdots + \varphi(u)}{u} & \text{if } T \in \mathrm{Q}^+. \end{cases}$$

[1] If $T \in \mathrm{R}^+ - \mathrm{Q}$ and $\varphi(x)$ satisfies an other suitable hypothesis the result is known (see [2, p. 48, Exercise 2.15]).

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Proof We suppose at first that $T \in \mathrm{R}^+ - \mathrm{Q}$. In this case, the sequence $\left\{\frac{\tau_n}{2T}\right\}$ is uniformly distributed in $[0,1]$ and therefore for a known fact (see for example, [1, p. 473] and [7, p. 3, Corollary 1.1]) we have

$$\lim_{n\to\infty} \frac{f\left(\frac{\tau_1}{2T}\right) + f\left(\frac{\tau_2}{2T}\right) + \cdots + f\left(\frac{\tau_n}{2T}\right)}{n} = \int_0^1 f(x)\,dx, \tag{1}$$

where $f(x) = \varphi(2Tx)$ is Riemann-integrable in $[0,1]$.

On the other hand we have also

$$\begin{aligned} \mathcal{A}_{n,\varphi} &= \frac{\varphi(1) + \varphi(2) + \cdots + \varphi(n)}{n} = \frac{\varphi(\tau_1) + \varphi(\tau_2) + \cdots + \varphi(\tau_n)}{n} \\ &= \frac{f\left(\frac{\tau_1}{2T}\right) + f\left(\frac{\tau_2}{2T}\right) + \cdots + f\left(\frac{\tau_n}{2T}\right)}{n}, \end{aligned} \tag{2}$$

and

$$\int_0^1 f(x)\,dx = \int_0^1 \varphi(2Tx)\,dx = \frac{1}{2T}\int_0^{2T} \varphi(x)\,dx. \tag{3}$$

Then, from (1), (2) and (3) the thesis follows easily.

If $T \in \mathrm{Q}^+$, $\forall n > u$, we have

$$\begin{aligned} \mathcal{A}_{n,\varphi} &= \frac{\varphi(1) + \varphi(2) + \cdots + \varphi(u) + \cdots + \varphi(n)}{n} \\ &= \frac{\left[\frac{n}{u}\right]\big(\varphi(1) + \varphi(2) + \cdots + \varphi(u)\big) + \varphi\left(\left[\frac{n}{u}\right]u + 1\right) + \cdots + \varphi(n)}{n}. \end{aligned} \tag{4}$$

Then, from (4) the desired conclusion follows, and the proof is completed. ■

Remark 1 Theorem 1 cannot be extended supposing that the function $\varphi(x)$ is Lebesgue-integrable. Indeed let us suppose that $f(x) \in \Phi_{2T}$ $(T \in \mathrm{R}^+ - \mathrm{Q})$ and that $f(x)$ is bounded, Lebesgue-integrable in $[0, 2T]$ and such that $\int_0^{2T} f(x)\,dx > 0$. Then let us consider the function $\varphi(x)$ obtained by extending by periodicity in R the function

$$g(x) = \begin{cases} f(x) & \text{if } x \in [0,2T] - E \\ 0 & \text{if } x \in E, \end{cases}$$

where E is the range of the sequence $\{\tau_n\}$. Therefore we have:

$$\varphi(x) \in \Phi_{2T}, \quad \varphi(n) = g(\tau_n) = 0 \quad \forall n \in \mathrm{N}, \quad \mathcal{A}_{n,\varphi} = 0 \quad \forall n \in \mathrm{N}$$

and

$$\int_0^{2T} \varphi(x)\,dx = \int_0^{2T} f(x)\,dx > 0.$$

Corollary 1 *Let $\varphi(x)$ satisfy the hypotheses of Theorem 1; moreover let $\varphi(x)$ satisfy the symmetry condition*[2] $\varphi(x) + \varphi(2T - x) = 2\varphi(T)$ $\forall x \in [0, T]$. *Then, $\forall T \in \mathrm{R}^+$ the sequence $\{\mathcal{A}_{n,\varphi}\}$ is convergent to $\varphi(0)$.*

[2]This condition is equivalent to saying that the function $\varphi(x) - \varphi(0)$ is an odd function.

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Proof We observe at first that from the symmetry condition it follows that $\forall k \in \mathrm{N}$

$$\varphi(kT) = \varphi(0)$$

and

$$\varphi(x) + \varphi(2kT - x) = 2\varphi(T) \quad \forall x \in [0, kT].$$

Therefore if $T \in \mathrm{Q}^+$ and u is even, taking into account that $u = 2Tv$, we have

$$\begin{aligned}\frac{\varphi(1) + \varphi(2) + \cdots + \varphi(u)}{u}& \\ &= \frac{\varphi(1) + \cdots + \varphi(vT-1) + \varphi(vT) + \varphi(vT+1) + \cdots + \varphi(2Tv-1) + \varphi(2Tv)}{u} \\ &= \varphi(T) = \varphi(0).\end{aligned}$$

If $T \in \mathrm{Q}^+$ and u is odd, we have

$$\begin{aligned}\frac{\varphi(1) + \varphi(2) + \cdots + \varphi(u)}{u}& \\ &= \frac{\varphi(1) + \cdots + \varphi\left(\frac{2vT-1}{2}\right) + \varphi\left(\frac{2vT+1}{2}\right) + \cdots + \varphi(2Tv-1) + \varphi(2Tv)}{u} \\ &= \varphi(T) = \varphi(0).\end{aligned}$$

Finally, if $T \in \mathrm{R}^+ - \mathrm{Q}$, we have easily

$$\int_0^{2T} \varphi(x)\, dx = 2T\varphi(T) = 2T\varphi(0),$$

and this completes the proof. ■

Corollary 2 *Let $\varphi(x) \geq 0\ \forall x \in \mathrm{R}$ and let $[\varphi(x)]^p$ ($p \in \mathrm{R}^+$) verify the hypotheses of Corollary 1. Then the sequence $\{\mathcal{M}^p_{n,\varphi}\}$ converges to $\varphi(0)$.*

Proof Since

$$\mathcal{M}^p_{n,\varphi} = \left(\frac{\big(\varphi(1)\big)^p + \big(\varphi(2)\big)^p + \cdots + \big(\varphi(n)\big)^p}{n}\right)^{\frac{1}{p}},$$

it is sufficient to notice that the sequence $\{\mathcal{A}_{n,\varphi^p}\}$ converges to $\big(\varphi(0)\big)^p$. ■

Corollary 3 *Let $\varphi(x) \in \Phi_{2T}$, $\inf \varphi(x) > 0$ and, if $T \in \mathrm{R}^+ - \mathrm{Q}$, let $\varphi(x)$ be bounded and Riemann-integrable in $[0, 2T]$. Then*

$$\lim_{n\to\infty} \mathcal{G}_{n,\varphi} = \begin{cases} e^{\frac{1}{2T}\int_0^{2T} \log \varphi(x)\, dx} & \text{if } T \in \mathrm{R}^+ - \mathrm{Q} \\ \sqrt[u]{\varphi(1)\cdot\varphi(2)\cdots\varphi(u)} & \text{if } T \in \mathrm{Q}^+. \end{cases}$$

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Proof It is sufficient to observe that the function $\log\varphi(x)$ satisfies the hypotheses of Theorem 1, and then the sequence $\{\mathcal{A}_{n,\log\varphi}\}$ converges. From this, indeed, and from the relation

$$\sqrt[n]{\varphi(1)\varphi(2)\cdots\varphi(n)} = e^{\frac{1}{n}\sum_{i=1}^{n}\log\varphi(i)} \quad \forall n \in \mathrm{N}$$

the desired conclusion follows immediately. ■

Corollary 4 *Let $\varphi(x)$ verify the hypotheses of Corollary 3; moreover let $\varphi(x)$ verify the symmetry condition*

$$\varphi(x)\varphi(2T - x) = \big(\varphi(T)\big)^2 \quad \forall x \in [0, T].$$

Then the sequence $\{\mathcal{G}_{n,\varphi}\}$ converges to $\varphi(0)$.

Proof It is sufficient to observe that the function $\log\varphi(x)$ satisfies the hypotheses of Corollary 1 and therefore the sequence $\{\mathcal{A}_{n,\log\varphi}\}$ converges to $\log\varphi(0)$. ■

Corollary 5 *Let $\varphi(x)$ verify the hypotheses of Corollary 4. Then the series* [3] $\sum_{\lambda}^{\varphi}$ *converges or diverges to $+\infty$ according as $\varphi(0) < 1$ or $\varphi(0) > 1$.*

Remark 2 Theorem 1 and the following corollaries are useful if the sequence $\{u_n\}$ is not convergent because in this case the well known theorems of Cesàro cannot be applicable.

This case really happens, for example, if $\varphi(x) \in \Phi_{2T} \cap C^{\circ}(\mathrm{R})$, $T \in \mathrm{R}^+ - \mathrm{Q}$ and $\varphi(x)$ is not constant. Indeed let $x_1, x_2 \in\,]0, 2T[$ and such that $\varphi(x_1) < \varphi(x_2)$. Then, by the continuity of $\varphi(x)$ there exists $\delta > 0$ such that

$$[x_1 - \delta, x_1 + \delta] \subseteq [0, 2T], \quad [x_2 - \delta, x_2 + \delta] \subseteq [0, 2T],$$
$$[x_1 - \delta, x_1 + \delta] \cap [x_2 - \delta, x_2 + \delta] = \varnothing$$

and moreover

$$\forall x' \in [x_1 - \delta, x_1 + \delta] \quad \text{and} \quad \forall x'' \in [x_2 - \delta, x_2 + \delta]$$

we have

$$\varphi(x'') - \varphi(x') > \frac{\varphi(x_2) - \varphi(x_1)}{2}. \tag{5}$$

On the other hand for Kronecker's theorem (see for example [5, p. 373]) $\forall\eta > 0$ there exist $n, m \in \mathrm{N} \cap [\eta, +\infty[$ such that

$$\tau_n \in [x_1 - \delta, x_1 + \delta] \quad \text{and} \quad \tau_m \in [x_2 - \delta, x_2 + \delta]. \tag{6}$$

[3] $\sum_{\lambda}^{\varphi}$ is the series whose terms are defined recursively by setting

$$\begin{cases} a_1 = \lambda \in \mathrm{R}^+ \\ a_{n+1} = \varphi(n)a_n \quad \forall n \in \mathrm{N}. \end{cases}$$

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From (5) and (6) it follows

$$\varphi(m) - \varphi(n) = \varphi(\tau_m) - \varphi(\tau_n) > \frac{\varphi(x_2) - \varphi(x_1)}{2},$$

and this proves, obviously, that the sequence $\{u_n\}$ is not regular.

Corollary 6 *Let $f(x)$ be bounded and Riemann-integrable in the interval $[a, b]$ and let $T \in \mathrm{R}^+ - Q$. Then*[4]

$$\int_a^b f(x)\,dx = (b-a) \lim_{n\to\infty} \mathcal{A}_{n,\varphi}, \tag{7}$$

where $\varphi(x)$ is obtained by extending by periodicity in R *the function $f\left(\frac{b-a}{2T}x + a\right)$, $x \in [0, 2T]$.*

Proof We have easily

$$\int_a^b f(x)\,dx = \frac{b-a}{2T} \int_0^{2T} f\left(\frac{b-a}{2T}t + a\right)\,dt = \frac{b-a}{2T} \int_0^{2T} \varphi(x)\,dx.$$

And from this relation the desired conclusion follows immediately by virtue of Theorem 1.

Example 1 Let $\varphi_1(x) = \sin x$. We see easily that $\varphi_1(x)$ satisfies the hypotheses of Corollary 1 (with $T = \pi$), then the sequence $\{\mathcal{A}_{n,\varphi_1}\}$ converges to 0.

We have so obtained, by other means, a well known result (see [6, p. 316, Example 5]).

Example 2 Let us consider the function

$$f(x) = \begin{cases} -x^2 + \pi x + \lambda & \text{for } x \in [0, \pi] \\ x^2 - 3\pi x + 2\pi^2 + \lambda & \text{for } x \in \,]\pi, 2\pi], \end{cases}$$

where $\lambda \in \mathrm{R}$.

Let $\varphi_2(x)$ be the function obtained by extending $f(x)$ by periodicity in R. We see easily that $\varphi_2(x)$ satisfies the hypotheses of Corollary 1 (with $T = \pi$), therefore the sequence $\{\mathcal{A}_{n,\varphi_2}\}$ converges to λ.

Example 3 Let us consider the function

$$\varphi_3(x) = ke^{\sin x},$$

where $k \in \mathrm{R}^+$.

We see easily that $\varphi_3(x)$ satisfies the hypotheses of Corollaries 4 and 5 (with $T = \pi$). Therefore the sequence $\{\mathcal{G}_{n,\varphi_3}\}$ converges to k and the series $\sum_\lambda^{\varphi_3}$ converges or diverges to $+\infty$ according as[5] $k < 1$ or $k > 1$.

[4] The formula (1) may be utilized to compute an approximate value of the integral $\int_a^b f(x)\,dx$.

[5] For $k = 1$ the series $\sum_\lambda^{\varphi_3}$ diverges to $+\infty$ because the general term does not tend to 0.

Example 4 Let

$$g(x) = \begin{cases} -x^2 + \pi x + k & \text{for } x \in [0, \pi] \\ \dfrac{k^2}{-x^2 + 3\pi x - 2\pi^2 + k} & \text{for } x \in]\pi, 2\pi], \end{cases}$$

where $k \in \mathrm{R}^+$, and let us consider the function $\varphi_4(x)$ obtained by extending $g(x)$ by periodicity in R. We see easily that the function $\varphi_4(x)$ satisfies the hypotheses of Corollaries 4 and 5 (with $T = \pi$). Therefore the sequence $\{\mathcal{G}_{n,\varphi_4}\}$ converges to k and the series $\sum_\lambda^{\varphi_4}$ converges or diverges to $+\infty$ according as[6] $k < 1$ or $k > 1$.

Example 5 Let

$$f(x) = \arctan x \quad x \in \left[0, \frac{3}{2}\right[,$$

and let us consider the function $\varphi_5(x)$ obtained by extending $f(x)$ by periodicity in R. We see easily that the function $\varphi_5(x)$ satisfies the hypotheses of Theorem 1 (with $2T = \frac{3}{2}$). Therefore the sequence $\{\mathcal{A}_{n,\varphi_5}\}$ converges to

$$\frac{\varphi_5(1) + \varphi_5(2) + \varphi_5(3)}{3} = \frac{\frac{\pi}{4} + \arctan\frac{1}{2}}{3}.$$

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[6]For $k = 1$ the problem of determining the behaviour of the series $\sum_\lambda^{\varphi_4}$ is open.

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